NOTES AND REFERENCES.

1. As to the history of Determinants, see Dr Muir's "List of Writings on Determinants," *Quart. Math. Jour.* vol. XVII. (1882), pp. 110—149; and the interesting analyses of the earlier papers in course of publication by him in the *R. S. E. Proceedings*, vol. XIII. (1885—86) *et seq.*

The (new?) theorem for the multiplication of two determinants was given by Binet in his "Mémoire sur un système de formules analytiques &c." *Jour. École Polyt.* t. X. (1815), pp. 29—112.

An expression for the relation between the distances of five points in space, but not by means of a determinant or in a developed form, is given by Lagrange in the Memoir "Solutions analytiques de quelques problèmes sur les pyramides triangulaires," *Mém. de Berlin*, 1773: the question was afterwards considered by Carnot in his work "Sur la relation qui existe entre les distances respectives de cinq points quelconques pris dans l'espace, suivi d'un essai sur la théorie des transversales," 4to Paris, 1806. Carnot projected four of the points on a spherical surface having for its centre the fifth point, and then, from the relation connecting the cosines of the sides and diagonals of the spherical quadrilateral, deduced the relation between the distances of the five points: this is given in a completely developed form, containing of course a large number of terms.

Connected with the question we have the theorem given by Staudt in the paper "Ueber die Inhalte der Polygone und Polyeder," *Crelle* t. XXIV. (1842), pp. 252—256; the product of the volumes of two polyhedra is expressible as a rational and integral function of the distances of the vertices of the one from those of the other polyhedron.

More general determinant-formulæ relating to the "powers" of circles and spheres have been subsequently obtained by Darboux, Clifford and Lachlan: see in particular Lachlan's Memoir, "On Systems of Circles and Spheres," *Phil. Trans.* vol. CLXXVII. (1886), pp. 481—625.

2 and 3. The investigation was suggested to me by a passage in the *Mécanique Analytique*, Ed. 2 (1811), t. I. p. 113 (Ed. 3, p. 106); after referring to a formula of Laplace, whereby it appeared that the attraction of an ellipsoid on an exterior point depends only on the quantities $B^2 - A^2$ and $C^2 - A^2$ which are the squares of

the eccentricities of the two principal sections through the major semiaxis A, Lagrange remarks that, starting from this result and making use of a theorem of his own in the Berlin Memoirs 1792—93, he was able to construct the series by means of the development of the radical $1 \div \sqrt{x^2 + y^2 + z^2 - 2by - 2cz + b^2 + c^2}$ in powers of b, c, preserving therein only the even powers of b and c, and transforming a term such as $Hb^{2m}c^{2n}$ into a determinate numerical multiple of $\frac{4}{3}\pi ABC \,.\, H(B^2 - A^2)^m (C^2 - A^2)^n$.

It occurred to me that Lagrange's series must needs be a series

$$\Sigma \zeta_p \left(A^2 \frac{d^2}{da^2} + B^2 \frac{d^2}{db^2} + C^2 \frac{d^2}{dc^2}\right)^p \phi(a,\ b,\ c),$$

reducible to his form as a function of $B^2 - A^2$, $C^2 - A^2$, in virtue of the equation $\left(\frac{d^2}{da^2} + \frac{d^2}{db^2} + \frac{d^2}{dc^2}\right) \phi(a,\ b,\ c) = 0$ satisfied by the function ϕ (I wrote this out some time before the Senate House Examination 1842, in an examination paper for my tutor, Mr Hopkins): and I was thus led to consider how the series in question could be transformed so as to identify it with the known expression for the attraction as a single definite integral.

I remark that my formulæ relate to the case of n variables: as regards ellipsoids the number of variables is of course $= 3$: in the earlier solutions of the problem of the attraction of ellipsoids there is no ready method of making the extension from 3 to n. The case of n variables had however been considered in a most able manner by Green in his Memoir "On the determination of the exterior and interior attractions of Ellipsoids of variable densities," *Camb. Phil. Trans.* vol. V. 1835, pp. 395—430 (and Mathematical Papers, 8vo London, 1871, pp. 187—222); and in the Memoir by Lejeune-Dirichlet, "Sur une nouvelle méthode pour la détermination des intégrales multiples," *Liouv.* t. IV. (1839), pp. 164—168, although the case actually treated is that of three variables, the method can be at once extended to the case of any number of variables: it is to be noticed also that the methods of Green and Lejeune-Dirichlet are each applicable to the case of an integral involving an integer or fractional negative power of the distance. This is far more general than my formulæ, for in them the negative exponent for the squared distance is $= \frac{1}{2}n$, and, by differentiation in regard to the coordinates $a, b, \ldots$ of the attracted point, we can only change this into $\frac{1}{2}n + p$, where p is a positive integer. But in 28, the radical contained in the multiple integral is $\frac{1}{\{(a_1 - x_1 t)^2 + \ldots\}^{\frac{1}{2}n - s}}$, where s is integer or fractional, and by a like process of expansion and summation I obtain a result depending on a single integral $\int_0^1 \frac{(1-u)^\sigma u^{i+\kappa-1} du}{\{(\zeta + h_1^2 u) \ldots\}^{\frac{1}{2}}}$. And in 29, retaining throughout the general function $\phi(a_1 - x_1, \ldots)$ and making the analogous transformation of the multiple integral itself, I express the integral

$$V = \int dx_1 \ldots dx_n \ x_1^{2a_1+1} \ldots x_{f+1}^{2a_f+1} \ldots \phi(a_1 - x_1, \ldots)$$

in terms of an integral $\int_0^1 T^{n+1} (1 - T^2)^{k+f} W dT$, where $W = \int dx_1 \ldots dx_n \phi(a_1 - x_1 T, \ldots)$.

I recall the fundamental idea of Lejeune-Dirichlet's investigation; starting with an integral $\iiint U dx dy dz$ over a given volume he replaces this by $\iiint \rho U dx dy dz$ where ρ is a discontinuous function, $=1$ for points inside, and $=0$ for points outside, the given volume; such a function is expressible as a definite integral (depending on the form of the bounding surface) in regard to a new variable θ: the limits for x, y, z, may now be taken to be ∞, $-\infty$ for each of the variables x, y, z, and it is in many cases possible to effect these integrations and thus to express the original multiple integral as a single definite integral in regard to θ.

I have not ascertained how far the wholly different method in Lejeune-Dirichlet's Memoir "Sur un moyen général de vérifier l'expression du potentiel relatif à une masse quelconque homogène ou hétérogène," *Crelle*, t. XXXII. (1846), pp. 80—84, admits of extension in regard to the number of variables, or the exponent of the radical.

4. As noticed p. 22, the investigation was suggested to me by Mr Greathead's paper, "Analytical Solutions of some problems in Plane Astronomy," *Camb. Math. Jour.* vol. I. (1839), pp. 182—187, giving the expression of the true anomaly in multiple sines of the mean anomaly. I am not aware that this remarkable expression has been elsewhere at all noticed except in a paper by Donkin, "On an application of the Calculus of Operations in the transformation of trigonometrical series," *Quart. Math. Journal*, vol. III. (1860), pp. 1—15; see p. 9, *et seq.*

5. In a terminology which I have since made use of:

The Postulandum or Capacity (□) of a curve of the order r is $=\frac{1}{2}r(r+3)$; and the Postulation (∇) of the condition that the curve shall pass through k given points is in general $=k$.

If however the k points are the mn intersections of two given curves of the orders m and n respectively, and if r is not less than m or n, and not greater than $m+n-3$, then the postulation for the passage through the mn points, instead of being $=mn$, is $=mn-\frac{1}{2}(m+n-r-1)(m+n-r-2)$.

Writing $\gamma=m+n-r$, and $\delta=\frac{1}{2}(\gamma-1)(\gamma-2)$, the theorem may be stated in the form, a curve of the order r passing through $mn-\delta$ of the mn points of intersection will pass through the remaining δ points. The method of proof is criticised by Bacharach in his paper, "Ueber den Cayley'schen Schnittpunktsatz," *Math. Ann.* t. 26 (1886), pp. 275—299, and he makes what he considers a correction, but which is at any rate an important addition to the theorem, viz. if the δ points lie in a curve of the order $\gamma-3$, then the curve of the order r through the $mn-\delta$ points does not of necessity nor in general pass through the δ points. See my paper "On the Intersection of Curves," *Math. Ann.* t. XXX. (1887), pp. 85—90.

6. The formulæ in Rodrigues' paper for the transformation of rectangular coordinates afterwards presented themselves to me in connexion with Quaternions, see 20; and again in connexion with the theory of skew determinants, see 52.

8. A correction to the theorem (18), p. 42, is made in my paper "Notes on Lagrange's theorem," *Camb. and Dubl. Math. Jour.* vol. VI. (1851), pp. 37—45.

10. This paper is connected with 5, but it is a particular investigation to which I attach little value. The like remark applies to 40.

12. The second part of this paper, pp. 75—80, relates to the functions obtained from n columns of symbolical numbers in such manner as a determinant is obtained from 2 columns, and which are consequently sums of determinants: they are the functions which have since been called Commutants; the term is due to Sylvester.

13. In modern language: Boole (in his paper "Exposition of a general theory of linear transformations," *Camb. Math. Jour.* vol. III. (1843), pp. 1—20 and 106—119) had previously shown that a discriminant was an invariant; and Hesse in the paper "Ueber die Wendepunkte der Curven dritter Ordnung," *Crelle*, t. XXVIII. (1844), pp. 68—96, had established certain covariantive properties of the ternary cubic function. I first proposed in this paper the general problem of invariants (that is, functions of the coefficients, invariantive for a linear transformation of the facients), treating it by what may be called the "tantipartite" theory: the idea is best seen from the example p. 89, viz. for the tripartite function

$$U = ax_1y_1z_1 + bx_2y_1z_1 + cx_1y_2z_1 + dx_2y_2z_1 + ex_1y_1z_2 + fx_2y_1z_2 + gx_1y_2z_2 + hx_2y_2z_2,$$

we have a function of the coefficients which is simultaneously of the forms

$$H\left|\begin{matrix} a, & b, & c, & d \\ e, & f, & g, & h \end{matrix}\right|, \quad H\left|\begin{matrix} a, & b, & e, & f \\ c, & d, & g, & h \end{matrix}\right|, \quad H\left|\begin{matrix} a, & c, & e, & g \\ b, & d, & f, & h \end{matrix}\right|,$$

and as such it is invariantive for linear transformations of the (x_1, x_2), (y_1, y_2), (z_1, z_2).

Passing from the tantipartite form to a binary form, I obtained for the binary quartic the quadrinvariant $(I=)\ ae - 4bd + 3c^2$: as noticed at the end of the paper, the remark that there is also the cubinvariant $(J=)\ ace - ad^2 - b^2e - c^3 + 2bcd$ was due to Boole. The two functions present themselves, but without reference to the invariantive property and not in an explicit form, in Cauchy's Memoir "Sur la détermination du nombre des racines réelles dans les équations algébriques," *Jour. École Polyt.* t. X. (1815), pp. 457—548.

In p. 92 it is assumed that the invariant called θu is the discriminant of the function $U = ax_1y_1z_1w_1 \ldots + px_2y_2z_2w_2$: but, as mentioned in [ ], the assumption was incorrect. This was shown by Schläfli in his Memoir, "Ueber die Resultante eines Systemes mehrerer algebraischen Gleichungen," *Wiener Denks.* t. IV. Abth. 2 (1852), pp. 1—74: see pp. 35 *et seq.* The discriminant is there found by actual calculation to be a function (not of the order 6 as is θu, but) of the order 24, not breaking up into factors; in the particular case where the coefficients $a, \ldots p$ are equal, 1, 4, 6, 4, 1 of them to a; b, c, d, e respectively, in such wise that changing only the variables the function becomes $= (a, b, c, d, e \between x, y)^4$, then the discriminant in question does break up into factors, the value in this case being $J^6(I^3-27J^2)$ of the order 24 as in the general case, but containing the factor I^3-27J^2 which is the discriminant of the binary quartic.

14. In this paper I developed what (to give it a distinctive name) may be called the "hyperdeterminant" theory, viz. the expressions considered are of the form

$$\overline{12}^\alpha\ \overline{13}^\beta\ \overline{23}^\gamma \ldots U_1U_2U_3\ldots,$$

where after the differentiations the variables (x_1, y_1), (x_2, y_2), ... are to be or may be put equal to each other: it is to be noticed that although in the examples I chiefly consider constant derivatives, or invariants, the memoir throughout relates as well to covariants as invariants. The theory is to be distinguished from Gordan's process of Ueberschiebung, or derivational theory, viz. this may be considered as dealing exclusively, or nearly so, with the single class of derivatives $(V, W)^\alpha, = \overline{12}^\alpha V_1W_2$: the theorem that all the covariants of a binary function can be obtained successively by operating in this manner on the function itself and a covariant of the next inferior degree was a very important one.

15. Eisenstein's theorem may be stated as follows: the function $a^2d^2+4ac^3-6abcd+4b^3d-3b^2c^2$ (which is the discriminant of the binary cubic $(a, b, c, d\between x, y)^3$) is automorphic, viz. it is converted into a power of itself when for a, b, c, d we substitute the differential coefficients $\frac{d\phi}{da}$, $\frac{d\phi}{db}$, $\frac{d\phi}{dc}$, $\frac{d\phi}{dd}$ of the function itself. It is remarkable, see 54, that the function is automorphic in a different manner, viz. the Hessian determinant formed with the second differential coefficients $\frac{d^2\phi}{da^2}$, &c., is also equal to a power of the function itself. The first part of the paper relates to the function $a^2h^2+b^2g^2+\ldots+4bceh$ which had presented itself to me, 13, in the theory of linear transformations, and which is in like manner automorphic for the change $a, b, \ldots$ into $\frac{d\phi}{da}$, $\frac{d\phi}{db}$, &c. The function however occurs in connexion with the arithmetical theory of the composition of quadratic forms, Gauss, *Disquisitiones Arithmeticæ* (1801), and see 92. The second part gives for the binary quartic covariant an automorphic formula analogous to those previously obtained by Hesse for the ternary cubic, viz. the Hessian of any linear function of the quartic and its Hessian, is itself a linear function of the quartic and its Hessian, the coefficients depending on the invariants I, J of the quartic form.

16. This is a mere reproduction of 13 and 14, and requires no remark.

19 and 23. These papers contain a mere sketch of the application of the doubly infinite product expression of the elliptic function sn u to the problem of transformation. As noticed in 23, I purposely abstained from any consideration of the infinite limiting values of m and n.

20. The discovery of the formula $q(ix+jy+kz)q^{-1}=ix'+jy'+kz'$, as expressing a rotation, was made by Sir W. R. Hamilton some months previous to the date of this paper. As appears by the paper itself, I was led to it by Rodrigues' formulæ, see 6. For the further development of the theory, see 68.

21. The system of imaginaries $i_1, i_2, \ldots i_7$ had presented itself to J. T. Graves about Christmas 1843, see his paper "On a Connection &c." *Phil. Mag.* vol. XXVI. (1845), pp. 315—320. They are called by him Octads, or Octaves.

24 and 25. These papers precede the researches of Eisenstein on the same subject. *Crelle,* t. XXXV. (1847). It was I think right that the theory of the doubly infinite products should have been investigated as in these papers: but the investigation is in some measure superseded by the beautiful theory of Weierstrass, viz. he takes for the element of the product (not a mere linear function of u, but) a linear function multiplied by an exponential factor,

$$\sigma u = u\Pi\left\{\left(1+\frac{u}{w}\right)e^{-\frac{u}{w}-\frac{1}{2}\frac{u^2}{w^2}}\right\},$$

$w=2m\omega+2m'\omega'$ where the ratio $\omega:\omega'$ is imaginary, and the product extends to all positive or negative integer values of m, m' (the simultaneous values 0, 0 excluded): in consequence of the introduction of the exponential factor the form of the bounding curve becomes immaterial, and the only condition is that it shall be ultimately everywhere at an infinite distance from the origin.

The general theory of Weierstrass in regard to the exponential factor is given in the Memoir, "Zur Theorie der eindeutigen analytischen Functionen," *Berlin. Abh.* 1876 (reprinted, *Abhandlungen aus der Functionenlehre,* 8° Berlin, 1886): and the application to Elliptic Functions is made in his lectures, edited by Schwarz, *Formeln und Lehrsätze* u. s. w. 4° Gött. 1883. See also Halphen, *Théorie des Fonctions Elliptiques,* Paris, 1887.

26. The geometrical results in regard to corresponding points on a cubic curve are many of them due to Maclaurin. See his "De linearum geometricarum proprietatibus generalibus tractatus," published as an Appendix to his Treatise on Algebra, 5 Ed. Lond. 1788. (See also De Jonquières' "Mélanges de Géométrie pure," 8° Paris, 1856.) But the theorem in the "Addition" was probably new: the curve of the third class touched by the line PP' is the curve called by me the Pippian (as represented by the contravariant equation $PU=0$), but which has since been called the Cayleyan of the cubic curve.

27. The theory of the conics of involution was so far as I am aware new.

28 and 29. See 2, 3.

30. As noticed in the paper, the investigation is directly founded upon that of Plücker for the singularities of a plane curve. It is to be observed that the definition of a "line through two points" *ligne menée par deux points (non consecutifs en général) du système,* does not exclude actual double points, for the line through an actual double point is a line through two points, coincident indeed, but not consecutive; but

it would have been proper to notice the distinction between actual and apparent double points (first made by Dr Salmon, to whom the term apparent double point, adp, is due): and the like in regard to the lines in two planes. In the translation of this paper, 83, there are two footnotes signed G. S. (Dr Salmon), giving for plane curves the formulæ $\iota - \kappa = 3(\nu - \mu)$ and $2(\tau - \delta) = (\nu - \mu)(\nu + \mu - 9)$: and for curves of double curvature and developable surfaces the analogous formulæ $\alpha - \beta = 2(n - m)$, $x - y = (n - m)$ and $2(g - h) = (n - m)(n + m - 7)$. Also in the second set of six equations, p. 210, the last three equations are replaced by

$$x = \tfrac{1}{2}m(m-1)(m^2 - m - 4) - \tfrac{1}{2}(2h + 3\beta)(2h + 3\beta + 1) - m(m-1)(2h + 3\beta) + 3h + 4\beta,$$

$$\alpha = 2m(3m - 7) - 3(4h + 5\beta),$$

$$g = \tfrac{1}{2}m(3m-7)(3m^2 - 5m - 7) + \tfrac{1}{2}(6h + 8\beta)(6h + 8\beta + 1) - 3m(m-2)(6h + 8\beta) + 19h + 24\beta,$$

viz. these are the equations serving to express x, α, g in terms of m, h, β.

For the discussion of some singularities not considered in the present paper see Zeuthen, "Sur les singularités ordinaires des courbes géométriques à double courbure," *Comptes Rendus*, t. LXVII. (1868), pp. 225—233, and "Sur les singularités ordinaires d'une courbe gauche et d'une surface développable," *Annali di Matem.* t. III. (1869-70), pp. 175—217.

40. See 10.

41 and 44. For demonstrations of Sir W. Thomson's theorem for the value of the definite integral $\int \frac{dx\ldots}{\{(x-a)^2 + \ldots + u^2\}^i (x^2 + \ldots + v^2)^{i+1}}$ see his papers "Démonstration d'un theorème d'Analyse" and "Extrait d'une lettre à M. Liouville," *Liouv.* t. x. (1845), pp. 137—147 and 364—367, also the paper "On certain definite integrals suggested by problems in the theory of Electricity," *Camb. Math. Jour.* t. II. (1845), pp. 109—121.

45. I am not aware that the equation $\sqrt{k}\,\mathrm{sn}\,u = \mathrm{H}(u) \div \Theta(u)$ had been previously demonstrated otherwise than by the circuitous process employed in the *Fundamenta Nova*.

The series $z = 1 + C_1 \frac{x^2}{1 \,.\, 2} \ldots + C_r \frac{x^r}{1 \,.\, 2 \ldots r} + \ldots$, which is the solution of the differential equation $x^2 z + \alpha x \cdot \frac{dz}{dx} + \frac{d^2 z}{dx^2} - 2(\alpha^2 - 4)\frac{dz}{dx} = 0$, and in which C_1, C_2, ... denote the coefficients of the highest powers of n in the expressions given p. 299, is in fact (as remarked by me in a later paper, *Liouv.* t. VII. (1862)) the Weierstrassian function $\mathrm{Al}(x)$.

47. The surface here considered, the Tetrahedroid, is the general homographic transformation of the wave surface. It is a special case of the 16-nodal quartic surface considered by Kummer in his Memoir, "Algebraische Strahlen-systeme," *Berl. Abh.* 1866, pp. 1—120, and in various papers in the *Berliner Monatsberichte*.

48. The expressions $f_2x = \Sigma (a-b)^2 (x-c)(x-d) \ldots$ for the Sturmian functions in terms of the roots, or (to use Sylvester's term), say the endoscopic expressions of these functions, were obtained by him, *Phil. Mag.* vol. XV. (1839).

It was interesting to express these in terms of the sums of powers S_1, S_2, &c., that is in terms of symmetrical functions of the coefficients, but for the actual expression of the Sturmian functions in terms of the coefficients the process is a very circuitous one, and the proper course is to start directly from the exoscopic expressions as linear functions of fx, $f'x$ also due to Sylvester (see his Memoir "On a theory of the syzygetic relations of two rational integral functions &c.," *Phil. Trans.* t. CXLIII. (1853), pp. 407—548), which is what is done in the subsequent paper 65.

49. I attach some value to this paper as a contribution to the theory of the Gamma function.

50, 55, 70, 95, 98. The general theorem of § I. was given in a very different and less suggestive notation, by an anonymous writer, "Théorèmes appartenant à la géométrie de la règle," *Gergonne* t. IX. (1818–19), pp. 289—291; viz. the statement is in effect as follows: considering in a plane or in space any n points 1, 2, 3 ... n; then joining these in order, take 12 any point in the line 1, 2; 23 any point in the line 2, 3, and so on to $n-1.n$. Take then 123 the intersection of the lines 1, 23 and 12, 3; 234 the intersection of the lines 2, 34 and 23, 4; and so on to $n-2.n-1.n$. Take then 1234 the intersection of the lines 1, 234; 12, 34; and 123, 4 (viz. these three lines will meet in a point): 2345 the intersection of the lines 2, 345; 23, 45; 234, 5 (viz. these three lines will meet in a point)... and so on to $n-3.n-2.n-1.n$. And so on for 12345, &c. up to 123 ... n; in the successive constructions we have four, five, ... and finally n lines which in each case meet in a point. A proof is given by Gergonne, t. XI. (1820–21).

A large part of this paper relates to the theory of the relations to each other of the 60 Pascalian lines derived from the hexagons which can be formed with the same six points upon a conic: the literature of the question is very extensive, and I hope to refer to it again in the Notes to another volume.

54. See for an addition which should have been printed with this paper, the last paragraph of 92.

63. Boole's theorem of integration which is here demonstrated is a very remarkable one, and it would be very interesting to investigate the general forms, or a larger number of particular forms, for the functions P, Q satisfying the condition mentioned in the theorem. It may be remarked that the demonstration is very closely connected with Lejeune-Dirichlet's method for the determination of certain definite integrals referred to in 2 and 3, we have a triple integral the real part of which is $=fP \div Q^{\frac{1}{2}n+q}$ or 0, according as P is or is not comprised between the limits 0 and 1. It includes Boole's formula mentioned in 44 and in 64.

65. See 48.

66. The method here employed of establishing the theory of Laplace's coefficients in n-dimensional space by means of rectangular coordinates, has I think some advantage

over the ordinary one, in which angular coordinates are introduced. The method is not given in Heine's *Kugelfunctionen*, Berlin 1878–81.

67, 93, and 99. These papers relate to the equation of differences, see p. 540, derived from Jacobi's equation $n(n-1)x^2z+(n-1)(\alpha x-2x^3)\frac{dz}{dx}+(1-\alpha x^2+x^4)\frac{d^2z}{dx^2}-2n(\alpha^2-4)\frac{dz}{d\alpha}=0$: the investigation seems to show that the integration cannot be effected in any tolerably simple form.

68. See 20.

69. This paper relates to the theory of the functions considered by Jacobi, and to which the name "Pfaffian" has since been given. It is shown that the definition of a determinant may be so extended as to include within it the Pfaffian: and it is proved that a symmetrical skew determinant is the square of a Pfaffian.

71. I doubt whether the theorem is true except in the cases $n=3$ and $n=4$.

76. As mentioned at the conclusion of the Memoir the whole subject was developed in a correspondence with Dr Salmon. Steiner's researches upon Cubic Surfaces are of later date, viz. we have his Memoir, "Ueber die Flächen dritten Grades" (read to the Berlin Academy 31 January 1856), *Crelle*, t. LIII. (1857), pp. 133—141 and *Werke*, t. II. pp. 651—659.

77. Contains the general definition of the 'order' of a system of equations.

81. Contains the remark that the Cartesian has a *cusp* at each of the two circular points at infinity.

82. The Problem of the fifteen school girls was proposed by Kirkman, *Lady's and Gentleman's Diary*, 1850: it is a particular case of the Prize-question proposed by him in the *Diary* for 1844. A great deal has been written on the subject.

83. See 30.

92. See 15.

94. This paper contains my fourfold formulæ for the addition of the elliptic functions sn, cn, and dn.

100. I remark that the terms covariant, invariant, here referred to as introduced by Sylvester, were first employed (together with many other valuable new terms) in the first part of his paper "On the Principles of the Calculus of Forms," *Camb. and Dubl. Math. Jour.* vol. VII. (1852), pp. 52—97. I hope to give in the next volume, in connexion with my Introductory Memoir on Quantics, a review of the earlier history of the subject.

END OF VOL. I.