86.

ON THE DEVELOPABLE DERIVED FROM AN EQUATION OF THE FIFTH ORDER.

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MÖBIUS, in his "Barycentrische Calcul," [Leipzig, 1827], has considered, or rather suggested for consideration, the family of curves of double curvature given by equations such as $x : y : z : w = A : B : C : D$, where A, B, C, D are rational and integral functions of an indeterminate quantity t. Observing that the plane $Ax + By + Cz + Dw = 0$ may be considered as the polar of the point determined by the system of equations last preceding, the reciprocal of the curve of Möbius is the developable, which is the envelope of a plane the coefficients in the equation of which are rational and integral functions of an indeterminate quantity t, or what is equivalent, homogeneous functions of two variables ξ, η. Such an equation may be represented by $U = a\xi^n + n\xi^{n-1}\eta + \ldots = 0$, (where a, b, &c. are linear functions of the coordinates); and we are thus led to the developables noticed, I believe for the first time, in my "Note sur les Hyper-determinants," *Crelle*, t. XXXIV. p. 148, [54]. I there remarked, that not only the equation of the developable was to be obtained by eliminating ξ, η from the first derived equations of $U = 0$; but that the second derived equations conducted in like manner to the edge of regression, and the third derived equations to the cusps or stationary points of the edge of regression. It followed that the order of the surface was $2(n-1)$, that of the edge of regression $3(n-2)$, and the number of stationary points $4(n-3)$. These values lead at once, as Mr Salmon pointed out to me, to the table,

$$\begin{aligned}
m &= 3(n-2),\\
n &= n,\\
r &= 2(n-1),\\
\alpha &= 0,\\
\beta &= 4(n-3),\\
g &= \tfrac{1}{2}(n-1)(n-2),\\
h &= \tfrac{1}{2}(9n^2 - 53n + 80),\\
x &= 2(n-2)(n-3),\\
y &= 2(n-1)(n-3),
\end{aligned}$$

where the letters in the first column have the same signification as in my memoir in Liouville, [30], translated in the last number of the *Journal*. The order of the nodal line is of course $2(n-2)(n-3)$; Mr Salmon has ascertained that there are upon this line $6(n-3)(n-4)$ stationary points and $\frac{4}{3}(n-3)(n-4)(n-5)$ real double points, (the stationary points lying on the edge of regression, and with the stationary points of the edge of regression forming the system of intersections of the nodal line and edge of regression, and the real double points being triple points upon the surface). Also, that the number of apparent double points of the nodal line is

$$(n-3)(2n^3-18n^2+57n-65).$$

The case of U a function of the second order gives rise to the cone $ac-b^2=0$. When U is a function of the third order, we have the developable

$$4(ac-b^2)(bd-c^2)-(ad-bc)^2=0,$$

which is the general developable of the fourth order having for its edge of regression the curve of the third order,

$$ac-b^2=0, \quad bd-c^2=0, \quad ad-bc=0,$$

which is likewise the most general curve of this order: there are of course in this case no stationary points on the edge of regression. In the case where U is of the fourth order we have the developable of the sixth order,

$$(ae-4bd+3c^2)^3-27(ace+2bcd-ad^2-b^2e-c^3)^2=0;$$

having for its edge of regression the curve of the sixth order,

$$ae-4bd+3c^2=0, \quad ace+2bcd-ad^2-b^2e-c^3=0,$$

with four stationary points determined by the equations

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}.$$

The form exhibiting the nodal line of the surface has been given in the *Journal* by Mr Salmon. I do not notice it here, but pass on to the principal subject of the present paper, which is to exhibit the edge of regression and the stationary points of this edge of regression for the developable obtained from the equation of the fifth order,

$$U=a\xi^5+5b\xi^4\eta+10c\xi^3\eta^2+10d\xi^2\eta^3+5e\xi\eta^4+f\eta^5=0;$$

viz. that represented by the equation

$$\square=0=a^4f^4+160a^3cef^2+\ldots-4000b^2c^3e^3,$$

[I do not reproduce here this expression for the discriminant of the binary quintic] a result for which I am indebted to Mr Salmon.

To effect the reduction of this expression, consider in the first place the equations which determine the stationary points of the edge of regression. Writing instead of $\xi : \eta$ the single letter t, these equations are

$$\begin{aligned} at^2 + 2bt + c &= 0, \\ bt^2 + 2ct + d &= 0, \\ ct^2 + 2dt + e &= 0, \\ dt^2 + 2et + f &= 0, \end{aligned}$$

write for shortness

$$\begin{aligned} A &= 2\,(bf - 4ce + 3d^2), \\ B &= \quad af - 3be + 2cd, \\ C &= 2\,(ae - 4bd + 3c^2), \end{aligned}$$

and let α, 3β, 3γ, δ represent the terms of

$$\left\| \begin{matrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{matrix} \right\|,$$

viz.

$$\begin{aligned} \alpha &= bdf - be^2 + 2cde - c^2 f \quad - d^3, \\ 3\beta &= adf - ae^2 - bcf \quad + bde \quad + c^2 e - cd^2, \\ 3\gamma &= acf - ade - b^2 f \quad + bd^2 \quad + bce - c^2 d, \\ \delta &= ace - ad^2 - b^2 e \quad + 2bcd - c^3\,; \end{aligned}$$

it is obvious at first sight that the result of the elimination of t from the four quadratic equations is the system (equivalent of course to three equations),

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0.$$

The system in question may however be represented under the more simple form (which shows at once that the number of stationary points is, as it ought to be, eight),

$$A = 0, \quad B = 0, \quad C = 0\,;$$

this appears from the identical equations,

$$\begin{aligned} &(2ct + 3d)\,(bt^2 + 2ct + d) \\ &- (2bt + 4c)\,(ct^2 + 2dt + e) \\ &\qquad + b\,(dt^2 + 2et + f) = \tfrac{1}{2}A\,; \end{aligned}$$

$$\begin{aligned} &(2ct + 3d)\,(at^2 + 2bt + c) \\ &\quad - c\,(bt^2 + 2ct + d) \\ &- (2at + 3b)\,(ct^2 + 2dt + e) \\ &\quad + a\,(dt^2 + 2et + f) = B\,; \end{aligned}$$

$$(2bt+3c)(at^2+2bt+c)$$
$$-(2at+4b)(bt^2+2ct+d)$$
$$+a(ct^2+2dt+e)=\tfrac{1}{2}C,$$

(formulæ the first and third of which are readily deduced from an equation given in the Note on Hyperdeterminants above quoted). The connexion between the quantities A, B, C and α, β, γ, δ, is given by

$$Aa-2Bb+Cc=-6\delta,$$
$$Ab-2Bc+Cd=-6\gamma,$$
$$Ac-2Bd+Ce=-6\beta,$$
$$Ad-2Be+Cf=-6\alpha.$$

The theory of the stationary points being thus obtained, the next question is that of finding the equations of the edge of regression. We have for this to eliminate t from the three cubic equations,

$$at^3+3bt^2+3ct+d=0,$$
$$bt^3+3ct^2+3dt+e=0,$$
$$ct^3+3dt^2+3et+f=0:$$

treating the quantities t^3, t^2, t^1, t^0 as if they were independent, we at once obtain

$$\beta t+\alpha=0,\quad \delta t+\gamma=0,\quad \gamma t^2-\alpha=0,\quad \delta t^2-\beta=0;$$

or as this system may be more conveniently written,

$$\beta t+\alpha=0,\quad \gamma t+\beta=0,\quad \delta t+\gamma=0.$$

But the most simple forms are obtained from the identical equations,

$$ft(at^3+3bt^2+3ct+d),$$
$$-(3et+f)(bt^3+3ct^2+3dt+e),$$
$$+(2dt+e)(ct^3+3dt^2+3et+f)=t^3(Bt+A);$$

$$(bt+c)(at^3+3bt^2+3ct+d),$$
$$-(at+3b)(bt^3+3ct^2+3dt+e),$$
$$+\quad a\quad(ct^3+3dt^2+3et+f)=\quad Ct+B;$$

equations which, combined with those which precede, give the complete system

$$\beta t+\alpha=0,\quad \gamma t+\beta=0,\quad \delta t+\gamma=0,\quad Bt+A=0,\quad Ct+B=0:$$

or the equations of the edge of regression are given by the system (equivalent of course to two equations),

$$\left\|\begin{matrix} \alpha, & \beta, & \gamma, & A, & B \\ \beta, & \gamma, & \delta, & B, & C \end{matrix}\right\| = 0.$$

The simplest mode of verifying *à posteriori* that the edge of regression is only of the ninth order, appears to be to consider this curve as the common intersection of the three surfaces of the seventh order:

$$A^3a - 3A^2Bb + 3AB^2c - B^3d = 0,$$
$$A^3b - 3A^2Bc + 3AB^2d - B^3e = 0,$$
$$A^3c - 3A^2Bd + 3AB^2e - B^3f = 0,$$

(which are at once obtained by combining the equation $Bt + A = 0$ with the cubic equations in t). It is obvious from a preceding equation that if the equations first given are multiplied by fA, $-3eA + fB$, $2dA - eB$, and added, an identical result is obtained. This shows that the curve of the forty-ninth order, the intersection of the first two surfaces, is made up of the curve in question, the curve of the fourth order $A = 0$, $B = 0$ (which reckons for thirty-six, as being a triple line on each surface), and the curve which is common to the two surfaces of the seventh order and the surface $2dA - eB = 0$. The equations of this last curve may be written,

$$e(af - 3be + 2cd) - 4d(bf - 4ce + 3d^2) = 0,$$
$$e^3a - 6e^2db + 12ed^2c - 8d^4 = 0,$$
$$e^3b - 6e^2dc + 4ed^3 = 0;$$

or, observing that these equations are

$$f(ae - 4bd) - 3\ (be^2 - 6ced + 4d^3) = 0,$$
$$e^2(ae - 4bd) - 2d(be^2 - 6ced + 4d^3) = 0,$$
$$e(be^2 - 6ced + 4d^3) = 0;$$

the last-mentioned curve is the intersection of

$$ae - 4bd = 0,$$
$$be^2 - 6ced + 4d^3 = 0,$$

where the second surface contains the double line $e = 0$, $d = 0$, which is also a single line upon the first surface. Omitting this extraneous line, the intersection is of the *fourth* order; and we may remark that, in passing, it is determined (exclusively of the double line) as the intersection of the three surfaces

$$ae - 4bd = 0,$$
$$be^2 - 6ced + 4d^3 = 0,$$
$$a^2d - 6abc + 4b^3 = 0,$$

being in fact of the species IV. 4, of Mr Salmon's paper, "On the Classification of Curves of Double Curvature" [*Journal*, vol. V. (1850), pp. 23—46]. But returning to the question in hand; since $49 = 9 + 4 + 36$, we see that the curve common to the three surfaces of the seventh order, or the edge of regression, is, as it ought to be, of the ninth order. It only remains to express the equation of the developable surface in terms of the functions A, B, C, α, β, γ, δ, which determine the stationary points and edge of regression; I have satisfied myself that the required formula is

$$\square = (AC - B^2)^2 - 1152\,\{A\,(\beta\delta - \gamma^2) - B\,(\alpha\delta - \beta\gamma) + C\,(\alpha\gamma - \beta^2)\} = 0,$$

where the quantities α, β, γ, δ may be replaced by their values in A, B, C; and it will be noticed that when this is done, the terms of $\square$ are each of them at least of the third order in the last-mentioned functions.

I propose to term the family of developables treated of in this paper, 'planar developables.' In general, the coefficients of the generating plane of a developable being algebraical functions of a variable parameter t, the equation rationalized with respect to the parameter belongs to a system of n different planes; the developable which is the envelope of such a system may be termed a 'multiplanar developable,' and in the particular case of n being equal to unity, we have a planar developable. It would be very desirable to have some means of ascertaining from the equation of a developable what the degree of its 'planarity' is.

P.S.—At the time of writing the preceding paper I was under the impression that the only surface of the fourth order through the edge of regression was that given by the equation $AC - B^2 = 0$; but Mr Salmon has since made known to me an entirely new form of the function $\square$, the component functions of which, equated to zero, give six different surfaces of the fourth order, each of them passing through the edge of regression. The form in question is

$$3\,\square = LL' + 64MM' - 64NN',$$

where

$$\begin{aligned}
L &= a^2f^2 + 225b^2e^2 - 32ace^2 - 32b^2df + 480bd^3 + 480c^3e - 34abef + 76acdf - 12bc^2f \\
&\qquad - 12ad^2e - 820bcde - 320c^2d^2, \\
L' &= 3a^2f^2 - 45b^2e^2 + 64ace^2 + 64b^2df - 22abef - 12acdf - 36bc^2f \\
&\qquad - 36ad^2e + 20bcde, \\
M &= 10bcd^2 - 18ad^3 - 15bc^2e + 32acde + 6b^2cf - 9ac^2f + 2abdf + a^2ef - 9abe^2, \\
M' &= 10c^2de - 18c^3f - 15bd^2e + 32bcdf + 6ade^2 - 9b^2ef + 2acef + abf^2 - 9ad^2f, \\
N &= 10b^2d^2 - 15b^2ce - 12acd^2 + 18ac^2e + abde - 2a^2e^2 + 6b^3f - 9abcf + 3a^2df, \\
N' &= 10c^2e^2 - 15bde^2 - 12c^2df + 18bd^2f + bcef - 2b^2f^2 + 6ae^3 - 9adef + 3acf^2,
\end{aligned}$$

where it should be noticed that

$$L + 3L' = -10\,(AC - B^2).$$

The expressions of L, L', M, M', N, N' as linear functions of A, B, C (also due to Mr Salmon) are

$$L = (11\,ae + 28bd - 39c^2)\,A + (\;af - 75be + 74cd)\,B + (\;11bf + 28ce - 39d^2)\,C,$$

$$L' = (-7ae + \;4bd + \;3c^2)\,A + (3af + 15be - 18cd)\,B + (-7bf + \;4ce + \;3d^2)\,C,$$

$$M = 3\,(bc - ad)\,A + 3\,(ae - c^2)\,B + \;(cd - af)\,C,$$

$$M' = \;(cd - af)\,A + 3\,(bf - d^2)\,B + 3\,(de - cf)\,C,$$

$$N = 3\,(b^2 - ac)\,A + 3\,(ad - bc)\,B + \;(bd - ae)\,C,$$

$$N' = \;(ce - bf)\,A + 3\,(cf - de)\,B + 3\,(e^2 - df)\,C.$$

I propose resuming the subject of these forms, and the general theory, in a subsequent paper. [This was never written.]