571.

A DEMONSTRATION OF DUPIN'S THEOREM.

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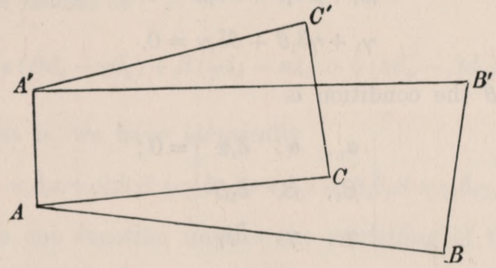
THE theorem is that three families of surfaces intersecting everywhere at right angles intersect along their curves of curvature. The following demonstration puts in evidence the geometrical ground of the theorem.

I remark that it was suggested to me by the perusal of a most interesting paper by M. Lévy, "Mémoire sur les coordonnées curvilignes orthogonales et en particulier sur celles qui comprennent une famille quelconque de surfaces de second degré," (*Jour. de l'École Polyt.*, Cah. 43 (1870), pp. 157—200). It was known that a family of surfaces $\rho=f(x, y, z)$ where the function is arbitrary, does not in general form part of an orthogonal system, but that ρ considered as a function of (x, y, z) must satisfy a partial differential equation of the third order. M. Lévy obtains a theorem which, in fact, enables the determination of this partial differential equation; he does not himself obtain it, although he finds what the equation becomes on writing therein $\frac{d\rho}{dx}=0$, $\frac{d\rho}{dy}=0$; but I have, in a recent communication to the French Academy, found this equation.

Proceeding to the consideration of Dupin's theorem, on a surface of the first family take a point A and through it two elements of length on the surface, AB, AC, at right angles to each other; draw at A, B, C the normals meeting the consecutive surface in A', B', C' and join $A'B'$, $A'C'$. It is to be shown that the condition in order that $B'A'C'$ may be a right angle is the same as the condition for the intersection of the normals AA' and BB' (or of the normals AA' and CC'); for this being so, since by hypothesis $B'A'C'$ is a right angle, it follows that AA', BB' intersect;

that is, that AB is an element of one of the curves of curvature through the point A of the surface. And, similarly, that AA', CC' intersect; that is, that AC is an element of the other of the curves of curvature through the point A on the surface.



Take x, y, z for the coordinates of the point A; α, β, γ for the cosine inclinations of AA'; α_1, β_1, γ_1 for those of AB; and α_2, β_2, γ_2 for those of AC. Write also

$$\delta = \alpha\, d_x + \beta\, d_y + \gamma\, d_z,$$
$$\delta_1 = \alpha_1 d_x + \beta_1 d_y + \gamma_1 d_z,$$
$$\delta_2 = \alpha_2 d_x + \beta_2 d_y + \gamma_2 d_z;$$

then it will be shown that the condition for the intersection of the normals AA', BB' is

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = 0,$$

the condition for the intersection of the normals AA', CC' is

$$\alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma = 0,$$

and that these are equivalent to each other, and to the condition for the angle $B'A'C'$ being a right angle.

Taking l, l_1, l_2 for the lengths AA', AB, AC, the coordinates of A', B, C measured from the point A are

$$(l\alpha,\ l\beta,\ l\gamma),\quad (l_1\alpha_1,\ l_1\beta_1,\ l_1\gamma_1),\quad (l_2\alpha_2,\ l_2\beta_2,\ l_2\gamma_2) \text{ respectively.}$$

The equations of the normal at A may be written

$$X = x + \theta\alpha,$$
$$Y = y + \theta\beta,$$
$$Z = z + \theta\gamma,$$

where X, Y, Z are current coordinates, and θ is a variable parameter. Hence for the normal at B, passing from the coordinates x, y, z to $x + l_1\alpha_1$, $y + l_1\beta_1$, $z + l_1\gamma_1$, the equations are

$$X = x + l_1\alpha_1 + l_1\delta_1(\theta\alpha),$$
$$Y = y + l_1\beta_1 + l_1\delta_1(\theta\beta),$$
$$Z = z + l_1\gamma_1 + l_1\delta_1(\theta\gamma),$$

and if the two normals intersect in the point (X, Y, Z), then

$$\alpha_1 + \alpha\delta_1\theta + \theta\delta_1\alpha = 0,$$
$$\beta_1 + \beta\delta_1\theta + \theta\delta_1\beta = 0,$$
$$\gamma_1 + \gamma\delta_1\theta + \theta\delta_1\gamma = 0,$$

viz. eliminating θ and $\delta_1\theta$ the condition is

$$\begin{vmatrix} \alpha_1, & \alpha, & \delta_1\alpha \\ \beta_1, & \beta, & \delta_1\beta \\ \gamma_1, & \gamma, & \delta_1\gamma \end{vmatrix} = 0;$$

or, since

$$\alpha_2,\ \beta_2,\ \gamma_2 = \beta\gamma_1 - \beta_1\gamma,\ \gamma\alpha_1 - \gamma_1\alpha,\ \alpha\beta_1 - \alpha_1\beta,$$

this is

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = 0.$$

Similarly the condition for the intersection of the normals AA', CC' is

$$\alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma = 0.$$

We have

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = \alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma;$$

in fact, this equation is

$$(\alpha_2\delta_1 - \alpha_1\delta_2)\,\alpha + (\beta_2\delta_1 - \beta_1\delta_2)\,\beta + (\gamma_2\delta_1 - \gamma_1\delta_2)\,\gamma = 0,$$

which I proceed to verify.

In the first term the symbol $\alpha_2\delta_1 - \alpha_1\delta_2$ is

$$\alpha_2(\alpha_1 d_x + \beta_1 d_y + \gamma_1 d_z) - \alpha_1(\alpha_2 d_x + \beta_2 d_y + \gamma_2 d_z),$$

viz. this is

$$(\alpha_2\beta_1 - \alpha_1\beta_2)\,d_y + (\gamma_1\alpha_2 - \gamma_2\alpha_1)\,d_z;$$

or, what is the same thing, it is

$$\beta d_z - \gamma d_y,$$

and the equation to be verified is

$$(\beta d_z - \gamma d_y)\,\alpha + (\gamma d_x - \alpha d_z)\,\beta + (\alpha d_y - \beta d_x)\,\gamma = 0,$$

viz. writing

$$\alpha,\ \beta,\ \gamma = \frac{X}{R},\ \frac{Y}{R},\ \frac{Z}{R},$$

where if $\rho = f(x, y, z)$ is the equation of the surface X, Y, Z are the derived functions $\dfrac{df}{dx}, \dfrac{df}{dy}, \dfrac{df}{dz}$, and $R = \sqrt{(X^2 + Y^2 + R^2)}$, the function on the left-hand consists of two parts; the first is

$$\frac{1}{R}\{(\beta d_z - \gamma d_y)\,X + (\gamma d_x - \alpha d_z)\,Y + (\alpha d_y - \beta d_x)\,Z\},$$

that is,

$$\frac{1}{R}\{\alpha(d_yZ-d_zY)+\beta(d_zX-d_xZ)+\gamma(d_xY-d_yX)\},$$

which vanishes; and the second is

$$-\frac{1}{R}\{\alpha(\beta d_z-\gamma d_y)+\beta(\gamma d_x-\alpha d_z)+\gamma(\alpha d_y-\beta d_x)\}R,$$

which also vanishes; that is, we have identically

$$\alpha_2\delta_1\alpha+\beta_2\delta_1\beta+\gamma_2\delta_1\gamma=\alpha_1\delta_2\alpha+\beta_1\delta_2\beta+\gamma_1\delta_2\gamma,$$

and the vanishing of the one function implies the vanishing of the other.

Proceeding now to the condition that the angle $B'A'C'$ shall be a right angle, the coordinates of B' are what those of A' become on substituting therein $x+l_1\alpha_1$, $y+l_1\beta_1$, $z+l_1\gamma_1$ in place of x, y, z; that is, these coordinates are

$$x+l\alpha+l_1\alpha_1+l_1\delta_1(l\alpha),\ \&c.,$$

or, what is the same thing, measuring them from A' as origin, the coordinates of B' are

$$\begin{aligned}&l_1(\alpha_1+l\delta_1\alpha+\alpha\,\delta_1 l),\\&l_1(\beta_1+l\delta_1\beta+\beta\delta_1 l),\\&l_1(\gamma_1+l\delta_1\gamma+\gamma\delta_1 l),\end{aligned}$$

and similarly those of C' measured from the same origin A' are

$$\begin{aligned}&l_2(\alpha_2+l\delta_2\alpha+\alpha\,\delta_2 l),\\&l_2(\beta_2+l\delta_2\beta+\beta\delta_2 l),\\&l_2(\gamma_2+l\delta_2\gamma+\gamma\,\delta_2 l).\end{aligned}$$

Hence the condition for the right angle is

$$\begin{aligned}&(\alpha_1+l\delta_1\alpha+\alpha\delta_1 l)(\alpha_2+l\delta_2\alpha+\alpha\delta_2 l)\\&+(\beta_1+l\delta_1\beta+\beta\delta_1 l)(\beta_2+l\delta_2\beta+\beta\delta_2 l)\\&+(\gamma_1+l\delta_1\gamma+\gamma\delta_1 l)(\gamma_2+l\delta_2\gamma+\gamma\delta_2 l)=0.\end{aligned}$$

Here the terms independent of l, $\delta_1 l$, $\delta_2 l$ vanish; and writing down only the terms which are of the first order in these quantities, the condition is

$$\begin{aligned}&\alpha_1(l\delta_2\alpha+\alpha\delta_2 l)+\alpha_2(l\delta_1\alpha+\alpha\delta_1 l)\\&+\beta_1(l\delta_2\beta+\beta\delta_2 l)+\beta_2(l\delta_1\beta+\beta\delta_1 l)\\&+\gamma_1(l\delta_2\gamma+\gamma\delta_2 l)+\gamma_2(l\delta_1\gamma+\gamma\delta_1 l)=0,\end{aligned}$$

where the terms in $\delta_1 l$, $\delta_2 l$ vanish; the remaining terms divide by l, and throwing out this factor, the condition is

$$(\alpha_1\delta_2\alpha+\beta_1\delta_2\beta+\gamma_1\delta_2\gamma)+(\alpha_2\delta_1\alpha+\beta_2\delta_1\beta+\gamma_2\delta_1\gamma)=0,$$

viz. by what precedes, this may be written under either of the forms

$$\alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma = 0,$$

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = 0,$$

and the theorem is thus proved.

It may be remarked that if we had simply the first surface, and two other surfaces, or say a second and a third surface, cutting the first surface and each other at right angles, that is, cutting each other in AA' the element of the normal at A, and cutting the first surface in the elements AB, AC at right angles to each other, then the tangent plane of the second surface will be the plane $A'AB$, not in general passing through B'; and the tangent plane of the third surface will be the plane $A'AC$, not in general passing through C'. The condition, that the elements $A'B'$ and $A'C'$ on the surface consecutive to the first surface are at right angles, makes CC' and BB' each intersect AA'; and we then have, the tangent plane of the second surface is the plane through the elements AA', BB', the tangent plane of the second surface is the plane through the elements AA', CC'.

As already remarked, a family of surfaces $\rho = f(x, y, z)$ where the function is arbitrary cannot form part of an orthogonal system. In fact, if the surfaces do belong to an orthogonal system, we have AA', BB' in the same plane, and consequently AB and $A'B'$ intersect; and, similarly, AC and $A'C'$ intersect; that is, if from a point A on a given surface of the family we pass along the normal to the point A' on the consecutive surface; and if the lines AB, AC are the tangents to the curves of curvature at A, and $A'B'$, $A'C'$ the tangents to the curves of curvature at A', then AB intersects $A'B'$, or, what is the same thing, AC intersects $A'C'$; and, conversely, when this condition is satisfied in general (that is, for every surface of the family and the surface consecutive thereto), then the family forms part of an orthogonal system; this is, in fact, the fundamental theorem of M. Lévy's memoir. The analytical form of the condition, viewed in this manner, is

$$\alpha_2\delta\alpha_1 + \beta_2\delta\beta_1 + \gamma_2\delta\gamma_1 = 0, \quad \text{or} \quad \alpha_1\delta\alpha_2 + \beta_1\delta\beta_2 + \gamma_1\delta\gamma_2 = 0;$$

or, as it is convenient to write it,

$$\alpha_2\delta\alpha_1 + \beta_2\delta\beta_1 + \gamma_2\delta\gamma_1 - (\alpha_1\delta\alpha_2 + \beta_1\delta\beta_2 + \gamma_1\delta\gamma_2) = 0;$$

and it was by means of it that I obtained the partial differential equation of the third order above referred to. The condition written in the form

$$X_2\delta X_1 + Y_2\delta Y_1 + Z_2\delta Z_1 = 0, \quad \text{or} \quad X_1\delta X_2 + Y_1\delta Y_2 + Z_1\delta Z_2 = 0,$$

presents itself in the proof of Dupin's Theorem by R. L. Ellis, (given in Gregory's *Examples*, Cambridge, 1841), but the geometrical signification of it is not explained.

Closely connected with Dupin's, we have the following theorem: if two surfaces intersect at right angles along a curve which is a curve of curvature of one of them, it is a curve of curvature of the other of them. I remark hereon as follows:

Let the intersection be a curve of curvature on the first surface; the successive normals intersect, giving rise to a developable, and the intersection of the two surfaces, say I, is an involute of the edge of regression of this developable, say of the curve C. The successive normals of the second surface are the lines at the different points of I at right angles to the planes of the developable, that is, to the osculating planes of C; or, what is the same thing, they are lines parallel to the binormals of C (the line at any point of a curve, at right angles to the osculating plane, is termed the "binormal"). But if the intersection I is a curve of curvature on the second surface, then the successive lines intersect; that is, starting from the curve C, the theorem in effect is that at each point of the involute drawing a line parallel to the binormal of the corresponding point of the curve, the successive lines intersect, giving rise to a developable. To prove this, let the arc s be measured from any fixed point of the curve, and the coordinates x, y, z be considered as functions of s; and let x', x'', x''' denote $\frac{dx}{ds}, \frac{d^2x}{ds^2}, \frac{d^3x}{ds^3}$, and the like as regards y and z. Measuring off on the tangent at the point (x, y, z) a length $l-s$, the locus of the extremity is the involute; that is, for the point (x, y, z) on the curve, the coordinates of the corresponding point on the involute are $x+(l-s)x'$, $y+(l-s)y'$, $z+(l-s)z'$. Moreover, the cosine inclinations of the binormal are as $y'z''-y''z'$, $z'x''-z''x'$, $x'y''-x''y'$. Hence taking X, Y, Z as current coordinates, the equations of the line parallel to the binormal may be written

$$\begin{aligned} X &= x+(l-s)x'+\theta(y'z''-y''z'),\\ Y &= y+(l-s)y'+\theta(z'x''-z''x'),\\ Z &= z+(l-s)z'+\theta(x'y''-x''y'), \end{aligned}$$

and the condition of intersection is therefore

$$\begin{vmatrix} x'', & y'z''-y''z', & (y'z''-y''z')' \\ y'', & z'x''-z''x', & (z'x''-z''x')' \\ z'', & x'y''-x''y', & (x'y''-x''y')' \end{vmatrix} = 0.$$

Form a minor out of the first and second columns, e.g.

$$y''(x'y''-x''y')-z''(z'x''-z''x'),$$

this is,

$$x'(x''^2+y''^2+z''^2)-x''(x'x''+y'y''+z'z''),$$

or the last term being $=0$, and the factor $x''^2+y''^2+z''^2$ being common, the minors are as $x' : y' : z'$. Moreover $(y'z''-y''z')'=y'z'''-y'''z'$, &c., hence the determinant is

$$x'(y'z'''-y'''z')+y'(z'x'''-z'''x')+z'(x'y'''-x'''y'),$$

viz. this is $=0$, or the theorem is proved.