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THEOREM IN REGARD TO THE HESSIAN OF A QUATERNARY FUNCTION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 193—197.]

I WISH to put on record the following expression for the Hessian of $P^k + \lambda P'^{k'}$, where P, P' are quaternary functions of (x, y, z, w) of the degrees l, l' respectively, and λ is a constant; the demonstration is tedious enough, but presents no particular difficulty.

I write (A, B, C, D) for the first derived functions of P; and $(a, b, c, d, f, g, h, l, m, n)$ for the second derived functions; and similarly for P'. The Hessian of P is thus

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix},$$

which is denoted by $(abcd)$; moreover, if in this determinant we substitute the accented letters for the letters of each line successively, the result is denoted by $(abcd')$; and so if we substitute the accented letters for the letters of each pair of lines successively, the result is denoted by $(abc'd')$. Observe that

$$abcd' = (a'\delta_a + b'\delta_b + \ldots)\, abcd \text{ and } abc'd' = \tfrac{1}{2}(a'\delta_a + b'\delta_b + \ldots)^2\, abcd.$$

The notation $(abcD'^2)$ is used to denote the determinant

$$- \begin{vmatrix} & A', & B', & C', & D' \\ A', & a, & h, & g, & l \\ B', & h, & b, & f, & m \\ C', & g, & f, & c, & n \\ D', & l, & m, & n, & d \end{vmatrix},$$

and from it we derive the expression $(abc'D'^2)$, viz.

$$abc'D'^2 = (a'\delta_a + b'\delta_b + \ldots)\, abcD'^2.$$

The final result is expressed in terms of the several functions $abcd$, $abcd'$, $abc'd'$, $ab'c'd'$, $a'b'c'd'$, $abcD'^2$, $a'b'c'D^2$, $abc'D'^2$, $a'b'cD^2$, viz. we have

$$
\begin{aligned}
\mathfrak{H}(P^k + \lambda P'^{k'}) = \;& k^4 (k-1)\left(\frac{1}{k-1} + \frac{l}{l-1}\right) P^{4k-4} \,.\, abcd \\
& + \lambda \left\{ \begin{array}{l} k^3 (k-1)\left(\dfrac{1}{k-1} + \dfrac{l}{l-1}\right) k' P^{3k-3} \left\{ \begin{array}{r} P'^{k'-1} \,.\, abcd' \\ + (k'-1) P'^{k'-2} \,.\, abcD'^2 \end{array} \right\} \\ - \dfrac{k^3 (k-1) k'}{(l-1)^2} [l'(l'-1) + l^2 (k-1)]\, P^{3k-4} P'^{k'} \,.\, abcd \end{array} \right\} \\
& + \lambda^2 \left\{ \begin{array}{l} k^2 k'^2 \qquad P^{2k-2} P'^{2k'-2}\, abc'd' \\ + k^2 k'^2 (k'-1)\, P^{2k-2} P'^{2k'-3}\, abc'D'^2 \\ + k^2 k'^2 (k-1)\, P^{2k-3} P'^{2k'-2}\, a'b'cD^2 \\ + k^2 k'^2 (k-1)(k'-1)\, P^{2k-3} P'^{2k'-3} \left\{ \begin{array}{l} \dfrac{l(l-1)}{(l'-1)^2}\, a'b'c'D^2 \,.\, P \\ + \dfrac{l'(l'-1)}{(l-1)^2}\, abcD'^2 \,.\, P' \\ - \dfrac{l^2}{(l'-1)^2}\, ab'c'd' \,.\, P^2 \\ + \dfrac{ll'}{(l-1)(l'-1)}\, abc'd' \,.\, PP' \\ - \dfrac{l'^2}{(l-1)^2}\, abcd' \,.\, P'^2 \end{array} \right\} \end{array} \right\} \\
& + \lambda^3 \left\{ \begin{array}{l} k'^3 (k'-1)\left(\dfrac{1}{k'-1} + \dfrac{l'}{l'-1}\right) k P'^{3k'-3} \left\{ \begin{array}{r} P^{k-1} \,.\, a'b'c'd \\ + (k-1) P^{k-2} \,.\, a'b'c'D^2 \end{array} \right\} \\ - \dfrac{k'^3 (k'-1) k}{(l'-1)^2} [l(l-1) + l'^2 (k'-1)]\, P^k P'^{3k'-4} \,.\, a'b'c'd' \end{array} \right\} \\
& + \lambda^4 \,.\, k'^4 (k'-1)\left(\frac{1}{k'-1} + \frac{l'}{l'-1}\right) P'^{4k'-4} \,.\, a'b'c'd'.
\end{aligned}
$$

In verification, I remark that, $\lambda = 0$, the formula becomes

$$\mathfrak{H}(P^k) = k^4 (k-1)\left(\frac{1}{k-1} + \frac{l}{l-1}\right) P^{4k-4} \,.\, abcd,$$

that is

$$= \frac{k^4 (kl-1)}{l-1} P^{4k-4} \,.\, abcd.$$

Hence, writing $P'=P$, which implies $k'=k$ and $l'=l$, we ought to have

$$\mathfrak{H}\,\{(1+\lambda)\,P^k\} = (1+\lambda)^4 \,.\, \frac{k^4(kl-1)}{l-1}\,P^{4k-4}\,.\,abcd.$$

But writing in the formula $P'=P$, it is to be observed that $abcd'=4abcd$, $abc'd'=6abcd$, $ab'c'd'=4abcd$, $a'b'c'd'=abcd$: moreover that $abcD'^2$ and $a'b'c'D^2$ are each $=abcD^2$, but that $abc'D'^2$ and $a'b'cD^2$ are each $=3abcD^2$, and (as is easily shown to be the case)

$$abcD^2 = \frac{l}{l-1}\,P\,.\,abcd.$$

Thus the whole coefficient of λ becomes

$$\left\{\begin{array}{l} k^4(k-1)\left(\dfrac{1}{k-1}+\dfrac{l}{l-1}\right)\left\{4+\dfrac{(k-1)\,l}{l-1}\right\} \\ -\,k^4\dfrac{(k-1)}{(l-1)^2}\{l\,(l-1)+l^2(k-1)\} \end{array}\right\} P^{4k-4}\,.\,abcd,$$

where the numerical factor is

$$= k^4(k-1)^2\left(\frac{1}{k-1}+\frac{l}{l-1}\right)\left(\frac{4}{k-1}+\frac{l}{l-1}-\frac{l}{l-1}\right)$$

$$= 4k^4(k-1)\left(\frac{1}{k-1}+\frac{l}{l-1}\right);$$

or, finally, it is

$$= \frac{4k^4(kl-1)}{l-1}.$$

The coefficient of λ^2 is

$$= k^4\left\{6-\frac{2l^2(k-1)^2}{(l-1)^2}\right\}P^{4k-4}\,.\,abcd$$

$$+\,k^4\left\{6\,(k-1)+\frac{2\,(k-1)^2\,l}{l-1}\right\}P^{4k-5}\,.\,abcD^2;$$

or, substituting for $abcD^2$ its value $=\dfrac{l}{l-1}P\,.\,abcd$, the expression is equal to $P^{4k-4}abcd$ into a numerical coefficient, which is

$$k^4\left\{6-\frac{2l^2(k-1)^2}{(l-1)^2}+\left(\frac{6\,(k-1)\,l}{l-1}+\frac{2\,(k-1)^2\,l^2}{(l-1)^2}\right)\right\},$$

viz. this is

$$6k^4\left\{1+\frac{(k-1)\,l}{l-1}\right\}$$

$$= 6\,\frac{k^4(kl-1)}{l-1},$$

and the coefficients of λ^3, and λ^4 are equal to those of λ and λ^0 respectively. Hence the formula gives, as it should do,

$$\mathfrak{H}\,\{(1+\lambda)\,P^k\} = (1+\lambda)^4\,\frac{k^4(kl-1)}{l-1}\,P^{4k-4}\,.\,abcd.$$

Attending only to the form of the result, and representing the numerical factors by A, B, &c., we may write

$$\begin{aligned}
\mathfrak{H}(P^k+\lambda P'^{k'}) = \quad & A \quad P^{4k-4}\,abcd \\
+\lambda\,.\quad & B \begin{Bmatrix} P^{3k-3}\,P'^{k'-1}\,abcd' \\ +(k'-1)\,P^{3k-3}\,P'^{k'-2}\,abcD'^2 \end{Bmatrix} \\
& +C \quad P^{3k-4}\,P'^{k'}\,abcd \\
+\lambda^2\,.\quad & D \quad P^{2k-2}\,P'^{2k'-2}\,abc'd' \\
& +E \quad P^{2k-2}\,P'^{2k'-3}\,abc'D'^2 \\
& +E' \quad P^{2k-3}\,P'^{2k'-2}\,a'b'cD^2 \\
& +F \quad P^{2k-3}\,P'^{2k'-3}\,(\Lambda P+\Lambda' P') \\
+\lambda^3\,.\quad & C' \quad P^{k}\,P'^{3k'-4}\,a'b'c'd' \\
& +B' \begin{Bmatrix} P^{k-1}\,P'^{3k'-3}\,a'b'c'd \\ +(k-1)\,P^{k-2}\,P'^{3k'-3}\,a'b'c'D^2 \end{Bmatrix} \\
+\lambda^4\,.\quad & A' \quad P'^{4k'-4}\,a'b'c'd',
\end{aligned}$$

where, for shortness, certain terms in λ^2 have been represented by $\Lambda P+\Lambda' P'$.

Suppose $k=k'=2$; then attending only to the terms of the lowest order in P, P' conjointly, we have

$$\begin{aligned}
\mathfrak{H}(P^2+\lambda P'^2) = \quad & \lambda B\ .\ P^3\,.\,abcD'^2 \\
& +\lambda^2\ \ .\ PP'(\Lambda P+\Lambda' P') \\
& +\lambda^3 B'\,.\,P'^3\,.\,a'b'c'D^2.
\end{aligned}$$

If the function operated upon with $\mathfrak{H}$ had been $UP^2+U'P'^2$, the lowest terms in P, P' would have been of the like form; and it thus appears that for a surface of the form $UP^2+U'P'^2=0$, the nodal curve $P=0$, $P'=0$ is a triple curve on the Hessian surface.

If $k=2$, $k'=3$, then attending only to the terms of the lowest order in P, P' conjointly, we have

$$\begin{aligned}
\mathfrak{H}(P^2+\lambda P'^3) = \quad & A\,.\,P^4\,.\,abcd \\
& +\lambda\,.\,2B\,.\,P^3P'\ .\,abcD'^2;
\end{aligned}$$

and the like result would be obtained if the function operated upon with $\mathfrak{H}$ had been $UP^2+U'P'^3$. It thus appears that for a surface of the form $UP^2+U'P'^3=0$, the cuspidal curve $P=0$, $P'=0$ is a 4-tuple curve on the Hessian surface, the form in the vicinity of this line, or direction of the tangent plane, being given by

$$P^3(A\,.\,P\,.\,abcd+2B\lambda\,.\,P'\,.\,abcD'^2)=0,$$

viz. there is a triple sheet $P^3=0$, coinciding with the direction of the surface in the vicinity of the cuspidal line; and a single sheet

$$A\,.\,P\,.\,abcd+2B\lambda\,.\,P'\,.\,abcD'^2=0.$$

At the points for which the osculating plane of the curve $P=0$, $P'=0$ coincides with the tangent plane of $P=0$ (or, what is the same thing, with that of the surface), we have $abcD'^2=0$, and the triple and single sheets then coincide in direction.