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ON STEINER'S SURFACE.

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I HAVE constructed a model and drawings of the symmetrical form of Steiner's Surface, viz. that wherein the four singular tangent planes form a regular tetrahedron, and consequently the three nodal lines (being the lines joining the mid-points of opposite edges) a system of rectangular axes at the centre of the tetrahedron. Before going into the analytical theory, I describe as follows the general form of the surface: take the tetrahedron, and inscribe in each face a circle (there will be, of course, two circles touching at the mid-point of each edge of the tetrahedron; each circle will contain, on its circumference at angular distances of 120°, three mid-points, and the lines joining these with the centre of the tetrahedron, produced beyond the centre, meet the opposite edges, and are in fact the before-mentioned lines joining the mid-points of opposite edges). Now truncate the tetrahedron by planes parallel to the faces so as to reduce the altitudes each to three-fourths of the original value, and from the centre of each new face round off symmetrically up to the adjacent three circles; and within each circle scoop down to the centre of the tetrahedron, the bounding surface of the excavation passing through the three right lines, and the sections (by planes parallel to the face) being in the neighbourhood of the face nearly circular, but as they approach the centre, assuming a trigonoidal form, and being close to the centre an indefinitely small equilateral triangle. We have thus the surface, consisting of four lobes united only by the lines through the mid-points of opposite edges, these lines being consequently nodal lines; the mid-points being pinch-points of the surface, and the faces singular planes, each touching the surface along the inscribed circle. The joining lines, produced indefinitely both ways, belong as nodal lines to the surface; but they are, outside the tetrahedron, mere acnodal lines not traversed by any real sheet of the surface.

We may imagine the tetrahedron placed in two different positions, (1) resting with one of its faces on the horizontal plane, (2) with two opposite edges horizontal, or say with the horizontal plane passing through the centre of the tetrahedron and being parallel to two opposite edges; or, what is the same thing, the nodal lines form a system of rectangular axes, one of them, say that of z, being vertical. And I proceed to consider, in the two cases respectively, the horizontal sections of the surface.

In the first case, the coordinates x, y, z, w may be taken to be the perpendicular distances of a point from the faces of the tetrahedron, w being the distance from the base. We have*, if the altitude be h,

$$x+y+z+w=h;$$

an equation which may be used to homogenize any equation not originally homogeneous; thus, for the plane $w=\lambda$, of altitude λ, we have

$$w=\frac{\lambda}{h}(x+y+z+w),$$

or, what is the same thing,

$$w=\frac{\lambda}{h-\lambda}(x+y+z).$$

The equation of the surface is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{w}=0,$$

and if we herein consider w as having the last-mentioned value, the equation will belong to the section by the plane $w=\lambda$. I remark that the section of the tetrahedron, by this plane, is an equilateral triangle, the side of which is to an edge of the tetrahedron as $h-\lambda : h$. For a point in the plane of the triangle, if X, Y, Z are the perpendiculars on the sides, then

$$X+Y+Z=P,$$

(if for a moment P is the perpendicular from a vertex on the opposite side of the triangle, viz. we have $P=\dfrac{h-\lambda}{h}p$, if p be the perpendicular for a face of the tetrahedron). And it is clear that x, y, z are proportional to X, Y, Z; we consequently have, for the equation of the section,

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}+\sqrt{\frac{\lambda}{h-\lambda}(X+Y+Z)}=0,$$

* I take the opportunity of remarking that in a regular tetrahedron, if s be the length of an edge, p the perpendicular from a summit on an edge (or altitude of a face), h the perpendicular from a summit on a face (or altitude of the tetrahedron), and q the distance between the mid-points of opposite edges, then

$$s=\frac{\sqrt{3}}{\sqrt{2}}h,\quad p=\frac{3}{2\sqrt{2}}h,\quad q=\frac{\sqrt{3}}{2}h.$$

The tetrahedron can, by means of planes through the mid-points of the edges at right angles thereto, be divided into four hexahedral figures (8 summits, 6 faces, 12 edges, each face a quadrilateral); viz. in each such figure there are, meeting in a summit of the tetrahedron, three edges, each $=\frac{1}{2}s$; meeting in the centre three edges, each $=\frac{1}{4}h$; and six other edges, each $=\frac{1}{3}p$.

where the coordinates X, Y, Z are the perpendicular distances from the sides of the triangle which is the section of the tetrahedron. To simplify, I write

$$\frac{\lambda}{h-\lambda} = 2q+1,$$

that is,

$$q = \frac{2\lambda - h}{2h - 2\lambda};$$

the equation then is

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} + \sqrt{(2q+1)(X+Y+Z)} = 0;$$

or, proceeding to rationalize, we have first

$$q(X+Y+Z) = \sqrt{YZ} + \sqrt{ZX} + \sqrt{XY},$$

and thence

$$q^2(X+Y+Z)^2 - (YZ+ZX+XY) = 2\sqrt{XYZ}\,(\sqrt{X}+\sqrt{Y}+\sqrt{Z});$$

and finally

$$\{q^2(X+Y+Z)^2 - YZ - ZX - XY\}^2 = 4(2q+1)\,XYZ\,(X+Y+Z).$$

This is a quartic curve, having for double tangents the four lines $X=0$, $Y=0$, $Z=0$, $X+Y+Z=0$, the last of these being the line infinity touching the curve in two imaginary points, since obviously the whole real curve lies within the triangle. This is as it should be: the double tangents are the intersections of the plane $w=\lambda$ by the singular planes of the surface.

To find the points of contact, writing for instance $Z=0$, the equation becomes

$$q^2(X+Y)^2 - XY = 0,$$

that is,

$$X^2 + \left(2 - \frac{1}{q^2}\right)XY + Y^2 = 0;$$

whence

$$X = \left(-1 + \frac{1}{2q^2} \pm \sqrt{\frac{1}{4q^4} - \frac{1}{q^2}}\right)Z,$$

giving the two points of contact equi-distant from the centre; these are imaginary if $q > \frac{1}{2}$, but otherwise real, which agrees with what follows. (See the Table afterwards referred to.)

The nodal lines of the surface are $(x-y=0,\ z-w=0)$, $(y-z=0,\ x-w=0)$, $(z-x=0,\ y-w=0)$. Considering the first of these, we have for its intersection with the plane $w=\lambda$,

$$X=Y,\quad Z = \frac{\lambda}{h-\lambda}(X+Y+Z),\ = (2q+1)(X+Y+Z),$$

and the last equation gives

$$Z = (2q+1)(2X+Z),$$

that is,

$$0 = (2q+1)X + qZ,$$

so that for the point in question we have $X : Y : Z = -q : -q : 2q+1$; and taking the perpendicular from the vertex on a side as unity, the values $-q$, $-q$, $2q+1$ will be absolute magnitudes. We thus see that the curve must have the three nodes $(2q+1, -q, -q)$, $(-q, 2q+1, -q)$, $(-q, -q, 2q+1)$; and it is easy to verify that this is so.

The curve will pass through the centre $X = Y = Z$, if

$$(9q^2 - 3)^2 = 12(2q+1),$$

that is, if

$$3(3q^2-1)^2 - 4(2q+1) = 0,$$

or if

$$(3q+1)^2(q-1) = 0.$$

If $q = 1$, that is, $\lambda = 3(h-\lambda)$, or $\lambda = \frac{3}{4}h$, the equation is

$$(X^2 + Y^2 + Z^2 + YZ + ZX + XY)^2 - 12XYZ(X+Y+Z) = 0,$$

where the curve is, in fact, a pair of imaginary conics meeting in the four real points $(3, -1, -1)$, $(-1, 3, -1)$, $(-1, -1, 3)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. To verify this, observe that, writing

$$A = (Y-Z)(2X + Y + Z),$$
$$B = (Z-X)(X + 2Y + Z),$$
$$C = (X-Y)(X + Y + 2Z),$$

and therefore

$$A + B + C = 0,$$

the function in (X, Y, Z) is $= \frac{1}{2}(A^2 + B^2 + C^2)$, and thus the equation may be written in the equivalent forms

$$B^2 + BC + C^2 = 0, \quad C^2 + CA + A^2 = 0, \quad A^2 + AB + B^2 = 0,$$

each of which shows that the curve breaks up into two imaginary conics. The foregoing value $q = 1$, or $\lambda = \frac{3}{4}h$, belongs to the summit or highest real point of the surface.

In the case $3q + 1 = 0$, that is,

$$q = -\tfrac{1}{3} = \frac{2\lambda - h}{2h - 2\lambda}, \quad \text{or } \lambda = \tfrac{1}{4}h,$$

the equation is

$$\{(X+Y+Z)^2 - 9(YZ + ZX + XY)\}^2 = 108XYZ(X+Y+Z),$$

which is, in fact, the equation of a curve having the centre, or point $X = Y = Z$, for a triple point.

To verify this, write

$$X = \beta - \gamma + u,$$
$$Y = \gamma - \alpha + u,$$
$$Z = \alpha - \beta + u;$$

also

$$2\Delta = (\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2,$$
$$\Omega = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Then we have

$$X + Y + Z = 3u,$$
$$YZ + ZX + XY = 3u^2 - \Delta,$$
$$XYZ = u^3 - u\Delta + \Omega,$$

and the equation is

$$\{9u^2 - 9(3u^2 - \Delta)\}^2 - 324u(u^3 - u\Delta + \Omega) = 0,$$

that is,

$$(-2u^2 + \Delta)^2 - 4u(u^3 - u\Delta + \Omega) = 0,$$

or finally

$$\Delta^2 - 4u\Omega = 0,$$

where the lowest terms in $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ are of the order 3, and the theorem is thus proved. The case in question, $q = -\frac{1}{3}$ or $\lambda = \frac{1}{4}h$, is where the plane passes through the centre of the tetrahedron.

When $q = \frac{1}{2} = \dfrac{2\lambda - h}{2h - 2\lambda}$, or $\lambda = \frac{2}{3}h$, the equation is

$$(X^2 + Y^2 + Z^2 - 2YZ - 2ZX - 2XY)^2 = 128XYZ(X + Y + Z).$$

Here each of the lines $X=0$, $Y=0$, $Z=0$ is an osculating tangent having with the curve a 4-pointic intersection.

When $q = 0 = \dfrac{2\lambda - h}{2h - 2\lambda}$, or $\lambda = \frac{1}{2}h$, the equation is

$$(YZ + ZX + XY)^2 - 4XYZ(X + Y + Z) = 0,$$

that is,

$$Y^2Z^2 + Z^2X^2 + X^2Y^2 - 2XYZ(X + Y + Z) = 0;$$

viz. each angle of the triangle is here a cusp.

When $q = -\frac{1}{2}$, or $\lambda = 0$, the curve is

$$\{X^2 + Y^2 + Z^2 - 2(YZ + ZX + XY)\}^2 = 0,$$

viz. the plane is here the base of the tetrahedron, and the section is the inscribed circle taken twice.

For tracing the curves, it is convenient to find the intersections with the lines $Y-Z=0$, $Z-X=0$, $X-Y=0$ drawn from the centre of the triangle to the vertices; each of these lines passes through a node, and therefore besides meets the curve in two points. Writing, for instance, $Y=X$, the equation becomes

$$\{q^2(2X+Z)^2 - 2XZ - X^2\}^2 - 4(2q+1)X^2Z(2X+Z) = 0;$$

viz. this is

$$\{qZ + (2q+1)X\}^2\{q^2Z^2 + (4q^2 - 2q - 4)XZ + (4q^2 - 4q + 1)X^2\} = 0,$$

where the first factor gives the node. Equating to zero the second factor, we have

$$\left\{qZ+\left(2q-1-\frac{2}{q}\right)X\right\}^2=X^2\left\{\left(2q-1-\frac{2}{q}\right)^2-4q^2+4q-1\right\}$$

$$=X^2\frac{4}{q^2}(1-q)(1+2q);$$

or, finally,

$$qZ=\left\{-2q+1+\frac{2}{q}\pm\frac{2}{q}\sqrt{(1-q)(1+2q)}\right\}X,$$

giving two real values for all values of q from $q=1$ to $q=-\frac{1}{2}$. (See the Table afterwards referred to.)

We may recapitulate as follows:

$q>1$, or $\lambda>\frac{3}{4}h$; the curve is imaginary, but with three real acnodes, answering to the acnodal parts of the nodal lines:

$q=1$, or $\lambda=\frac{3}{4}h$; the summit appears as a fourth acnode:

$q<1>\frac{1}{2}$, or $\lambda<\frac{3}{4}h>\frac{2}{3}h$; the curve consists of three acnodes and a trigonoid lying within the triangle and having the sides of the triangle for bitangents of imaginary contact:

$q=\frac{1}{2}$, or $\lambda=\frac{2}{3}h$; the curve consists of three acnodes and a trigonoid having the sides of the triangle for osculating tangents:

$q<\frac{1}{2}>0$, or $\lambda<\frac{2}{3}h>\frac{1}{2}h$; the curve consists of three conjugate points and an indented trigonoid having the sides of the triangle for bitangents of real contact:

$q=0$, or $\lambda=\frac{1}{2}h$; curve has the summits of the triangle for cusps:

$q<0>-\frac{1}{3}$, or $\lambda<\frac{1}{2}h>\frac{1}{4}h$; curve has three crunodes, or say it is a cis-centric trifolium:

$q=-\frac{1}{3}$, or $\lambda=\frac{1}{4}h$; curve has a triple point, or say it is a centric trifolium:

$q<-\frac{1}{3}>-\frac{1}{2}$, or $\lambda<\frac{1}{4}h>0$; curve has three crunodes, or say it is a trans-centric trifolium:

$q=-\frac{1}{2}$, or $\lambda=0$; curve is a two-fold circle:

$q<-\frac{1}{2}$, or $\lambda<0$; the curve becomes again imaginary, consisting of three acnodes answering to the acnodal parts of the nodal lines.

For the better delineation of the series of curves, I calculated the following Table, wherein the first column gives a series of values of $\lambda : h$; the second the corresponding values of $q, =\dfrac{2\lambda-h}{2h-2\lambda}$; the third the positions of the point of contact, say with the side $Z=0$, the value of $X : Y$ being calculated from the foregoing formula,

$$X\div Y=-1+\frac{1}{2q^2}\pm\sqrt{\frac{1}{4q^4}-\frac{1}{q^2}};$$

and the fourth the apsidal distances, say for the radius vector $X = Y$, the value of $Z : X$ being calculated from the foregoing formula

$$Z \div X = -2 + \frac{1}{q} + \frac{2}{q^2} \pm \frac{2}{q}\sqrt{\left(\frac{1}{q} - 1\right)\left(2 + \frac{1}{q}\right)}.$$

The Table is:

	$\lambda : h$	q	Contact, $Z=0$, $X : Y=$		Apses, $X=Y$; $X : Z=$	
*	·75	1·00	imposs.		1·	
	·70	·6666	imposs.		·032 or	7·968
*	·666	·5	1·		0·	16·
	·65	·4285	·320 or	3·124	·006	22·44
	·6	·25	·0721	13·9279	·059	67·941
	·55	·1111	·011	78·988	·15	337·85
*	·5	0·	0	or ∞	·25	∞
	·45	− ·0909	·005	118·99	·39	457·61
	·4	− ·1666	·029	33·971	·496	127·504
	·35	− ·2308	·060	16·72	·648	61·79
*	·3	− ·2857	·099	10·151	·812	37·187
*	·25	− ·3333	·141	6·854	1·	25
	·2	− ·375	·207	4·904	1·218	17·89
	·15	− ·4118	·276	3·622	1·48	13·25
	·10	− ·4444	·372	2·690	1·813	9·937
	·05	− ·4737	·515 or	1·941	3·15 or	6·46
*	·0	− ·5	1·		4·	

where the asterisks show the critical values of $\lambda : h$.

It is worth while to transform the equation to new coordinates X', Y', Z' such that $X' = 0$, $Y' = 0$, $Z' = 0$ represent the sides of the triangle formed by the three nodes. Writing for shortness $X + Y + Z = P$, $YZ + ZX + XY = Q$, $XYZ = R$, the equation is

$$(q^2P - Q)^2 = 4(2q + 1)PR.$$

The expressions of X, Y, Z in terms of the new coordinates are of the form $X' + \theta P'$, $Y' + \theta P'$, $Z' + \theta P'$, where $P' = X' + Y' + Z'$; writing also $Q' = Y'Z' + Z'X' + X'Y'$, $R' = X'Y'Z'$, then the values of P, Q, R are

$$(1 + 3\theta)P', \quad Q' + (2\theta + 3\theta^2)P'^2, \quad R' + \theta P'Q' + (\theta^2 + \theta^3)P',$$

and the transformed equation is

$$[\{q^2(1+3\theta)^2-2\theta-3\theta^2\}P'^2-Q']^2=4(2q+1)(1+3\theta)P\{(\theta^2+\theta^3)P'^3+\theta P'Q'+R'\},$$

which is satisfied by $Q'=0$, $R'=0$, if only

$$\{q^2(1+3\theta)^2-2\theta-3\theta^2\}^2=4(2q+1)(1+3\theta)(\theta^2+\theta^3),$$

or, if for a moment $q(1+3\theta)=\Omega$, the equation is

$$(\Omega^2-2\theta-3\theta^2)^2=4(\theta^2+\theta^3)(2\Omega+1+3\theta),$$

that is,

$$\Omega^4+\Omega^2(-6\theta^2-4\theta)+\Omega(-8\theta^3-8\theta^2)-3\theta^4-4\theta^3=0,$$

that is,

$$(\Omega+\theta)^2(\Omega^2-2\theta\Omega-3\theta^2-4\theta)=0.$$

If the new axes pass through the nodes, then $\Omega+\theta=0$; that is, $q(1+3\theta)+\theta=0$, which equation gives the value of θ for which the new axes have the position in question; substituting in the first instance for q the value $\frac{-\theta}{3\theta+1}$, the equation becomes

$$\{2\theta(1+\theta)P'^2+Q'\}^2=4(1+\theta)P'\{\theta^2(1+\theta)P'^3+\theta P'Q'+R'\},$$

that is,

$$Q'^2=4(1+\theta)P'R';$$

or, finally, substituting for θ its value in terms of q, the required equation is,

$$Q'^2=4\frac{2q+1}{3q+1}P'R',$$

that is,

$$(Y'Z'+Z'X'+X'Y')^2=4\frac{2q+1}{3q+1}X'Y'Z'(X'+Y'+Z').$$

In particular, for $q=0$ the equation is

$$(Y'Z'+Z'X'+X'Y')^2-4X'Y'Z'(X'+Y'+Z')=0,$$

which is right, since, in the case in question (the tricuspidal curve), we have

$$X,\ Y,\ Z=X',\ Y',\ Z'.$$

I remark, in passing, that, taking the equation to be

$$(Y'Z'+Z'X'+X'Y')^2=mX'Y'Z'(X'+Y'+Z'),$$

we may write herein

$$Z'=\tfrac{1}{3}-\ x,$$

$$X'=\tfrac{1}{3}+\tfrac{1}{2}x-\frac{\sqrt{3}}{2}y,$$

$$Y'=\tfrac{1}{3}+\tfrac{1}{2}x+\frac{\sqrt{3}}{2}y;$$

where

$$x = \frac{2\sqrt{m(m-3)}}{9}\cos\theta - \frac{2(m-3)}{9}\cos 2\theta,$$

$$y = \frac{2\sqrt{m(m-3)}}{9}\sin\theta + \frac{2(m-3)}{9}\sin 2\theta,$$

which are the formulæ for the description of the trinodal quartic as a unicursal curve.

I consider now the second position; viz. the horizontal plane now passes through the centre of the tetrahedron, and is parallel to two opposite edges. The equations of the nodal lines are here $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$; and if for convenience we assume the distance of the mid-points of opposite edges to be $=2$, or the half of this $=1$, then the equations of the faces are

$$\begin{aligned} X &= x+y+z-1=0,\\ Y &= -x-y+z-1=0,\\ Z &= x-y-z-1=0,\\ W &= -x+y-z-1=0, \end{aligned}$$

and the equation of the surface is

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}+\sqrt{W}=0.$$

Proceeding to rationalise, this is

$$X+Y+2\sqrt{XY}=Z+W+2\sqrt{ZW},$$

viz.

$$2z+\sqrt{XY}=\sqrt{ZW};$$

we thence have

$$4z^2+4z\sqrt{XY}+XY=ZW;$$

or, since

$$ZW-XY=4z+4xy,$$

this is

$$z+xy-z^2=z\sqrt{XY};$$

whence

$$(z+xy-z^2)^2=z^2\{(z-1)^2-(x+y)^2\};$$

or reducing,

$$2xyz+y^2z^2+z^2x^2+x^2y^2=0,$$

a form which puts in evidence the nodal lines. Considering z as constant, we have the equation of the section; this is a quartic having the node $(x=0,\ y=0)$, and two other nodes at infinity on the two axes respectively; moreover, the curve has for bitangents the intersections of its plane with the faces of the tetrahedron; or what is the same thing, attributing to z its constant value, the equations of the bitangents are

$$\begin{aligned} x+y+z-1&=0,\\ -x-y+z-1&=0,\\ x-y-z-1&=0,\\ -x+y-z-1&=0. \end{aligned}$$

These lines form a rectangle which is the section of the tetrahedron; observe that this is inscribed in the square the corners of which are $x = \pm 1$, $y = \pm 1$; viz. $z = +1$ (highest section), this is the dexter diagonal (considered as an indefinitely thin rectangle), and as z diminishes, the longer side decreases and the shorter increases until for $z = 0$ (central section) the rectangle becomes a square; after which, for z negative it again becomes a rectangle in the conjugate direction, and finally, for $z = -1$ (lowest section) it becomes the sinister diagonal (considered as an indefinitely thin rectangle). But on account of the symmetry it is sufficient to consider the upper sections for which z is positive. The sides $\pm(x+y)+z-1=0$ parallel to the dexter diagonal of the square may for convenience be termed the dexter sides, and the others the sinister sides. In what follows I write c to denote the constant value of z.

We require to know whether the bitangents have real or imaginary contacts; and for this purpose to find the coordinates of the points of contact.

Take first a dexter bitangent $x+y+c-1=0$; the coordinates of any point hereof are

$$x = \tfrac{1}{2}(1-c+\theta), \quad y = \tfrac{1}{2}(1-c-\theta),$$

where θ is arbitrary; and substituting in the equation of the curve, we should have for θ a twofold quadric equation, giving the values of θ for the two points of contact respectively. We have

$$x^2+y^2 = \tfrac{1}{2}\{(1-c)^2+\theta^2\}, \quad xy = \tfrac{1}{4}\{(1-c)^2-\theta^2\},$$

and thence

$$8c^2\{(1-c)^2+\theta^2\} + 8c\{(1-c)^2-\theta^2\} + \{(1-c)^2-\theta^2\}^2 = 0,$$

viz. this equation is

$$\{\theta^2-(1-c)(1+3c)\}^2 = 0,$$

a twofold quadric equation, as it should be; and the values of θ being $= \pm\sqrt{(1-c)(1+3c)}$, we see that these, and therefore the contacts, are real from $c=1$ to $c=-\tfrac{1}{3}$.

In exactly the same way for a sinister bitangent $\pm(x-y)-c-1=0$, we have

$$x = \tfrac{1}{2}(1+c+\phi), \quad -y = \tfrac{1}{2}(1+c-\phi), \text{ and } \phi = \pm\sqrt{(1-3c)(1+c)},$$

viz. the values of ϕ, and therefore the contacts, are real from $c=\tfrac{1}{3}$ to $c=-1$.

That is,

	Contacts of Dexter Bitangents.	Contacts of Sinister Bitangents.
$c = 1$ to $\tfrac{1}{3}$	real,	imaginary,
$c = \tfrac{1}{3}$ to $-\tfrac{1}{3}$	real,	real,
$c = -\tfrac{1}{3}$ to -1	imaginary,	real;

or say $c=1$ to $\tfrac{1}{3}$, the contacts are real, imaginary; but $c=\tfrac{1}{3}$ to 0, they are real, real. In the transition case, $c=\tfrac{1}{3}$, the sinister bitangents become osculating (4-pointic) tangents touching at points on the dexter diagonal. This can be at once verified.

Observe that when $c=1$, we have

$$(x+y)^2+x^2y^2=0,$$

so that the only real point is $x=0$, $y=0$; viz. this is a tacnode, having the real tangent $x+y=0$. For $c=0$ (central section) the equation becomes $x^2y^2=0$; viz. the curve is here the two nodal lines each twice.

It is now easy to trace the changes of form.

$c=1$; curve is a tacnode, as just mentioned, tangent the dexter diagonal.

$c<1>\frac{1}{3}$; curve is a figure of 8 inside the rectangle, having real contacts with the dexter sides, but imaginary contacts with the sinister sides.

$c=\frac{1}{3}$; curve is a figure of 8 having real contacts with the dexter sides, and osculating (4-pointic) contacts with the sinister sides.

$c<\frac{1}{3}>0$; curve is an indented figure of 8 having real contacts as well with the sinister as the dexter sides.

$c=0$; curve is squeezed up into a finite cross, being the crunodal parts of the nodal lines; and joined on to these we have the acnodal parts, so that the whole curve consists of the lines $x=0$, $y=0$ each as a twofold line.

For tracing the curve, it is convenient to turn the axes through an angle of 45°; viz. writing $\frac{y+x}{\sqrt{2}}$, $\frac{y-x}{\sqrt{2}}$ in place of x, y respectively, the equation becomes

$$c(y^2-x^2)+c^2(y^2+x^2)+\tfrac{1}{4}(y^2-x^2)^2=0;$$

$$x=0 \text{ gives } y^2=0 \text{ or } y^2=-4c(1+c),^*$$

$$y=0 \text{ gives } x^2=0 \text{ or } x^2=4c(1-c).$$

Moreover, we have

$$4(c-c^2)(y^2-x^2)+8c^2y^2+(y^2-x^2)^2=0,$$

viz.

$$(x^2-y^2+2c^2-2c)^2=4c^2\{(c-1)^2-2y^2\},$$

and similarly

$$(y^2-x^2+2c^2+2c)^2=4c^2\{(c+1)^2-2x^2\},$$

putting in evidence the bitangents, now represented by the equations $c-1=\pm y\sqrt{2}$ and $c+1=\pm x\sqrt{2}$ respectively. And for the first of these, or $c-1=\pm y\sqrt{2}$, we have for the points of contact $x^2=\frac{1}{2}(1-c)(1+3c)$; and for the second of them, or $c+1=\pm x\sqrt{2}$, the points of contact are $y^2=\frac{1}{2}(1+c)(1-3c)$.

I consider the circumscribed cone having its vertex at a point $(0, 0, \gamma)$ on the nodal line $(x=0, y=0)$. Writing in the equation of the surface $x=\lambda(z-\gamma)$, $y=\mu(z-\gamma)$, the equation, throwing out the factor $(z-\gamma)^2$, becomes

$$2\lambda\mu z+(\lambda^2+\mu^2)z^2+\lambda^2\mu^2(z-\gamma)^2=0,$$

that is,

$$\begin{aligned}&(\lambda^2\mu^2+\lambda^2+\mu^2)z^2\\ &+2(-\gamma\lambda\mu+1)z\lambda\mu\\ &+\quad\gamma^2\quad.\quad\lambda^2\mu^2=0;\end{aligned}$$

* y always imaginary when c is positive.

and equating to zero the discriminant in regard to z, we have

$$\gamma^2(\lambda^2\mu^2+\lambda^2+\mu^2)-(-\gamma\lambda\mu+1)^2=0,$$

that is,

$$\gamma^2(\lambda^2+\mu^2)+2\gamma\lambda\mu-1=0;$$

and substituting herein the values $\lambda=\dfrac{x}{z-\gamma}$ and $\mu=\dfrac{y}{z-\gamma}$, we have the equation of the cone, viz. this is

$$\gamma^2(x^2+y^2)+2\gamma xy-(z-\gamma)^2=0,$$

or, what is the same thing,

$$\gamma^2(x^2+y^2-1)+2\gamma(xy+z)-z^2=0;$$

viz. this is a quadric cone having for its principal planes $z-\gamma=0$, $x+y=0$, $x-y=0$, these last being the planes through the nodal line and the two edges of the tetrahedron. In the particular case $\gamma=\infty$, the cone becomes the circular cylinder $x^2+y^2-1=0$.

The cone intersects the plane $z=0$ in the conic

$$\gamma^2(x^2+y^2-1)+2\gamma xy=0,$$

which is a conic passing through the corners of the square $(x=0,\ y=\pm 1)$, $(x=\pm 1,\ y=0)$. For $\gamma>1$, that is, for an exterior point, the conic is an ellipse having for the squares of the reciprocals of the semi-axes $1+\dfrac{1}{\gamma}$, $1-\dfrac{1}{\gamma}$ (this at once appears by writing in the equation $\dfrac{x+y}{\sqrt{2}}$, $\dfrac{x-y}{\sqrt{2}}$ in place of x, y respectively). In particular, for $\gamma=\infty$, the curve becomes the circle $x^2+y^2-1=0$. We have thus the apparent contour of the surface as seen from the point $z=\gamma$ on the nodal line, projected on the plane $z=0$ of the other two nodal lines.

To find the curve of contact of the cone and surface, or say the surface-contour from the same point, write for a moment

$$V=\gamma(x^2+y^2-1)+2\gamma(xy+z)-z^2,$$
$$U=(xy+z)^2+z^2(x^2+y^2-1);$$

then, substituting for x^2+y^2-1 its value in terms of V from the first equation, we find

$$U=\left(xy+z-\frac{z^2}{\gamma}\right)^2+\frac{z^2}{\gamma^2}V,$$

and the equations $U=0$, $V=0$ give therefore $xy+z-\dfrac{z^2}{\gamma}=0$, or say $\gamma(xy+z)-z^2=0$. The cone and surface therefore touch along the quadriquadric curve

$$\gamma^2(x^2+y^2-1)+2\gamma(xy+z)-z^2=0,$$
$$\gamma(xy+z)-z^2=0,$$

equations which may be replaced by

$$\gamma\,(x^2+y^2-1)+xy+z=0,$$
$$\gamma^2(x^2+y^2-1)+z^2\quad=0.$$

In the case $\gamma=\infty$, the equations are $x^2+y^2-1=0$, $xy+z=0$, viz. the curve is the intersection of the hyperbolic paraboloid $xy+z=0$ by the cylinder $x^2+y^2-1=0$.