88.

ON THE TRANSFORMATION OF AN ELLIPTIC INTEGRAL.

[From the *Cambridge and Dublin Mathematical Journal*, vol. v. (1850), pp. 204—206.]

THE following is a demonstration of a formula proved incidentally by Mr Boole (*Journal*, vol. II. [1847] p. 7), in a paper "On the Attraction of a Solid of Revolution on an External Point."

Let $$U = \int_{-1}^{1} \frac{dx}{\sqrt{[(1-x^2)\{1-(mx+n)^2\}]}};$$

then, assuming

$$ix = \frac{\alpha + iy}{1 - i\alpha y},$$

(so that $x = \pm 1$ gives $y = \pm 1$), we obtain

$$1 - x^2 = \frac{(1+\alpha^2)(1-y^2)}{(1-i\alpha y)^2},$$

$$mx + n = \frac{(n - im\alpha) + (m - in\alpha)\, y}{1 - i\alpha y}.$$

Assume therefore

$$i\alpha + (n - im\alpha)(m - in\alpha) = 0,$$

whence

$$-i\alpha = \frac{(1 - m^2 - n^2) + \Delta}{2mn} \qquad (\Delta^2 = 1 + m^4 + n^4 - 2m^2 - 2n^2 - 2m^2n^2),$$

we find

$$1-(mx+n)^2=\frac{1-(n-im\alpha)^2}{(1-i\alpha y)^2}\{1-(m-in\alpha)^2 y^2\},$$

and also

$$dx=\frac{(1+\alpha^2)\,dy}{(1-i\alpha y)^2},$$

whence

$$U=\sqrt{\left\{\frac{1+\alpha^2}{1-(n-im\alpha)^2}\right\}}\int_{-1}^{1}\frac{dy}{\sqrt{[(1-y^2)\{1-(m-in\alpha)^2 y^2\}]}},$$

that is

$$U=2\sqrt{\left\{\frac{1+\alpha^2}{1-(n-im\alpha)^2}\right\}}\int_{0}^{1}\frac{dy}{\sqrt{[(1-y^2)\{1-(m-in\alpha)^2 y^2\}]}}.$$

But since

$$n-im\alpha=\frac{1-m^2+n^2+\Delta}{2n},$$

$$m-in\alpha=\frac{1+m^2-n^2+\Delta}{2m},$$

we have

$$1-(n-im\alpha)^2=-\frac{\Delta}{2n^2}\;(\Delta+1-m^2+n^2),$$

$$1+\alpha^2=-\frac{\Delta}{2m^2n^2}(\Delta+1-m^2-n^2);$$

and therefore

$$\frac{1+\alpha^2}{1-(n-im\alpha)^2}=\frac{1}{m^2}\;\frac{\Delta+1-m^2-n^2}{\Delta+1-m^2+n^2}$$

$$=\frac{1}{m^2}\;\frac{(1-m^2-n^2+\Delta)\,(1-m^2+n^2-\Delta)}{(1-m^2+n^2+\Delta)\,(1-m^2+n^2-\Delta)}=\frac{2\,(1+m^2-n^2+\Delta)}{4m^2};$$

consequently

$$U=\frac{1}{m}\sqrt{\{2\,(1+m^2-n^2+\Delta)\}}\int_0^1\frac{dy}{\sqrt{\left[(1-y^2)\left\{1-\left(\frac{1+m^2-n^2+\Delta}{2m}\right)^2 y^2\right\}\right]}}.$$

Write

$$k=\frac{1+m^2-n^2+\Delta}{2m},\qquad \lambda^2=\frac{4m}{(1+m)^2-n^2};$$

then

$$U=\frac{4\sqrt{k}}{\lambda}\;\frac{1}{\sqrt{\{(1+m)^2-n^2\}}}\int_0^1\frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}};$$

where λ and k are connected by the relation that exists for the transformation of the second order, viz.

$$\lambda=\frac{2\sqrt{k}}{1+k},$$

as may be immediately verified; hence, assuming

$$y=\frac{\lambda z}{\sqrt{k}}\sqrt{\left(\frac{1-z^2}{1-\lambda^2z^2}\right)},$$

which gives

$$\int_0^1\frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}}=\frac{\lambda}{\sqrt{k}}\int_0^1\frac{dz}{\sqrt{\{(1-z^2)(1-\lambda^2z^2)\}}},$$

we find

$$U=\frac{4}{\sqrt{\{(1+m)^2-n^2\}}}\int_0^1\frac{dz}{\sqrt{\left\{(1-z^2)\left(1-\frac{4m}{(1+m)^2-n^2}z^2\right)\right\}}};$$

that is

$$\int_{-1}^1\frac{dx}{\sqrt{[(1-x^2)\{1-(mx+n)\}^2]}}=4\int_0^1\frac{dz}{\sqrt{(1-z^2)[\{(1+m)^2-n^2\}-4mz^2]}}.$$

Writing here

$$x=\cos\theta,\quad z=\cos\tfrac{1}{2}\phi,$$

then

$$\int_0^\pi\frac{d\theta}{\sqrt{\{1-(m\cos\theta+n)^2\}}}=\int_0^\pi\frac{d\phi}{\sqrt{(1+m^2-n^2-2m\cos\phi)}},$$

or if

$$m=\frac{r}{a},\qquad n=-\frac{iz}{a},$$

then finally

$$\int_0^\pi\frac{d\theta}{\sqrt{\{a^2+(z+ir\cos\theta)^2\}}}=\int_0^\pi\frac{d\phi}{\sqrt{(a^2+r^2+z^2-2ar\cos\phi)}},$$

the formula in question.