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ON WRONSKI'S THEOREM.

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THE theorem, considered by the author as an answer to the question "En quoi consistent les Mathématiques? N'y aurait-il pas moyen d'embrasser par un seul problème, tous les problèmes de ces sciences et de resoudre généralement ce problème universel?" is given without demonstration in his *Réfutation de la Théorie de Fonctions Analytiques de Lagrange,* Paris, 1812, p. 30, and reproduced (with, I think, a demonstration) in the *Philosophie de la Technie,* Paris, 1815; and it is also stated and demonstrated in the *Supplément à la Réforme de la Philosophie,* Paris, 1847, p. CIX *et seq.*; the theorem, but without a demonstration, is given in *Montferrier's Encyclopédie Mathématique* (Paris, no date), t. III. p. 398.

The theorem gives the development of a function Fx of the root of an equation

$$0 = fx + x_1 f_1 x + x_2 f_2 x + \&c.,$$

but it is not really more general than that for the particular case $0 = fx + x_1 f_1 x$; or say when the equation is $0 = \phi x + \lambda fx$.* Considering then this equation

$$\phi x + \lambda fx = 0,$$

let a be a root of the equation $\phi x = 0$; the theorem is

$$\begin{aligned} Fx = {} & F \\ & - \frac{\lambda}{1}\,\frac{1}{\phi'}\,\left| \, (\textstyle\int fF')' \, \right| \\ & + \frac{\lambda^2}{1\,.\,2}\,\frac{1}{\phi'^3} \begin{vmatrix} \phi', & (\textstyle\int f^2F')' \\ \phi'', & (\textstyle\int f^2F')'' \end{vmatrix} \begin{matrix} \frac{1}{1} \\ \end{matrix} \end{aligned}$$

* For in the result, as given in the text, instead of λfx write $x_1 f_1 x + x_2 fx + \&c.$, then expanding the several powers of this quantity, each determinant is replaced by a sum of determinants of the same order, and we have the expansion of Fx in powers of $x_1, x_2, \ldots$.

$$-\frac{\lambda^3}{1.2.3}\frac{1}{\phi'^6}\begin{vmatrix}\phi', & (\phi^2)', & (\int f^3F')' \\ \phi'', & (\phi^2)'', & (\int f^3F')'' \\ \phi''', & (\phi^2)''', & (\int f^3F')'''\end{vmatrix}\frac{1}{1.1.2}$$

$$+ \&c.,$$

where F, f, F', &c. denote $Fa, fa, F'a$, &c. and the accents denote differentiation in regard to a; the integral sign $\int$ is written instead of $\int_a$; this is introduced for symmetry only, and obviously disappears; in fact, we may equally well write

$$\begin{aligned}Fx &= F \\ &-\frac{\lambda}{1}\frac{1}{\phi'}fF' \\ &+\frac{\lambda^2}{1.2}\frac{1}{\phi'^3}\begin{vmatrix}\phi', & f^2F' \\ \phi'', & (f^2F')'\end{vmatrix}\frac{1}{1} \\ &-\frac{\lambda^3}{1.2.3}\frac{1}{\phi'^6}\begin{vmatrix}\phi', & (\phi^2)', & f^3F' \\ \phi'', & (\phi^2)'', & (f^3F')' \\ \phi''', & (\phi^2)''', & (f^3F')''\end{vmatrix}\frac{1}{1.1.2} \\ &+ \&c.\end{aligned}$$

I stop for a moment to remark that Laplace's theorem is really equivalent to Lagrange's; viz. in the first mentioned theorem we have $x = \phi(a + \lambda fx)$, that is $\phi^{-1}x = a + \lambda f\phi \,.\, \phi^{-1}x$, and then $Fx = F\phi \,.\, \phi^{-1}x$, viz. by Lagrange's theorem

$$Fx = F\phi + \frac{\lambda}{1}F\phi' \,.\, f\phi + \frac{\lambda^2}{1.2}\{F\phi' \,.\, (f\phi)^2\}' + \&c.,$$

where on the right hand $F\phi$ and $f\phi$ are each regarded as one symbol, the argument being always a and the accents denoting differentiation in regard to a, thus $F\phi'$ is

$$d_a \,.\, F\phi a = F'\phi a \,.\, \phi' a, \;\&c.,$$

viz. this is Laplace's theorem.

Suppose in Wronski's theorem $\phi x = x - a$; that is, let the equation be

$$x - a + \lambda\phi x = 0,$$

then each determinant reduces itself to a single term: thus the determinant of the third order is

$$\begin{vmatrix}(x-a)', & \{(x-a)^2\}', & .f^3F' \\ (x-a)'', & \{(x-a)^2\}'', & (f^3F')' \\ (x-a)''', & \{(x-a)^2\}''', & (f^3F')''\end{vmatrix},$$

where in the first and second columns the accents denote differentiation in regard to x, which variable is afterwards put $= a$; the determinant is thus

$$= \begin{vmatrix}1, & *, & * \\ 0, & 1.2, & * \\ 0, & 0, & (f^3F')''\end{vmatrix},$$

viz. it is

$$=1\,.\,1\,.\,2\,(f^3F')'',$$

and so in other cases; the formula is thus

$$Fx=F-\frac{\lambda}{1}fF'+\frac{\lambda^2}{1\,.\,2}(f^2F')'-\frac{\lambda^3}{1\,.\,2\,.\,3}(f^3F')''+\text{\&c.},$$

agreeing with Lagrange's theorem.

Suppose in general $\phi x=(x-a)\,\psi x$, or let the equation be

$$(x-a)\,\psi x+\lambda fx=0,$$

that is,

$$x-a+\lambda\,\frac{fx}{\psi x}=0:$$

we have then by Lagrange's theorem

$$Fx=F-\frac{\lambda}{1}F'\frac{f}{\psi}+\frac{\lambda^2}{1\,.\,2}\left\{F'\left(\frac{f}{\psi}\right)^2\right\}'-\frac{\lambda^3}{1\,.\,2\,.\,3}\left\{F'\left(\frac{f}{\psi}\right)^3\right\}''+\text{\&c.}$$

Consider for example the term $\left\{F'\left(\dfrac{f}{\psi}\right)^3\right\}''$; this is

$$=\left\{F'x\,.\,\frac{(x-a)^3\,(fx)^3}{(\phi x)^3}\right\}'',$$

the accents denoting differentiation in regard to x, and x being ultimately put $=a$; or, what is the same thing, it is

$$=\left(\frac{d}{d\theta}\right)^2\left[F'(a+\theta)\,\frac{\theta^3\,\{f(a+\theta)\}^3}{\{\phi\,(a+\theta)\}^3}\right],$$

the accents now denoting differentiation in regard to θ, and this being ultimately put $=0$. This is

$$\left(\frac{d}{d\theta}\right)^2\left[F'(a+\theta)\,\frac{\{f(a+\theta)\}^3}{\left(\phi'a+\dfrac{\theta}{1\,.\,2}\,\phi''a+\dots\right)^3}\right].$$

This may be written $\left(F'f^3\,\dfrac{1}{A^3}\right)''$, where

$$A=\phi'+\tfrac{1}{2}\theta\phi''+\tfrac{1}{6}\theta^2\phi''+\dots,$$

it being understood that as regards $F'f^3$, which is expressed as a function of a only (θ having been therein put $=0$), the exterior accents denote differentiations in respect to a, whereas in regard to A, $=\phi'+\frac{1}{2}\theta\phi''+\text{\&c.}$, they denote differentiation in regard to θ, which is afterwards put $=0$. And the theorem thus is

$$Fx=F-\frac{\lambda}{1}\left(F'f\,.\,\frac{1}{A}\right)+\frac{\lambda^2}{1\,.\,2}\left(F'f^2\,.\,\frac{1}{A^2}\right)'-\frac{\lambda^3}{1\,.\,2\,.\,3}\left(F'f^3\,.\,\frac{1}{A^3}\right)''+\text{\&c.}$$

This must be equivalent to Wronski's theorem; it is in a very different, and, I think, a preferable form; but the results obtained from the comparison are very interesting, and I proceed to make this comparison.

Taking the foregoing coefficient $\left(F'f^3\frac{1}{A^3}\right)''$ this should be equal to Wronski's term

$$\frac{1}{1.1.2}\frac{1}{\phi'^6}\begin{vmatrix} \phi', & (\phi^2)', & f^3F' \\ \phi'', & (\phi^2)'', & (f^3F')' \\ \phi''', & (\phi^2)''', & (f^3F')'' \end{vmatrix};$$

or, what is the same thing, the determinant should be

$$=1.1.2\phi'^6\left(\frac{1}{A^3}f^3F'\right)''$$

$$=1.1.2\phi'^6\left\{f^3F'\left(\frac{1}{A^3}\right)''+2(f^3F')'\left(\frac{1}{A^3}\right)'+(f^3F')''\frac{1}{A^3}\right\},$$

that is, the values of

$$1.1.2\phi'^6\frac{1}{A^3},\quad 1.1.2\phi'^6 2\left(\frac{1}{A^3}\right)',\quad 1.1.2\phi'^6\left(\frac{1}{A^3}\right)''$$

should be

$$=\phi'(\phi^2)''-\phi''(\phi^2)',\quad \phi'''(\phi^2)'-\phi'(\phi^2)''',\quad \phi''(\phi^2)'''-\phi'''(\phi^2)''$$

respectively. Or, what is the same thing, if

$$\frac{1}{\left(\phi'+\frac{\theta}{2}\phi''+\frac{\theta^2}{2.3}\phi'''+\dots\right)^3}=A_0+\frac{1}{1}A_1\theta+\frac{1}{1.2}A_2\theta^2+\dots,$$

then the last mentioned functions should be

$$1.1.2\phi'^6A_0,\quad 1.1.2\phi'^6 2A_1,\quad 1.1.2\phi'^6A_2.$$

We have

$$A_0=\frac{1}{\phi'^3},\quad A_1=-\frac{3}{2}\frac{\phi''}{\phi'^4},\quad A_2=-\frac{\phi'''}{\phi'^4}+\frac{3\phi''^2}{\phi'^5},$$

or the identities are

$$\begin{aligned} 2\phi'^3 &= \phi'(\phi^2)''-\phi''(\phi^2)', &&= \phi'(2\phi\phi''+2\phi'^2)-\phi''.2\phi\phi', \\ -6\phi''\phi'^2 &= \phi'''(\phi^2)'-\phi'(\phi^2)''', &&= \phi'''.2\phi\phi'-\phi'(2\phi\phi''+6\phi'\phi''), \\ +6\phi''^2\phi'-2\phi'''\phi'^2 &= \phi''(\phi^2)'''-\phi'''(\phi^2)'', &&= \phi''(2\phi\phi'''+6\phi'\phi'')-\phi'''(2\phi\phi''+2\phi'^2), \end{aligned}$$

which is right. And in like manner to verify the coefficient of λ^4, we should have to compare the first four terms of the expansion of

$$\frac{1}{\left(\phi'+\frac{\theta}{2}\phi''+\frac{\theta^2}{2.3}\phi'''+\dots\right)^4}.$$

with the determinants formed out of the matrix

$$\begin{vmatrix} \phi', & \phi'', & \phi''', & \phi'''' \\ (\phi^2)', & (\phi^2)'', & (\phi^2)''', & (\phi^2)'''' \\ (\phi^3)', & (\phi^3)'', & (\phi^3)''', & (\phi^3)'''' \end{vmatrix}.$$

The series of equalities may be presented as follows, writing as above A to denote the function

$$\phi' + \frac{\theta}{2}\phi'' + \frac{\theta^2}{2.3}\phi''' + \ldots,$$

$$\frac{1}{A} = \frac{1}{\phi'}.1,$$

$$\frac{1}{A^2} = \frac{-1}{\phi'^3}\begin{vmatrix} \theta, & 1 \\ \phi', & \phi'' \end{vmatrix}.\frac{1}{1},$$

$$\frac{1}{A^3} = \frac{+1}{\phi'^6}\begin{vmatrix} \tfrac{1}{2}\theta^2, & \tfrac{1}{2}\theta, & 1 \\ \phi', & \phi'', & \phi''' \\ (\phi^2)', & (\phi^2)'', & (\phi^2)''' \end{vmatrix}.\frac{1}{1.1.2},$$

$$\frac{1}{A^4} = \frac{-1}{\phi'^{10}}\begin{vmatrix} \tfrac{1}{6}\theta^3, & \tfrac{1}{6}\theta^2, & \tfrac{1}{3}\theta, & 1 \\ \phi', & \phi'', & \phi''', & \phi'''' \\ (\phi^2)', & (\phi^2)'', & (\phi^2)''', & (\phi^2)'''' \\ (\phi^3)', & (\phi^3)'', & (\phi^3)''', & (\phi^3)'''' \end{vmatrix}.\frac{1}{1.1.2.1.2.3},$$

&c.,

where in each case the function on the left hand is to be expanded only as far as the power of θ which is contained in the determinant: the numerical coefficients in the top-lines of the several determinants are the reciprocals of

$$n(n-1)\ldots 2.1,\quad n(n-1)\ldots 2,\quad n(n-1),\quad n,\quad 1,$$

where n is the index of the highest power of θ. The demonstration of Wronski's theorem therefore ultimately depends on the establishment of the foregoing equalities As a verification, in the fourth formula, write $\phi = e^a\,(a=0)$, we have

$$\left(\frac{\theta}{e^\theta - 1}\right)^4 \text{ or } \frac{1}{(1+\frac{1}{2}\theta+\frac{1}{6}\theta^2+\frac{1}{24}\theta^3+\ldots)^4} = -\tfrac{1}{12}\begin{vmatrix} \tfrac{1}{6}\theta^3, & \tfrac{1}{6}\theta^2, & \tfrac{1}{3}\theta, & 1 \\ 1, & 1, & 1, & 1 \\ 2, & 4, & 8, & 16 \\ 3, & 9, & 27, & 81 \end{vmatrix},$$

where the right hand is

$$\begin{aligned} &= -\tfrac{1}{12}(-1.12 + \tfrac{1}{3}\theta.72 - \tfrac{1}{6}\theta^2.132 + \tfrac{1}{6}\theta^3.72) \\ &= \qquad 1 - 2\theta + \tfrac{11}{6}\theta^2 - \theta^3, \end{aligned}$$

and expanding the left hand as far as θ^3, this is

$$\begin{array}{llll} = & 1 & = 1 & \\ - & 4\left(\tfrac{1}{2}\theta + \tfrac{1}{6}\theta^2 + \tfrac{1}{24}\theta^3\right) & -2\theta - & \tfrac{2}{3}\theta^2 - \tfrac{1}{6}\theta^3 \\ + & 10\left(\quad \tfrac{1}{4}\theta^2 + \tfrac{1}{6}\theta^3\right) & & + \tfrac{5}{2}\theta^2 + \tfrac{5}{3}\theta^3 \\ - & 20\left(\quad\quad \tfrac{1}{8}\theta^3\right) & & - \tfrac{5}{2}\theta^3 \\ \hline & & 1 - 2\theta + & \tfrac{11}{6}\theta^2 - \theta^3, \end{array}$$

which agrees.

Reverting to the above equations, and expanding the several terms $(\phi^2)' = 2\phi\phi'$, $(\phi^2)'' = 2\phi\phi'' + 2\phi'^2$, &c., then, since in each case the left-hand side contains ϕ', ϕ'', ϕ''', &c. but not ϕ, it is clear that on the right-hand side the terms involving ϕ must disappear of themselves; and assuming that this is so, the equality takes the more simple form obtained by writing in the foregoing expressions $\phi = 0$, viz. we thus have $(\phi^2)' = 0$, $(\phi^2)'' = 2\phi'^2$, &c. In order to simplify the formulæ, I replace the series ϕ', $\tfrac{1}{2}\phi''$, $\tfrac{1}{6}\phi'''$, $\tfrac{1}{24}\phi''''$, &c. by b, c, d, e, &c., and I thus find that they assume the following simple form, viz. writing

$$\Theta = b + c\theta + d\theta^2 + e\theta^3 + \&c.,$$

then we have

$$\frac{1}{\Theta} = \frac{1}{b}\,.\,1,$$

$$\frac{1}{\Theta^2} = -\frac{2}{b^3}\begin{vmatrix} \theta, & \tfrac{1}{2} \\ b, & c \end{vmatrix},$$

$$\frac{1}{\Theta^3} = +\frac{3}{b^6}\begin{vmatrix} \theta^2, & \tfrac{1}{2}\theta, & \tfrac{1}{3} \\ b\,, & c\,, & d \\ & b^2, & 2bc \end{vmatrix},$$

$$\frac{1}{\Theta^4} = -\frac{4}{b^{10}}\begin{vmatrix} \theta^3, & \tfrac{1}{2}\theta^2, & \tfrac{1}{3}\theta\,, & \tfrac{1}{4} \\ b\,, & c\,, & d\,, & e \\ & b^2, & 2bc, & 2bd + c^2 \\ & & b^3, & 3b^2c \end{vmatrix},$$

viz. for Θ^{-n} the right-hand gives the development as far as θ^{n-1}. It will be observed, that in the determinants the several lines are the coefficients in the expansions of Θ, Θ^2, Θ^3, &c. respectively.

The demonstration is very easy; it will be sufficient to take the equation for $\frac{1}{\Theta^4}$. Assume

$$\frac{1}{\Theta^4} = \dots r\theta^6 + q\theta^5 + p\theta^4 + \beta\theta^3 + \tfrac{1}{2}\gamma\theta^2 + \tfrac{1}{3}\delta\theta + \tfrac{1}{4}\epsilon,$$

where clearly $\epsilon = \frac{4}{b^4}$, and write also

$$\begin{aligned} \Theta &= B_1 + C_1\theta + D_1\theta^2 + E_1\theta^3 + \dots, \\ \Theta^2 &= \qquad\; B_2 + C_2\theta \;+ D_2\theta^2 + \dots, \\ \Theta^3 &= \qquad\qquad\quad B_3 \;\; + C_3\theta \;+ \dots, \end{aligned}$$

where $B_1 = b$, $B_2 = b^2$, $B_3 = b^3$; we wish to show that

$$\begin{aligned} \beta B_1 + \gamma C_1 + \delta D_1 + \epsilon E_1 &= 0, \\ \gamma B_2 + \delta C_2 + \epsilon D_2 &= 0, \\ \delta B_3 + \epsilon C_3 &= 0, \end{aligned}$$

for this being the case, neglecting the terms in θ^4, θ^5, &c., and writing

$$\beta\theta^3 + \tfrac{1}{2}\gamma\theta^2 + \tfrac{1}{3}\delta\theta + \epsilon\left(\tfrac{1}{4} - \frac{1}{\epsilon\Theta^4}\right) = 0,$$

then eliminating β, γ, δ, ϵ, we have

$$\left|\begin{array}{cccc} \theta^3, & \frac{1}{2}\theta^2, & \frac{1}{3}\theta, & \frac{1}{4} - \frac{1}{\epsilon\Theta^4} \\ B_1, & C_1, & D_1, & E_1 \\ & B_2, & C_2, & D_2 \\ & & B_3, & C_3 \end{array}\right| = 0,$$

in which equation the term which contains

$$\frac{1}{\Theta^4} \text{ is } + \frac{1}{\epsilon} B_1 B_2 B_3 \frac{1}{\Theta^4}, \quad = \tfrac{1}{4} b^{10} \frac{1}{\Theta^4};$$

and the equation thus is $\frac{1}{\Theta^4} = -\frac{4}{b^{10}}$ multiplied by the determinant without the term in question (that is, with $\frac{1}{4}$ for its corner term).

To prove the subsidiary theorems, multiply the expression of $\frac{1}{\Theta^4}$ by $\frac{1}{\theta^4}$, and differentiate in regard to θ, we have

$$\frac{4\,(\theta\Theta)'}{(\theta\Theta)^5} = \ldots - 2r\theta - q + \frac{\beta}{\theta^2} + \frac{\gamma}{\theta^3} + \frac{\delta}{\theta^4} + \frac{\epsilon}{\theta^5}.$$

Multiplying by

$$\theta\Theta = B_1\theta + C_1\theta^2 + D_1\theta^3 + E_1\theta^4,$$

we see that $B_1\beta + C_1\gamma + D_1\delta + E_1\epsilon$ is the coefficient of $\frac{1}{\theta}$ in $\frac{4\,(\theta\Theta)'}{(\theta\Theta)^4}$; and similarly $B_2\gamma + C_2\delta + E_2\epsilon$ is the coefficient of $\frac{1}{\theta}$ in $\frac{4\,(\theta\Theta)'}{(\theta\Theta)^3}$, and $B_3\delta + C_3\epsilon$ that of $\frac{1}{\theta}$ in $\frac{4\,(\theta\Theta)'}{(\theta\Theta)^2}$. Now, m being any positive integer, $\frac{1}{(\Theta\theta)^m}$ expanded in ascending powers of θ contains negative and positive powers of θ, but of course no logarithmic term; hence differentiating in regard to θ, $\frac{(\Theta\theta)'}{(\Theta\theta)^{m+1}}$ contains no term in $\frac{1}{\theta}$;* and the expressions in question are thus each $=0$; which completes the demonstration.

The foregoing formulæ giving the expansion of $\frac{1}{\Theta^n}$ up to θ^{n-1} in terms of the coefficients in the expansions of Θ, Θ^2, ... Θ^{n-1} are I think interesting.

* This is a well-known method made use of by Jacobi and Murphy.