558.

A GEOMETRICAL INTERPRETATION OF THE EQUATIONS OBTAINED BY EQUATING TO ZERO THE RESULTANT AND THE DISCRIMINANTS OF TWO BINARY QUANTICS.

[From the *Proceedings of the London Mathematical Society*, vol. v. (1873—1874), pp. 31—33. Read March 12, 1874.]

CONSIDER the equations

$$U=(a\,,\ b\,,\ldots\between t,\ 1)^{\lambda}=0,$$
$$U'=(a',\ b',\ldots\between t,\ 1)^{\lambda'}=0\,;$$

and equating to zero the discriminants of the two functions respectively, and also the resultant of the two functions, let the equations thus obtained be

$$\Delta=(a\,,\ b\,,\ldots)^{2\lambda-2}=0,$$
$$\Delta'=(a',\ b',\ldots)^{2\lambda'-2}=0,$$
$$R=(a\,,\ b\,,\ldots)^{\lambda'}(a,\ b,\ldots)^{\lambda}=0.$$

I take $(a,\ b,\ldots)$, $(a',\ b',\ldots)$ to be linear functions of the coordinates $(x,\ y,\ z)$; and t to be an indeterminate parameter. Hence $U=0$ represents a line the envelope whereof is the curve $\Delta=0$, or, what is the same thing, the equation $U=0$ represents any tangent of the curve $\Delta=0$; this is a unicursal curve of the order $2\lambda-2$ and class λ, with $3(\lambda-2)$ cusps and $\tfrac{1}{2}(\lambda-2)(\lambda-3)$ nodes. Similarly $U'=0$ represents a line the envelope of which is the curve $\Delta'=0$: this is a unicursal curve of the order $2\lambda'-2$ and class λ', with $3(\lambda'-2)$ cusps and $\tfrac{1}{2}(\lambda'-2)(\lambda'-3)$ nodes; the equation $U'=0$ represents any tangent of this curve.

The equations $U=0$, $U'=0$ considered as existing simultaneously with the same value of t, establish a (1, 1) correspondence between the tangents (or if we please, between the points) of the two curves. The locus of the intersection of the corresponding tangents is the curve $R = 0$, a unicursal curve of the order $\lambda + \lambda'$, with $\frac{1}{2}(\lambda + \lambda' - 1)(\lambda + \lambda' - 2)$ nodes and no cusps; consequently of the class $2(\lambda + \lambda' - 1)$.

It is to be shown that the curve $R = 0$ touches the curve $\Delta = 0$ in $\lambda' + 2\lambda - 2$ points, and similarly the curve $\Delta' = 0$ in $2\lambda' + \lambda - 2$ points.

In fact, consider any tangent T' of the curve Δ'; let this meet the curve Δ in a point A, and let Q be the tangent at A to the curve Δ; suppose, moreover, that T is the tangent of Δ corresponding to the tangent T' of Δ'. Then if Q and T coincide, the corresponding tangent of T' will be Q, and the curve R will pass through A. It is easy to see that in this case the curves R, Δ will touch at A. Again, if P be a tangent from A to the curve Δ, then, if P and T coincide, the corresponding tangent of T' will be P, and the curve R will pass through A; but in this case the point A will be a mere intersection, not a point of contact, of the two curves.

The tangents T, Q each correspond to T', and they consequently correspond to each other. For a given position of T we have a single position of T', and therefore $2\lambda - 2$ positions of A, or, what is the same thing, of Q; that is, for a given position of T we have $2\lambda - 2$ positions of Q. Again, to a given position of Q corresponds a single position of A, therefore λ' positions of T', therefore also λ' positions of T; that is, for a given position of Q we have λ' positions of T. The correspondence between T, Q is thus a $(\lambda', 2\lambda - 2)$ correspondence, and the number of united tangents is therefore $\lambda' + 2\lambda - 2$, or the curves R, Δ touch in $\lambda' + 2\lambda - 2$ points.

The tangents T, P each correspond to T', and they therefore correspond to each other. For a given position of T we have a single position of T', and therefore $2\lambda - 2$ positions of A, and thence $(2\lambda - 2)(\lambda - 2)$ positions of P; that is, for a given position of T we have $(2\lambda - 2)(\lambda - 2)$ positions of P. Again, to a given position of P correspond $2\lambda - 4$ positions of A, therefore $(2\lambda - 4)\lambda'$ positions of T' or of T; that is, for a given position of P we have $(2\lambda - 4)\lambda'$ positions of T. The correspondence between T, P is thus a $[2\lambda'(\lambda - 2), 2(\lambda - 1)(\lambda - 2)]$ correspondence, and the number of united tangents is $2(\lambda + \lambda' - 1)(\lambda - 2)$; or the curves R, Δ meet in $2(\lambda + \lambda' - 1)(\lambda - 2)$ points.

Reckoning the contacts twice, the total number of intersections of R, Δ is

$$2\lambda' + 4\lambda - 4 + 2(\lambda + \lambda' - 1)(\lambda - 2), \quad = (\lambda + \lambda')(2\lambda - 2),$$

as it should be.

In the particular case $\lambda = \lambda' = 2$, the curves Δ, Δ' are conics, and the curve R is a quartic curve touching each of the conics 4 times; this is at once verified, since the equations here are

$$ac - b^2 = 0, \quad a'c' - b'^2 = 0, \quad 4(ac - b^2)(a'c' - b'^2) - (ac' + a'c - 2bb')^2 = 0.$$