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NOTE ON THE CARTESIAN.

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THE following are doubtless known theorems, but the form of statement, and the demonstration of one of them, may be interesting.

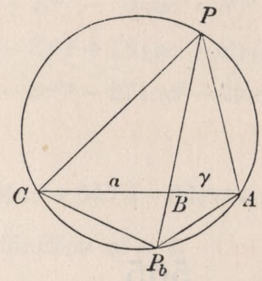
A point P on a Cartesian has three "opposite" points on the curve, viz. if the axial foci are A, B, C, then the opposite points are P_a, P_b, P_c where

P_a is intersection of line PA with circle PBC,
P_b „ „ PB „ PCA,
P_c „ „ PC „ PAB.

And, moreover, supposing in the three circles respectively, the diameters at right angles to PA, PB, PC are $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ respectively, then the points α, α', β, β', γ, γ' lie by threes in two lines passing through P, viz. one of these, say $P\alpha\beta\gamma$, is the tangent, and the other $P\alpha'\beta'\gamma'$ the normal, at P; and then the tangents and normals at the opposite points are $P_a\alpha$ and $P_a\alpha'$, $P_b\beta$ and $P_b\beta'$, $P_c\gamma$, and $P_c\gamma'$ respectively.

There exists a second Cartesian with the same axial foci A, B, C, and passing through the points P, P_a, P_b, P_c (which are obviously opposite points in regard thereto); the tangent at P is $P\alpha'\beta'\gamma'$ and the normal is $P\alpha\beta\gamma$; and the tangent and the normal at the other points are $P_a\alpha'$ and $P_a\alpha$, $P_b\beta'$ and $P_b\beta$, $P_c\gamma'$ and $P_c\gamma$ respectively: viz. the two curves cut at right angles at each of the four points.

Starting with the foci A, B, C and the point P, the points P_a, P_b, P_c are constructed as above, without the employment of the Cartesian; there are through P with the foci A, B, C two and only two Cartesians; and if it is shown that these pass through one of the opposite points, say P_b, they must, it is clear, pass through the other two points P_a, P_c. I propose to find the two Cartesians in question. To fix the ideas, let the points C, B, A be situate in order as shown in the figure, their



distances from a fixed point O being a, b, c, so that writing $\alpha, \beta, \gamma = b-c, c-a, a-b$ respectively, we have $\alpha+\beta+\gamma=0$, and α, γ will represent the positive distances CB and BA respectively, and $-\beta$ the positive distance AC. Suppose, moreover, that the distances PA, PB, PC regarded as positive are R, S, T respectively; and that the distances P_bA, P_bB, P_bC regarded as positive are R', S', T' respectively.

Suppose that for a current point Q the distances QA, QB, QC regarded as indifferently positive, or negative, are r, s, t respectively; then the equation of a bicircular quartic having the points A, B, C for axial foci is

$$lr+ms+nt=0,$$

where l, m, n are constants; and this will be a Cartesian if only

$$\frac{l^2}{\alpha}+\frac{m^2}{\beta}+\frac{n^2}{\gamma}=0.$$

We have the same curve whatever be the signs of l, m, n, and hence making the curve pass through P, we may, without loss of generality, write

$$lR+mS+nT=0,$$

R, S, T denoting the positive distances PA, PB, PC as above. We have thus for the ratios $l:m:n$, two equations, one simple, the other quadric; and there are thus two systems of values, that is, two Cartesians with the foci A, B, C, and passing through P.

I proceed to show that for one of these we have $-lR'+mS'+nT'=0$, and for the other $lR'+mS'-nT'=0$, or, what is the same thing, that the values of $l:m:n$ are

$$l:m:n=-(ST'+S'T):TR'+T'R:RS'-R'S,$$

and

$$l:m:n=\ (ST'-S'T):-(TR'+T'R):RS'+R'S;$$

viz. that the equations of the two Cartesians are

$$\begin{vmatrix} r, & s, & t \\ R, & S, & T \\ -R', & S', & T' \end{vmatrix}=0, \text{ and } \begin{vmatrix} r, & s, & t \\ R, & S, & T \\ R', & S', & -T' \end{vmatrix}=0,$$

respectively; this being so each of the Cartesians will, it is clear, pass through the point P_b, and therefore also through P_a and P_c.

The geometrical relations of the figure give

$$\begin{aligned} \alpha R^2 + \beta S^2 + \gamma T^2 &= -\alpha\beta\gamma, \\ \alpha R'^2 + \beta S'^2 + \gamma T'^2 &= -\alpha\beta\gamma, \\ RT' + R'T &= -\beta(S + S'), \\ \gamma\alpha &= SS', \\ \gamma TT' &= \alpha RR', \end{aligned}$$

to which might be joined

$$\begin{aligned} R'^2 S + \gamma^2 (S + S') + R^2 S' &= SS'(S + S'), \\ T'^2 S + \alpha^2 (S + S') + T^2 S' &= SS'(S + S'), \\ SR'T' &= S'RT, \\ SP'R' &= S'PR, \end{aligned}$$

but these are not required for the present purpose.

As regards the first Cartesian, we have to verify that

$$\frac{(ST' + S'T)^2}{\alpha} + \frac{(TR' + T'R)^2}{\beta} + \frac{(RS' - R'S)^2}{\gamma} = 0.$$

The left-hand side is

$$\frac{S^2T'^2 + S'^2T^2 + 2\gamma\alpha TT'}{\alpha} + \frac{\beta^2(S^2 + S'^2 + 2\gamma\alpha)}{\beta} + \frac{S^2R'^2 + S'^2R^2 - 2\gamma\alpha RR'}{\gamma},$$

viz. this is

$$= S^2\left(\frac{T'^2}{\alpha} + \beta + \frac{R'^2}{\gamma}\right) + S'^2\left(\frac{T^2}{\alpha} + \beta + \frac{R^2}{\gamma}\right) + 2\alpha\beta\gamma + 2(\gamma TT' - \alpha RR'),$$

which is

$$= S^2\left(\frac{-\beta S'^2}{\gamma\alpha}\right) + S'^2\left(\frac{-\beta S^2}{\gamma\alpha}\right) + 2\alpha\beta\gamma + 2(\gamma TT' - \alpha RR'),$$

and since the first and second terms are together $= -2\dfrac{\beta}{\gamma\alpha}S^2S'^2$, that is, $= -2\alpha\beta\gamma$, the whole is as it should be $= 0$.

In precisely the same manner we have

$$\frac{(ST' - S'T)^2}{\alpha} + \frac{(TR' + T'R)^2}{\beta} + \frac{(RS' + R'S)^2}{\gamma} = 0,$$

which is the condition for the second Cartesian: and the theorem in question is thus proved.