78.

NOTE ON THE MOTION OF ROTATION OF A SOLID OF REVOLUTION.

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USING the notation employed in my former papers on the subject of rotation (*Cambridge Math. Journal*, vol. III. pp. 224—232, [6]; and *Cambridge and Dublin Math. Journal*, vol. I. pp. 167—264, [37]), suppose $B = A$, then r is constant, equal to n suppose; and writing

$$\frac{(A-C)\,n}{A} = \nu, \quad \text{or } C = \left(1 - \frac{\nu}{n}\right) A\,;$$

also putting

$$\theta = \nu t + \gamma,$$

(where γ is an arbitrary constant) the values of p, q, r are easily seen to be given by the equations

$$\begin{cases} p = M \sin\theta, \\ q = M\cos\theta, \\ r = n, \end{cases}$$

(where M is arbitrary). And consequently

$$\begin{aligned} h &= A\ \ \{M^2 + n\,(n-\nu)\}, \\ k^2 &= A^2\,\{M^2 + \ \ (n-\nu)^2\}. \end{aligned}$$

Also, since $\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 = k^2$, we may write

$$\begin{aligned} \mathrm{a} &= -k\sin i\cos j, \\ \mathrm{b} &= \quad k\cos i\cos j, \\ \mathrm{c} &= \quad k\sin j\,; \end{aligned}$$

k having the value above given, and the angles i, j being arbitrary.

From the equations (12) and (15) in the second of the papers quoted, we deduce

$$2v = k\{k + A(n-\nu)\sin j + MA\cos j\cos(\theta+i)\},$$
$$\Phi = k\{n\sin j + M\cos j\cos(\theta+i)\},$$
$$\nabla = -\nu kMA\cos j\sin(\theta+i),$$

(values which verify as they should do the equation (19)). Hence, from the equation (27), writing $\frac{2dv}{\nabla} = dt = \frac{1}{\nu}\,d\theta$, we have

$$2\tan^{-1}\frac{\Omega}{k} = \delta + \frac{1}{\nu}\int d\theta\,\frac{h + kn\sin j + kM\cos j\cos(\theta+i)}{k + A(n-\nu)\sin j + MA\cos j\cos(\theta+i)}.$$

This is easily integrated; but the only case which appears likely to give a simple result is when the quantity under the integral sign is constant, or

$$A(h + kn\sin j) = k\{k + A(n-\nu)\sin j\},$$

or

$$Ah - k^2 + Ak\nu\sin j = 0;$$

that is,

$$A(n-\nu) + k\sin j = 0:$$

whence

$$\sin j = -\frac{A(n-\nu)}{k},\quad \cos j = \frac{AM}{k},\ \text{or}\ \tan j = \frac{-(n-\nu)}{M} = -\frac{C}{A}\,\frac{n}{M}.$$

Observing that $\frac{1}{2}\pi - j$ is the inclination of the axis of z to the normal to the invariable plane, this equation shows that the supposition above is not any restriction upon the generality of the motion, but amounts only to supposing that the axis of z (which is a line fixed in space) is taken upon the surface of a certain right cone having for its axis the perpendicular to the invariable plane. Resuming the solution of the problem, we have

$$2\tan^{-1}\frac{\Omega}{k} = \delta + \frac{k}{\nu A}\,\theta,$$

which may also be written under the form

$$2\tan^{-1}\frac{\Omega}{k} = \delta_1 + \frac{kt}{A},$$

$\left(\text{where } \delta_1 = \delta + \frac{k\gamma}{\nu A}\right)$. And hence

$$\Omega = k\tan\tfrac{1}{2}\left(\delta_1 + \frac{kt}{A}\right) = k\tan\psi,$$

where

$$\psi = \tfrac{1}{2}\left(\delta_1 + \frac{kt}{A}\right).$$

Substituting these values,

$$\mathrm{a} = -MA\sin i, \quad \mathrm{b} = MA\cos i, \quad \mathrm{c} = -A(n-\nu),$$

$$2v = A^2M^2\{1+\cos(\theta+i)\} = 2M^2A^2\cos^2\tfrac{1}{2}(\theta+i);$$

and substituting in the equations (14) the values of λ, μ, ν reduce themselves to

$$\begin{cases} \lambda = \dfrac{1}{MA\cos\frac{1}{2}(\theta+i)}\{k\tan\psi\sin\tfrac{1}{2}(\theta-i) - A(n-\nu)\cos\tfrac{1}{2}(\theta-i)\}, \\ \mu = \dfrac{1}{MA\cos\frac{1}{2}(\theta+i)}\{k\tan\psi\cos\tfrac{1}{2}(\theta-i) + A(n-\nu)\sin\tfrac{1}{2}(\theta-i)\}, \\ \nu = \quad \tan\tfrac{1}{2}(\theta+i); \end{cases}$$

where, recapitulating, $\theta = \nu t + \gamma$, $2\psi = \dfrac{kt}{A} + \delta_1$.

I may notice, in connexion with the problem of rotation, a memoir, "Specimen Inaugurale de motu gyratorio corporis rigidi &c.," by A. S. Rueb (Utrecht, 1834), which contains some very interesting developments of the ordinary solution of the problem, by means of the theory of elliptic functions.