75.

ON THE ATTRACTION OF AN ELLIPSOID.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 50—65.]

PART I.—ON LEGENDRE'S SOLUTION OF THE PROBLEM OF THE ATTRACTION OF AN ELLIPSOID ON AN EXTERNAL POINT.

I PROPOSE in the following paper to give an outline of Legendre's investigation of the attraction of an ellipsoid upon an exterior point, ["Mémoire sur les Intégrales Doubles," Paris, Mem. Acad. Sc. for 1788, published 1791, pp. 454—486], one of the earliest and (notwithstanding its complexity) most elegant solutions of the problem. It will be convenient to begin by considering some of the geometrical properties of a system of cones made use of in the investigation.

§ 1. The equation of the ellipsoid referred to axes parallel to the principal axes, and passing through the attracted point, may be written under the form

$$l(x-a)^2+m(y-b)^2+n(z-c)^2-k=0,$$

(where $\sqrt{\dfrac{k}{l}}$, $\sqrt{\dfrac{k}{m}}$, $\sqrt{\dfrac{k}{n}}$ are the semiaxes, and a, b, c are the coordinates of the attracted point referred to the principal axes). Or putting $la^2+mb^2+nc^2-k=\delta$, this equation becomes

$$lx^2+my^2+nz^2-2(lax+mby+ncz)+\delta=0.$$

The cones in question are those which have the same axes and directions of circular section as the cone having its vertex in the attracted point and circumscribed about the ellipsoid. The equation of the system of cones (containing the arbitrary parameter ω) is

$$(lx^2+my^2+nz^2)\,\delta-(lax+mby+ncz)^2+\omega^2(x^2+y^2+z^2)=0\,;$$

or as it may also be written,

$$(\omega^2+l\delta-l^2a^2)\,x^2+(\omega^2+m\delta-m^2b^2)\,y^2+(\omega^2+n\delta-n^2c^2)\,z^2-2mnbcyz-2nlcazx-2lmabxy=0.$$

For $\omega = 0$, the cone coincides with the circumscribed cone; as ω increases, the aperture of the cone gradually diminishes, until for a certain value, $\omega = \Omega$, the cone reduces itself to a straight line (the normal of the confocal ellipsoid through the attracted point). It is easily seen that Ω^2 is the positive root of the equation

$$\frac{l^2a^2}{\Omega^2 + l\delta} + \frac{m^2b^2}{\Omega^2 + m\delta} + \frac{n^2c^2}{\Omega^2 + n\delta} = 1,$$

a different form of which may be obtained by writing $\Omega^2 = \dfrac{k\delta}{\xi}$, ξ being then determined by means of the equation

$$\frac{la^2}{k + l\xi} + \frac{mb^2}{k + m\xi} + \frac{nc^2}{k + n\xi} = 1;$$

that is,
$$\sqrt{\left(\frac{k}{l} + \xi\right)}, \quad \sqrt{\left(\frac{k}{m} + \xi\right)}, \quad \sqrt{\left(\frac{k}{n} + \xi\right)}$$

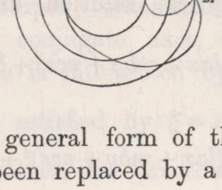
are the semiaxes of the confocal ellipsoid through the attracted point.

In the case where ω remains indeterminate, it is obvious that the cone intersects the ellipsoid in the curve in which the ellipsoid is intersected by a certain hyperboloid of revolution of two sheets, having the attracted point for a focus, and the plane of contact of the ellipsoid with the circumscribed cone (that is the polar plane of the attracted point) for the corresponding directrix plane: also the excentricity of the hyperboloid is $\dfrac{1}{\omega}\sqrt{(l^2a^2 + m^2b^2 + n^2c^2)}$, which suffices for its complete determination. For $\omega = 0$, the hyperboloid reduces itself to the plane of contact of the ellipsoid with the circumscribed cone, and for $\omega = \Omega$, the hyperboloid and the ellipsoid have a double contact, viz. at the points where the ellipsoid is intersected by the normal to the confocal ellipsoid through the attracted point.

If ω remains constant while k is supposed to vary, that is, if the ellipsoid vary in magnitude (the position and proportion of its axes remaining unaltered), the locus of the intersection of the cone and the ellipsoid is a surface of the fourth order defined by the equation

$$(lx^2 + my^2 + nz^2 - lax - mby - ncz)^2 = \omega^2(x^2 + y^2 + z^2),$$

and consisting of an exterior and an interior sheet meeting at the attracted point, which is a conical point on the surface, viz. a point where the tangent plane is



replaced by a tangent cone. The general form of this surface is easily seen from the figure, in which the ellipsoid has been replaced by a sphere, and the surface in question is that generated by the revolution of the curve round the line CM. The surface of the fourth order being once described for any particular value of ω, the cone corresponding to any one of the series of similar, similarly situated, and concentric ellipsoids is at once determined by means of the intersection of the ellipsoid in question with the surface of the fourth order. It is clear too that there is always one of these ellipsoids which has a double contact with the surface of the fourth order, viz. at the points where this ellipsoid is intersected by the normal to the confocal ellipsoid through the attracted point; thus there is always an ellipsoid for which the cone corresponding to a given value of ω reduces itself to a straight line.

Consider the attracting ellipsoid, which for distinction may be termed the ellipsoid S, and the two cones C, C', which correspond to the values ω, $\omega - d\omega$ of the variable parameter. Legendre shows that the attraction of the portion of the ellipsoid S included between the two cones C, C' is independent of the quantity k, which determines the magnitude of the ellipsoid: that is, if there be any other ellipsoid T similarly situated and concentric to and with the ellipsoid S, and two cones D, D', which for the ellipsoid T correspond to the same values ω, $\omega - d\omega$ of the variable parameter; then the attraction of the portion of the ellipsoid S, included between the two cones C, C', is equal to the attraction of the portion of the ellipsoid T included between the two cones D and D'. By taking for the ellipsoid T the ellipsoid for which the cone D reduces itself to a straight line, the aperture of the cone D' is indefinitely small, and the attraction of the portion of the ellipsoid T included within the cone D' is at once determined; and thus the attraction of the portion of the ellipsoid S included between the cones C, C' is obtained in a finite form. Hence the attraction of the portion of the ellipsoid S included between any two cones $C_{,}$, $C_{,,}$ corresponding to the values $\omega_{,}$, $\omega_{,,}$ of the variable parameter, is expressed by means of a single integral, and by extending the integration from $\omega = 0$ to $\omega = \sqrt{\left(\frac{k\delta}{\xi}\right)}$, the attraction of the whole ellipsoid is obtained in the form of a single integral readily reducible to that given by the ordinary solutions. It is clear too that the attraction of the portion of the ellipsoid S included between any two cones $C_{,}$, $C_{,,}$, is equal to that of the portion of the ellipsoid T included between the corresponding cones $D_{,}$ and $D_{,,}$. Hence also, assuming for the ellipsoid T, that for which the cone $D_{,,}$ reduces itself to a straight line, and supposing that the cones $C_{,}$ and $D_{,}$ coincide with the circumscribing cones, the attraction of the portion of the ellipsoid S exterior to the cone $C_{,,}$ is equal to the attraction of the entire ellipsoid T. More generally, the attraction of the portion of the ellipsoid S included between the cones $C_{,}$ and $C_{,,}$ is equal to the attraction of the shell included between the surfaces of the two ellipsoids, for which the cones $D_{,}$ and $D_{,,}$ respectively reduce themselves to straight lines.

§ 2. Proceeding to the analytical solution, and resuming the equation of the ellipsoid

$$lx^2 + my^2 + nz^2 - 2(lax + mby + ncz) + \delta = 0,$$

and that of the cone

$$(lx^2 + my^2 + nz^2)\,\delta - (lax + mby + ncz)^2 + \omega^2(x^2 + y^2 + z^2) = 0;$$

consider a radius vector on the conical surface such that the cosines of its inclinations to the axes are

$$\frac{P}{\Theta},\ \frac{Q}{\Theta},\ \frac{R}{\Theta},\ \{\Theta = \sqrt{(P^2+Q^2+R^2)}\},$$

P, Q, R and Θ being functions of the parameter ω, and of another variable ϕ, which determines the position of the radius vector upon the conical surface. Also let ρ be the length of the portion of the radius vector which lies within the ellipsoid; then representing by dS the spherical angle corresponding to the variations of ω and ϕ, the attraction in the direction of the axis of x is given by the formula

$$A = \iint \rho \frac{P}{\Theta} dS.$$

Also by a known formula

$$dS = \frac{1}{\Theta^3}\left\{P\left(\frac{dQ}{d\phi}\frac{dR}{d\omega} - \frac{dR}{d\phi}\frac{dQ}{d\omega}\right) + Q\left(\frac{dR}{d\phi}\frac{dP}{d\omega} - \frac{dR}{d\omega}\frac{dP}{d\phi}\right) + R\left(\frac{dP}{d\phi}\frac{dQ}{d\omega} - \frac{dP}{\delta\omega}\frac{dQ}{d\phi}\right)\right\},$$

and it is easy to obtain

$$\rho = \frac{2\omega\Theta^2}{lP^2 + mQ^2 + nR^2}.$$

The quantities P, Q, R have now to be expressed as functions of ω, ϕ, so that their values substituted for x, y, z, may satisfy identically the equation of the cone. This may be done by assuming

$$P = p,$$

$$Q = mb(\omega^2 + n\delta)\left(la + \frac{D}{U}\cos\phi\right) - \frac{D\sqrt{p}}{U} nc \sin\phi,$$

$$R = nc(\omega^2 + m\delta)\left(la + \frac{D}{U}\cos\phi\right) + \frac{D\sqrt{p}}{U} mb \sin\phi,$$

where

$$p = (\omega^2 + m\delta)(\omega^2 + n\delta) - m^2b^2(\omega^2 + n\delta) - n^2c^2(\omega^2 + m\delta),$$

$$U^2 = m^2b^2(\omega^2 + n\delta) + n^2c^2(\omega^2 + m\delta),$$

$$D^2 = (\omega^2 + l\delta)(\omega^2 + m\delta)(\omega^2 + n\delta)\left\{\frac{l^2a^2}{\omega^2 + l\delta} + \frac{m^2b^2}{\omega^2 + m\delta} + \frac{n^2c^2}{\omega^2 + n\delta} - 1\right\};$$

a system of values which, in point of fact, depend upon the following geometrical considerations: by treating x as a constant in the equation of the cone, that is, in effect by considering the sections of the cone by planes parallel to that of yz, the equation of the cone becomes that of an ellipse; transforming first to a set of axes through the centre and then to a set of conjugate axes, one of which passes through the point where the plane of the ellipse is intersected by the axis of x, then the equation takes the form $\frac{\xi^2}{A^2} + \frac{\eta^2}{B^2} = 1$, and is satisfied by $\xi = A\cos\phi$, $\eta = B\sin\phi$, and $\frac{y}{x}$, $\frac{z}{x}$ being of course linear functions of these values, the preceding expressions may be obtained.

The substitution of the above values of P, Q, R (a somewhat tedious one which does not occur in the process actually made use of by Legendre) gives the very simple result,

$$dS = \frac{P^{\frac{1}{2}}\omega\, d\omega\, d\phi}{\Theta};$$

and the formula for the attraction becomes

$$A = 2\iint \frac{P^{\frac{3}{2}}\omega^2\, d\omega\, d\phi}{lP^2 + mQ^2 + nR^2},$$

which is of the form $A = 2\int I\omega^2 d\omega$, where

$$I = \int \frac{P^{\frac{3}{2}} d\phi}{lP^2 + mQ^2 + nR^2},$$

which last integral, taken between the limits $\phi = 0$ and $\phi = 2\pi$, and multiplied by $2\omega^2 d\omega$, expresses the attraction of the portion of the ellipsoid included between two consecutive cones. The integration is evidently possible, but the actual performance of it is the great difficulty of Legendre's process. The result, as before mentioned, is independent of the quantity k, or, what comes to the same thing, of the quantity δ: assuming this property (an assumption which in fact resolves itself into the consideration of the ellipsoid for which the cone reduces itself to a straight line, as before explained), the integral is at once obtained by writing $\delta = \Delta$ where Δ represents the positive root of the equation

$$\frac{l^2a^2}{\omega^2 + l\Delta} + \frac{m^2b^2}{\omega^2 + m\Delta} + \frac{n^2c^2}{\omega^2 + n\Delta} - 1 = 0.$$

This gives

$$P = \frac{l^2a^2(\omega^2 + m\Delta)(\omega^2 + n\Delta)}{\omega^2 + l\Delta},$$

$$Q = lmab\,(\omega^2 + n\Delta),$$

$$R = lnac\;(\omega^2 + m\Delta),$$

values independent of ϕ, or the value of I is found by multiplying the quantity under the integral sign by 2π: and hence we have

$$A = 4\pi la \int \frac{\omega^2(\omega^2 + l\Delta)^{\frac{1}{2}}(\omega^2 + m\Delta)^{\frac{3}{2}}(\omega^2 + n\Delta)^{\frac{3}{2}}\, d\omega}{l^3a^2(\omega^2 + m\Delta)^2(\omega^2 + n\Delta)^2 + m^3b^2(\omega^2 + n\Delta)^2(\omega^2 + l\Delta)^2 + n^3c^2(\omega^2 + n\Delta)^2(\omega^2 + l\Delta)^2},$$

where of course Δ is to be considered as a function of ω. By integrating from $\omega = \omega_{,}$ to $\omega = \omega_{,,}$, we have the attraction of the portion of the ellipsoid included between any two of the series of cones, and to obtain the attraction of the whole ellipsoid we must integrate from $\omega = 0$ to $\omega = \sqrt{\left(\frac{k\delta}{\xi}\right)}$, where ξ is determined as before by the equation

$$\frac{la^2}{k + l\xi} + \frac{mb^2}{k + m\xi} + \frac{nc^2}{k + n\xi} = 1;$$

and it is obvious that for this value of ω we have $\Delta = \delta$. The expression for the attraction is easily reduced to a known form by writing $y = \dfrac{k\Delta}{\omega^2}$; this gives

$$A = 4\pi la \int \frac{k^{\frac{1}{2}}(k+ly)^{\frac{1}{2}}(k+my)^{\frac{3}{2}}(k+ny)^{\frac{3}{2}}\omega d\omega}{l^3a^2(k+my)^2(k+ny)^2 + m^3b^2(k+ny)^2(k+ly)^2 + n^3c^2(k+ly)^2(k+my)^2}.$$

Also

$$\omega^2 = k\left(\frac{l^2a^2}{k+ly} + \frac{m^2b^2}{k+my} + \frac{n^2c^2}{k+ny}\right);$$

whence

$$\omega d\omega = -\frac{k\,[l^3a^2(k+my)^2(k+ny)^2 + m^3b^2(k+ny)^2(k+ly)^2 + n^3c^2(k+ly)^2(k+my)^2]}{2(k+ly)^2(k+my)^2(k+ny)^2},$$

and thus
$$A = 2\pi k^{\frac{3}{2}} la \int \frac{dy}{(k+ly)^{\frac{3}{2}}(k+my)^{\frac{1}{2}}(k+ny)^{\frac{1}{2}}},$$

where for the entire ellipsoid the integral is to be taken from $y = \xi$ to $y = \infty$. A better known form is readily obtained by writing $x^2 = \dfrac{k+l\xi}{k+ly}$, in which case the limits for the entire ellipsoid are $x = 0$, $x = 1$.

It may be as well to indicate the first step of the reduction of the integral I, viz. the method of resolving the denominator into two factors. We have identically,

$$(\Delta - \delta)(lP^2 + mQ^2 + nR^2) = \omega^2(P^2 + Q^2 + R^2) + \Delta(lP^2 + mQ^2 + nR^2) - (laP + mbQ + ncR)^2,$$

and the second side of this equation is resolvable into two factors independently of the particular values of P, Q, R. Representing this second side for a moment in the notation of a general quadratic function, or under the form

$$AP^2 + BQ^2 + CR^2 + 2FQR + 2GRP + 2HPQ,$$

we have the required solution,

$lP^2 + mQ^2 + nR^2 =$

$$\frac{1}{A}\,[AP + \{H + \sqrt{(-\mathfrak{C})}\}\,Q + \{G + \sqrt{(-\mathfrak{B})}\}\,R]\,[AP + \{H - \sqrt{(-\mathfrak{C})}\}\,Q + \{G - \sqrt{(-\mathfrak{B})}\}\,R];$$

where, as usual, $\mathfrak{B} = CA - G^2$, $\mathfrak{C} = AB - H^2$, and the roots must be so taken that $\sqrt{(-\mathfrak{B})}\,\sqrt{(-\mathfrak{C})} = \mathfrak{F}\;\{\mathfrak{F} = (GH - AF)\}$.

I have purposely restricted myself so far to the problem considered by Legendre: the general transformation, of which the preceding is a particular case, and also a simpler mode of effecting the integration, are given in the next part of this paper.

PART II.—ON A FORMULA FOR THE TRANSFORMATION OF CERTAIN MULTIPLE INTEGRALS.

Consider the integral

$$V = \int F(x, y, \ldots)\, dx\, dy \ldots,$$

where the number of variables $x, y, \ldots$ is equal to n, and $F(x, y, \ldots)$ is a homogeneous function of the order μ.

Suppose that $x, y, \ldots$ are connected by a homogeneous equation $\psi(x, y, \ldots) = 0$ containing a variable parameter ω (so that ω is a homogeneous function of the order zero in the variables $x, y, \ldots$). Then, writing

$$r^2 = x^2 + y^2 + \ldots, \quad x = r\alpha, \quad y = r\beta, \ldots$$

the quantities $\alpha, \beta, \ldots$ are connected by the equations

$$\alpha^2 + \beta^2 + \ldots = 1, \quad \psi(\alpha, \beta, \ldots) = 0,$$

and we may therefore consider them as functions of ω and of $(n-2)$ independent variables θ, ϕ, &c.; whence

$$dx\, dy \ldots = r^{n-1} \nabla\, dr\, d\omega\, d\theta \ldots,$$

where

$$\nabla = \begin{vmatrix} \alpha, & \beta, & \ldots \\ \dfrac{d\alpha}{d\omega}, & \dfrac{d\beta}{d\omega}, & \\ \dfrac{d\alpha}{d\theta}, & \dfrac{d\beta}{d\theta}, & \\ \vdots & & \end{vmatrix}$$

Also

$$F(x, y, \ldots) = r^{\mu} F(\alpha, \beta \ldots),$$

and therefore

$$V = \int r^{\mu+n-1} F(\alpha, \beta, \ldots) \nabla\, dr\, d\omega\, d\theta \ldots,$$

or, integrating with respect to r,

$$\int r^{\mu+n-1}\, dr = \frac{1}{\mu+n} r^{\mu+n},$$

which, taken between the proper limits, is a function of $\alpha, \beta, \ldots$, equal $f(\alpha, \beta, \ldots)$ suppose; this gives

$$V = \int f(\alpha, \beta, \ldots) F(\alpha, \beta, \ldots) \nabla\, d\omega\, d\theta \ldots,$$

in which I shall assume that the limits of ω are constant. If, in order to get rid of the condition $\alpha^2 + \beta^2 \ldots = 1$, we assume

$$\alpha = \frac{p}{\rho}, \quad \beta = \frac{q}{\rho}, \ldots, \quad \rho^2 = p^2 + q^2 + \ldots$$

the preceding expression for V becomes

$$V=\int f\left(\frac{p}{\rho},\ \frac{q}{\rho},\ldots\right) F(p,\ q,\ldots)\,\frac{1}{\rho^{\mu+n}}\,D\,d\omega\,d\theta\ldots,$$

in which

$$D=\left|\begin{matrix} p\ , & q\ , & \ldots \\ \dfrac{dp}{d\omega}, & \dfrac{dq}{d\omega}, & \\ \dfrac{dp}{d\theta}, & \dfrac{dq}{d\theta}. & \\ \vdots & & \end{matrix}\right|$$

Assume

$$p=P\xi+P'\eta+P''\zeta\ldots$$
$$q=Q\xi+Q'\eta+Q''\zeta\ldots$$
$$\vdots$$

where the number of variables $\xi,\ \eta,\ \zeta\ldots$ (functions in general of $\omega,\ \theta$, &c.) is n, and where the coefficients $P,\ Q$, &c. are supposed to be functions of ω only. We have

$$\frac{dp}{d\theta}=P\frac{d\xi}{d\theta}+P'\frac{d\eta}{d\theta}+P''\frac{d\zeta}{d\theta}+\ldots,$$

$$\frac{dq}{d\theta}=Q\frac{d\xi}{d\theta}+Q'\frac{d\eta}{d\theta}+Q''\frac{d\zeta}{d\theta}+\ldots,$$

and, substituting these values as well as those of $p,\ q$, &c., but retaining the terms $\dfrac{dp}{d\omega},\ \dfrac{dq}{d\omega}$, &c. in their original form, the determinant D resolves itself into the sum of a series of products,

$$\left|\begin{matrix} \dfrac{dp}{d\omega}, & \dfrac{dq}{d\omega}, \ldots \\ P', & Q', \\ P'', & Q'', \\ \vdots & \end{matrix}\right| \left|\begin{matrix} 1 & \cdot & \\ \cdot & \eta\ , & \zeta\ , \\ \cdot & \dfrac{d\eta}{d\theta}, & \dfrac{d\zeta}{d\theta}, \\ \vdots & & \end{matrix}\right|$$

Let Ψ be the function to which $\psi\ (p,\ q,\ldots)$ is changed by the substitution of the above values of $p,\ q,\ldots$ so that Ψ is a homogeneous function of $\xi,\ \eta,\ \zeta,\ldots$ and we have the relation $\Psi=0$. (It will be convenient to consider $\xi,\ \eta,\ \zeta,\ldots$ as functions of $\omega,\ \theta$, &c., such as to satisfy identically this last equation.) We deduce

$$\frac{1}{X}\left|\begin{matrix} 1, & \cdot & \cdot & \ldots \\ \cdot & \eta\ , & \zeta\ , & \\ \cdot & \dfrac{d\eta}{d\theta}, & \dfrac{d\zeta}{d\theta}, & \\ \vdots & & & \end{matrix}\right| = \frac{1}{Y}\left|\begin{matrix} \cdot & 1, & \cdot & \ldots \\ \xi\ , & \cdot & \zeta\ , & \\ \dfrac{d\xi}{d\theta}, & \cdot & \dfrac{d\zeta}{d\theta}, & \\ \vdots & & & \end{matrix}\right| = \&\text{c.} = S \text{ suppose,}$$

where for shortness $X = \dfrac{d\Psi}{d\xi}$, $Y = \dfrac{d\Psi}{d\eta}$, &c. The substitution of these values gives

$$D = \begin{vmatrix} & \dfrac{dp}{d\omega}, & \dfrac{dq}{d\omega}, & \dots \\ X, & P\ , & Q\ , & \\ Y, & P'\ , & Q'\ , & \\ Z, & P''\ , & Q''\ , & \\ \vdots & & & \end{vmatrix} S,$$

where it will be remarked that the successive horizontal lines (after the first) of the determinant are the differential coefficients of Ψ, p, q, ... with respect to ξ, with respect to η, &c. In general, if ψ denote any function of p, q, ... these quantities being themselves functions of ω, ξ, η, ..., and ξ, η, ... containing ω; also if Ψ be what ψ becomes when for p, q, ... we substitute their values in ω, ξ, η, ...; then we have identically

$$\begin{vmatrix} \dfrac{d\psi}{d\omega}, & \dfrac{dp}{d\omega}, & \dfrac{dq}{d\omega}, & \dots \\ X\ , & P\ , & Q\ , & \\ Y\ , & P'\ , & Q'\ , & \\ Z\ , & P''\ , & Q''\ , & \\ \vdots & & & \end{vmatrix} = 0. \text{ [1]}$$

In the present case however, writing for shortness $\psi(p, q, \dots) = \psi$, this function ψ contains ω explicitly as well as implicitly through p, q, &c. The formula is still true if for $\dfrac{d\psi}{d\omega}$ we substitute $\dfrac{d(\psi)}{d\omega} - \dfrac{d\psi}{d\omega}$, $\dfrac{d\psi}{d\omega}$ on the second side denoting a partial differential coefficient taken only so far as ω is explicitly contained in ψ. And considering p, q, ... as functions of ω, ξ, η, ... (ξ, η, ... themselves functions of ω and of other variables which need not here be considered), ψ or Ψ vanishes identically, and we have $\dfrac{d(\psi)}{d\omega} = 0$. Hence, in the last formula, we have to write $-\dfrac{d\psi}{d\omega}$ instead of $\dfrac{d\psi}{d\omega}$, and we thus derive

$$\frac{d\psi}{d\omega} \begin{vmatrix} P, & Q, & \dots \\ P', & Q', & \\ \vdots & & \end{vmatrix} = \begin{vmatrix} . & \dfrac{dp}{d\omega}, & \dfrac{dq}{d\omega}, & \dots \\ X, & P\ , & Q\ , & \\ Y, & P'\ , & Q'\ , & \\ Z, & P''\ , & Q''\ , & \\ \vdots & & & \end{vmatrix},$$

[1] This formula, or one equivalent to it, is given in Jacobi's memoir "De Determinantibus Functionalibus," *Crelle*, t. XXII. [1841] p. 319.

whence D becomes

$$D = \begin{vmatrix} P, & Q, \dots \\ P', & Q', \\ \vdots & \end{vmatrix} \frac{d\psi}{d\omega} S,$$

which is the value to be made use of in the equation

$$V = \int f\left(\frac{p}{\rho}, \frac{q}{\rho}, \dots\right) F(p, q, \dots) \frac{1}{\rho^{\mu+n}} D\, d\omega\, d\theta \dots$$

The principal use of the formula is where ψ is a homogeneous function of the second order of $p, q, \dots$. Thus, suppose

$$\psi = \tfrac{1}{2}(Ap^2 + Bq^2 \dots + 2Hpq + \dots),$$

also, for the sake of conformity to the usual notation in the theory of transformation of quadratic functions, writing $\alpha, \alpha', \dots \beta, \beta', \dots$ instead of $P, P', \dots Q, Q', \dots$ and putting after the differentiations $\xi = 1$, we have

$$\begin{aligned} p &= \alpha + \alpha'\eta + \alpha''\zeta + \dots, \\ q &= \beta + \beta'\eta + \beta''\zeta + \dots, \\ &\vdots \end{aligned}$$

values which we may assume to give rise to the equation

$$(Ap^2 + Bq^2 \dots + 2Hpq \dots) = (1 - \eta^2 - \zeta^2 - \dots),$$

(where $\eta, \zeta, \dots$ are taken to be functions of θ, &c. such as to satisfy identically the equation $1 = \eta^2 + \zeta^2 + \dots$).

Hence, by a well-known property, if

$$\begin{vmatrix} A, & H, \dots \\ H, & B, \\ \vdots & \end{vmatrix} = \kappa,$$

we have

$$\begin{vmatrix} \alpha, & \beta, \dots \\ \alpha', & \beta', \\ \vdots & \end{vmatrix} = \sqrt{\left\{\frac{(-)^{n-1}}{\kappa}\right\}};$$

so that, observing that in the present case $X = 1$, and therefore

$$S = \begin{vmatrix} \eta, & \zeta, \dots \\ \dfrac{d\eta}{d\theta}, & \dfrac{d\zeta}{d\theta}, \\ \vdots & \end{vmatrix}$$

we have

$$V = \int \frac{(-)^{\frac{1}{2}(n-1)}}{\kappa^{\frac{1}{2}}} f\left(\frac{p}{\rho}, \frac{q}{\rho}, \dots\right) F(p, q, \dots) \frac{1}{\rho^{\mu+n}} \frac{d\psi}{d\omega} S\, d\omega\, d\theta \dots$$

The remainder of the process of integration may in many cases be effected by the method made use of by Jacobi in the memoir "De binis quibuslibet functionibus &c." *Crelle*, t. XII. [1834] p. 1, viz. the coefficients α, α', &c., β, &c., may in addition to the conditions which they are already supposed to satisfy, be so determined as to reduce any homogeneous function of $p, q, r, \ldots$ entering into the integral to a form containing the squares only of the variables. This method is applied in the memoir in question to the integrals of n variables, analogous to those which give the attraction of an ellipsoid; and that directly without effecting an integration with respect to the radius vector. I proceed to show how the preceding investigations lead to Legendre's integral, and how the method in question effects with the utmost simplicity the integration which Legendre accomplished by means of what Poisson has spoken of as inextricable calculations.

Consider in particular the formula

$$V = \int \frac{(x,\ y\ \ldots)^h\, dx\, dy \ldots}{(x^2 + y^2 \ldots)^{\frac{3}{2} - i}},$$

the number of variables being as before n, and $(x, y, \ldots)^h$ denoting a homogeneous function of the order h. The equation for the limits is assumed to be

$$l\,(x-a)^2 + m\,(y-b)^2 + \ldots = k.$$

Assume

$$\psi\,(x,\ y\ ,\ldots) = \omega^2\,(x^2 + y^2 + \ldots) + \delta\,(lx^2 + my^2 + \ldots) - (lax + mby + \ldots)^2,$$

(where δ, $= la^2 + mb^2 \ldots - k$, is taken to be positive); or more simply,

$$\psi\,(x,\ y\ \ldots) = (\omega^2 + l\delta - l^2a^2)\,x^2 + (\omega^2 + m\delta - m^2b^2)\,y^2 + \ldots - 2lm\,ab\,xy - \ldots$$

Here $\mu = h + 2i - 3$. Also, putting for shortness

$$lap + mbq + \ldots = \Lambda, \quad lp^2 + mq^2 + \ldots = \Phi,$$

it is easy to obtain

$$f\left(\frac{p}{\rho},\ \frac{q}{\rho}, \ldots\right) = \frac{1}{h + 2i + n - 3}\,\frac{\rho^{h+2i+n-3}\,[(\Lambda + \omega\rho)^{h+2i+n-3} - (\Lambda - \omega\rho)^{h+2i+n-3}]}{\Phi^{h+2i+n-3}}.$$

Also
$$F(p,\ q, \ldots) = \frac{(p,\ q, \ldots)^h}{\rho^{3-2i}}, \quad \frac{d\psi}{d\omega} = 2\omega\rho^2,$$

$$\kappa = (\omega^2 + l\delta)\,(\omega^2 + m\delta) \ldots \left(1 - \frac{l^2a^2}{\omega^2 + l\delta} - \frac{m^2b^2}{\omega^2 + m\delta} - \&\text{c.}\right),$$

values which give

$$V = \frac{2}{h + 2i + n - 3} \int \frac{(-)^{\frac{1}{2}(n-1)}}{\kappa^{\frac{1}{2}}}\ \frac{\omega\rho^{2i-1}\,[(\Lambda + \omega\rho)^{h+2i+n-3} - (\Lambda - \omega\rho)^{h+2i+n-3}]\,(p,\ q, \ldots)^h\, Sd\omega\, d\theta \ldots}{\Phi^{h+2i+n-3}},$$

where it will be remembered that $p, q, \ldots$ are linear functions (with constant terms) of $(n-1)$ variables $\eta, \zeta, \ldots$, these last mentioned quantities being themselves functions of $(n-2)$ variables θ, &c. such that $1-\eta^2-\zeta^2-\ldots=0$ identically. If besides we suppose that $\Phi=lp^2+mq^2+\ldots$ reduces itself to the form $\frac{1}{P}-\frac{\eta^2}{Q}-$ &c., we have, by the formula of the paper "On the Simultaneous Transformation of two Homogeneous Equations of the Second Order," [74],

$$\left(1-\frac{\lambda}{P}\right)\left(1-\frac{\lambda}{Q}\right)\ldots=$$

$$\frac{1}{\kappa}(\omega^2+l\delta-l\lambda)(\omega^2+m\delta-m\lambda)\ldots\left(1-\frac{l^2a^2}{\omega^2+l\delta-l\lambda}-\frac{m^2b^2}{\omega^2+m\delta-m\lambda}-\ldots\right),$$

which is true, whatever be the value of λ.

It seems difficult to proceed further with the general formula, and I shall suppose $n=3$, $i=0$, $h=1$, $(x, y\ldots)^h=x$, or write

$$V=\int''\frac{x\,dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

the equation of the limits being

$$l(x-a)^2+m(y-b)^2+n(z-c)^2=k.$$

Here we may assume $\eta=\cos\theta$, $\zeta=\sin\theta$, (values which give $S=1$). And we have

$$V=2\int\frac{\omega^2 d\omega}{\kappa^{\frac{1}{2}}}\int\frac{(\alpha+\alpha'\cos\theta+\alpha''\sin\theta)\,d\theta}{\frac{1}{P}-\frac{\cos^2\theta}{Q}-\frac{\sin^2\theta}{R}},$$

from $\theta=0$ to $\theta=2\pi$; or, what comes to the same thing,

$$V=8\int\frac{\omega^2\,d\omega}{\kappa^{\frac{1}{2}}}\int\frac{\alpha\,d\theta}{\frac{1}{P}-\frac{\cos^2\theta}{Q}-\frac{\sin^2\theta}{R}},$$

from $\theta=0$ to $\theta=\frac{1}{2}\pi$. Hence

$$V=4\pi\int\frac{\alpha\omega^2 P\sqrt{(QR)}\,d\omega}{\kappa^{\frac{1}{2}}\sqrt{\{(P-Q)(P-R)\}}};$$

we have from the formulæ of the paper before quoted,

$$\alpha^2=\frac{QR}{\kappa}\frac{(B-mP)(C-nP)-F^2}{(P-Q)(P-R)},$$

B, C, F, being the coefficients of y^2, z^2, yz in $\psi(x, y, z)$, viz.

$$B=\omega^2+m\delta-m^2b^2,\quad C=\omega^2+n\delta-n^2c^2,\quad F=mnbc;$$

and consequently

$$V=4\pi\int\omega^2\,d\omega\,\frac{PQR}{\kappa}\,\frac{\{(B-mP)(C-nP)-F^2\}^{\frac{1}{2}}}{(P-Q)(P-R)}.$$

Also from the equation

$$\left(1-\frac{\lambda}{P}\right)\left(1-\frac{\lambda}{Q}\right)\left(1-\frac{\lambda}{R}\right)=$$

$$\frac{1}{\kappa}(\omega^2+l\delta-l\lambda)(\omega^2+m\delta-m\lambda)(\omega^2+n\delta-n\lambda)\left(1-\frac{l^2a^2}{\omega^2+l\delta-l\lambda}-\frac{m^2b^2}{\omega^2+m\delta-m\lambda}-\frac{n^2c^2}{\omega^2+n\delta-n\lambda}\right);$$

differentiating with respect to λ, and writing $\lambda=P$,

$$\frac{-\frac{1}{P}\left(1-\frac{P}{Q}\right)\left(1-\frac{P}{R}\right)}{(\omega^2+l\delta-lP)(\omega^2+m\delta-mP)(\omega^2+n\delta-nP)}$$

$$=-\frac{1}{\kappa}\left\{\frac{l^3a^2}{(\omega^2+l\delta-lP)^2}+\frac{m^3b^2}{(\omega^2+m\delta-mP)^2}+\frac{n^3c^2}{(\omega^2+n\delta-nP)^2}\right\},$$

or, as this may be written,

$$\frac{PQR}{\kappa(P-Q)(P-R)}$$

$$=\frac{1}{(\omega^2+l\delta-lP)(\omega^2+m\delta-mP)(\omega^2+n\delta-nP)\left\{\frac{l^3a^2}{(\omega^2+l\delta-lP)^2}+\frac{m^3b^2}{(\omega^2+m\delta-mP)^2}+\frac{n^3c^2}{(\omega^2+n\delta-nP)^2}\right\}},$$

and from the values first written down, for B, C, F, we obtain $(B-mP)(C-nP)-F^2$

$$=(\omega^2+m\delta)(\omega^2+n\delta)-m^2b^2(\omega^2+n\delta)-n^2c^2(\omega^2+m\delta)-mP(\omega^2+n\delta-n^2c^2)-nP(\omega^2+m\delta-m^2b^2)+mnP^2$$

$$=(\omega^2+m\delta-mP)(\omega^2+n\delta-nP)-m^2b^2(\omega^2+n\delta-nP)-n^2c^2(\omega^2+m\delta-mP)$$

$=\dfrac{l^2a^2(\omega^2+m\delta-mP)(\omega^2+n\delta-nP)}{\omega^2+l\delta-lP}$, the last reduction being effected by means of the equation

$$1-\frac{l^2a^2}{\omega^2+l\delta-lP}-\frac{m^2b^2}{\omega^2+m\delta-mP}-\frac{n^2c^2}{\omega^2+n\delta-nP}=0.$$

Hence

$$\frac{PQR}{\kappa}\frac{\{(B-mP)(C-nP)-F^2\}^{\frac{1}{2}}}{(P-Q)(P-R)}$$

$$=\frac{la}{(\omega^2+l\delta-lP)^{\frac{3}{2}}(\omega^2+m\delta-mP)^{\frac{1}{2}}(\omega^2+n\delta-nP)^{\frac{1}{2}}\left\{\frac{l^3a^2}{(\omega^2+l\delta-lP)^2}+\frac{m^3b^2}{(\omega^2+m\delta-mP)^2}+\frac{n^3c^2}{(\omega^2+n\delta-nP)^2}\right\}}.$$

Substituting this value, and multiplying out the fractions in the denominator,

$$V=4\pi la\times$$

$$\int\frac{\omega^2(\omega^2+l\delta-lP)^{\frac{1}{2}}(\omega^2+m\delta-mP)^{\frac{3}{2}}(\omega^2+n\delta-nP)^{\frac{3}{2}}\,d\omega}{l^3a^2(\omega^2+m\delta-mP)^2(\omega^2+n\delta-nP)^2+m^3b^2(\omega^2+n\delta-nP)^2(\omega^2+l\delta-lP)^2+n^3c^2(\omega^2+l\delta-lP)^2(\omega^2+m\delta-mP)^2},$$

the reduction of which integral has been already treated of in the former part of this present memoir.