570.

ON THE SUPERLINES OF A QUADRIC SURFACE IN FIVE-DIMENSIONAL SPACE.

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In ordinary or three-dimensional space a quadric surface has upon it two singly infinite systems of lines, such that each line of the one system intersects each line of the other system, but that two lines of the same system do not intersect.

In five-dimensional space* a quadric surface has upon it two triply infinite systems of superlines, such that each superline of either system intersects each superline of the same system; a superline of the one system does not in general intersect a superline of the opposite system, but it may do so, and then it intersects it not in a mere point, but in a line.

The theory will be established by an independent analysis, but it is, in fact, a consequence of the correspondence which exists between the lines of ordinary space and the points of a quadric surface in five-dimensional space. Thus the correspondence is

In ordinary space.	In five-dimensional space.
Line.	Point on quadric surface.
Lines meeting a given line.	Points which lie in tangent plane at given point.
Pair of intersecting lines.	Two points such that each lies in the tangent plane at the other, or say, pair of harmonic points.
Lines meeting each of two given lines.	Points lying in the sub-plane common to the tangent planes at two given points.

* In explanation of the nomenclature, observe that in 5 dimensional geometry we have: space, surface, subsurface, supercurve, curve, and point-system, according as we have between the six coordinates 0, 1, 2, 3, 4, or 5 equations: and so when the equations are linear, we have: space, plane, subplane, superline, line, and point. Thus in the text a quadric surface is the locus determined by a single quadric equation between the coordinates; and the superline and line are the loci determined by three linear equations and four linear equations respectively.

But in ordinary space if the two given lines intersect, then the system of lines meeting these, breaks up into two systems, viz. that of the lines which pass through the point of intersection, and that of the lines which lie in the common plane of the two given lines. It follows that in the five-dimensional space the intersection of the quadric surface by the subplane common to the tangent planes at two harmonic points must break up into a pair of superlines, viz. that we have on the quadric two systems of superlines; a superline of the one kind answering in ordinary space to the lines which pass through a given point, and a superline of the other kind answering to the lines which lie in a given plane. (Observe that, as regards the five-dimensional geometry, this is no distinction of nature between the two kinds of superlines, they are simply correlative to each other, like the two systems of generating lines of a quadric in ordinary space.)

Moreover, considering two superlines of the first kind, then answering thereto in ordinary space we have the lines through one given point, and the lines through another given point; and these systems have a common line, that joining the two given points; whence the two superlines have a common point. And, similarly, two superlines of the second kind have a common point. But taking two superlines of opposite kinds, then in ordinary space we have the lines through a given point, and the lines in a given plane: and the two systems have not in general any common line; that is, the two superlines have no common point. If, however, the given point lies in the given plane, then there is not one common line, but a singly infinite series of common lines, viz. all the lines in the given plane and through the given point; and corresponding hereto we have as the intersection of the two superlines, not a mere point, but a line.

Passing now to the independent theory, I consider, for comparison, first the case of the lines on a quadric surface in ordinary space; the equation of the surface may be taken to be

$$u^2 + v^2 - x^2 - y^2 = 0,$$

(u, v, x, y ordinary quadriplanar coordinates) and the equations of a line on the surface are

$$u = \alpha x + \beta y,$$
$$v = \alpha' x + \beta' y,$$

where α, β, α', β' are coefficients of a rectangular transformation, viz. we have $\alpha^2 + \beta^2 = 1$, $\alpha'^2 + \beta'^2 = 1$, $\alpha\alpha' + \beta\beta' = 0$; and therefore $(\alpha\beta' - \alpha'\beta)^2 = 1$, consequently $\alpha\beta' - \alpha'\beta = \pm 1$; and the lines will be of one or the other kind, according as the sign is + or −. It is rather more convenient to assume always $\alpha\beta' - \alpha'\beta = +1$, and write the equations

$$u = \alpha x + \beta y,$$
$$v = k(\alpha' x + \beta' y),$$

k denoting ± 1, and the lines being of the one kind or of the other kind, according as the sign is + or −.

Thus considering any two lines, the equations may be written

$$u = \alpha x + \beta y, \quad u = - \ (ax + by),$$
$$v = \alpha' x + \beta' y, \quad v = -k(a'x + b'y),$$

where the lines will be of the same kind or of different kinds, according as k is $=+1$ or $=-1$. Observe that k is introduced into one equation only; if it had been introduced into both, there would be no change of kind. If the lines intersect we have

$$(\alpha + a)x + (\beta + b)y = 0,$$
$$(\alpha' + ka')x + (\beta' + kb')y = 0,$$

viz. the condition of intersection is

$$\begin{vmatrix} \alpha + a, & \beta + b \\ \alpha' + ka', & \beta' + kb' \end{vmatrix} = 0,$$

that is,

$$\alpha\beta' - \alpha'\beta + k(\alpha b' - \alpha' b) + a\beta' - a'\beta + k(ab' - a'b) = 0,$$

or, what is the same thing,

$$1 + a\beta' - a'\beta + k(1 + \alpha b' - \alpha' b) = 0.$$

But we have, say

$$\alpha = \ \cos\theta, \quad \beta = \sin\theta, \quad a = \ \cos\phi, \quad b = \sin\phi,$$
$$\alpha' = -\sin\theta, \quad \beta' = \cos\theta, \quad a' = -\sin\phi, \quad b' = \cos\phi,$$

and thence

$$a\beta' - a'\beta = \cos(\theta - \phi) = \alpha b' - \alpha' b,$$

and the equation is

$$(1 + k)\{1 + \cos(\theta - \phi)\} = 0,$$

viz. this is satisfied if $k = -1$, i.e. if the lines are of opposite kinds, but not if $k = +1$. And it is important to remark that there is no exception corresponding to the other factor, viz. if $k = +1$, and $1 + \cos(\theta - \phi) = 0$, for we then have $\theta - \phi = \pi$, $\cos\phi = -\cos\theta$, $\sin\phi = -\sin\theta$, and consequently the two sets of equations for u, v become identical; that is, for lines of the same kind a line meets itself only.

Passing to the five-dimensional space, the equation of the quadric surface may be taken to be

$$u^2 + v^2 + w^2 - x^2 - y^2 - z^2 = 0,$$

and for a superline on the surface we have

$$u = \alpha x \ + \beta y \ + \gamma z \,,$$
$$v = \alpha' x \ + \beta' y \ + \gamma' z \,,$$
$$w = \alpha'' x + \beta'' y + \gamma'' z,$$

where (α, β, γ), &c., are the coefficients of a rectangular transformation; the determinant formed with these coefficients is $= \pm 1$, and the superline is of the one kind or the other, according as the sign is + or −. It is more convenient to take the determinant to be always +, and to write the equations in the form

$$\begin{aligned} u &= k(\alpha x + \beta y + \gamma z), \\ v &= k(\alpha' x + \beta' y + \gamma' z), \\ w &= k(\alpha'' x + \beta'' y + \gamma'' z), \end{aligned}$$

where $k = \pm 1$, and the superline is of the one or the other kind, according as the sign is + or −.

Now considering two superlines, we may write

$$\begin{aligned} u &= \alpha x + \beta y + \gamma z, & u &= -k(ax + by + cz), \\ v &= \alpha' x + \beta' y + \gamma' z, & v &= -k(a'x + b'y + c'z), \\ w &= \alpha'' x + \beta'' y + \gamma'' z, & w &= -k(a''x + b''y + c''z). \end{aligned}$$

If the superlines intersect, then

$$\begin{aligned} (\alpha + ka)\, x + (\beta + kb)\, y + (\gamma + kc)\, z &= 0, \\ (\alpha' + ka')\, x + (\beta' + kb')\, y + (\gamma' + kc')\, z &= 0, \\ (\alpha'' + ka'')\, x + (\beta'' + kb'')\, y + (\gamma'' + kc'')\, z &= 0, \end{aligned}$$

viz. the determinant formed with these coefficients must be $= 0$. The condition is at once reduced to

$$1 + k^3 + (k + k^2)(a\alpha + b\beta + c\gamma + a'\alpha' + b'\beta' + c'\gamma' + a''\alpha'' + b''\beta'' + c''\gamma'') = 0,$$

viz. it is satisfied when $k = -1$, that is, when the superlines are of the *same* kind; but not in general when $k = +1$.

If $k = +1$ the condition will be satisfied if

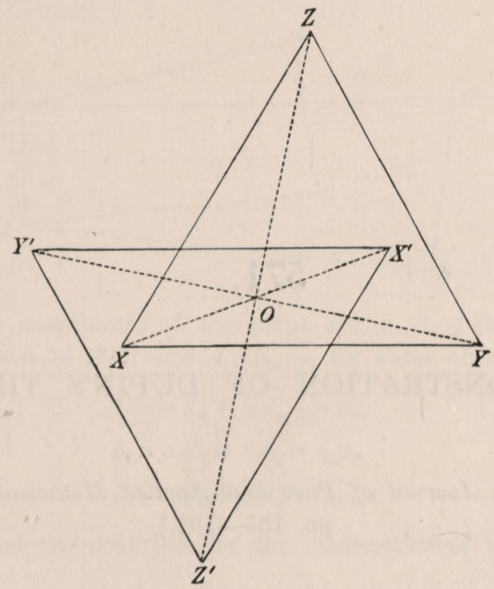
$$1 + a\alpha + b\beta + c\gamma + a'\alpha' + b'\beta' + c'\gamma' + a''\alpha'' + b''\beta'' + c''\gamma'' = 0,$$

and it is to be shown that then the three equations reduce themselves not to two equations, but to a single equation.

It is allowable to take the second set of equations to be simply $u = -kx$, $v = -ky$, $w = -kz$; for this comes to replacing the analytically rectangular system $ax + by + cz$, $a'x + b'y + c'z$, $a''x + b''y + c''z$ by x, y, z. Writing also $k = +1$, the theorem to be proved is that the equations

$$\begin{aligned} (\alpha + 1)\, x + \beta y + \gamma z &= 0, \\ \alpha' x + (\beta' + 1)\, y + \gamma' z &= 0, \\ \alpha'' x + \beta'' y + \gamma'' z &= 0, \end{aligned}$$

reduce themselves to a single equation, provided only $1 + \alpha + \beta' + \gamma'' = 0$; or, what is the same thing, we have to prove that the expressions $\beta'' - \gamma'$, $\gamma - \alpha''$, $\alpha' - \beta$ each vanish, provided only $1 + \alpha + \beta' + \gamma'' = 0$. This is a known theorem depending on the theory of the resultant axis, viz. the rotation round the resultant axis is then 180°, and we have $OX = OX'$, $OY = OY'$, $OZ = OZ'$, and thence we have evidently $YZ' = Y'Z$, $ZX' = Z'X$, $XY' = X'Y$.



But to prove it analytically, writing P, Q, R for $\beta'' - \gamma'$, $\gamma - \alpha''$, $\alpha' - \beta$ respectively, and Ω for $1 + \alpha + \beta' + \gamma''$, observe that we have identically

$$
\begin{aligned}
(\beta'' + \gamma\ )\,\Omega &= QR,\\
(\gamma\ + \alpha'\ )\,\Omega &= RP,\\
(\alpha'\ + \beta'')\,\Omega &= PQ,\\
(\beta'' + \gamma')\,P = (\gamma + \alpha'')\,Q &= (\alpha' + \beta)\,R,\\
(\alpha\ - 1)\,\Omega &= -\quad \gamma Q \quad + \quad \beta R \quad ,\\
\alpha'\Omega &= -\quad \gamma Q \quad + (1 + \beta')\,R,\\
\alpha''\Omega &= -(1 + \gamma'')\,Q + \quad \beta'' R \quad ,\\
\beta\Omega &= -(1 + \alpha\ )\,R + \quad \gamma P \quad ,\\
(\beta' - 1)\,\Omega &= -\quad \alpha' R \quad + \quad \gamma' P \quad ,\\
\beta''\Omega &= -\quad \alpha'' R \quad + (1 + \gamma'')\,P,\\
\gamma\Omega &= -\quad \beta P \quad + (1 + \alpha\ )\,Q,\\
\gamma'\Omega &= -(1 + \beta')\,P + \quad \alpha' Q \quad ,\\
(\gamma'' - 1)\,\Omega &= -\quad \beta'' P \quad + \quad \alpha'' Q \quad ,
\end{aligned}
$$

whence Ω being $= 0$, we have also $P = 0$, $Q = 0$, $R = 0$. The final conclusion is that the two superlines of opposite kinds, when they intersect, intersect in a line.