74.

ON THE SIMULTANEOUS TRANSFORMATION OF TWO HOMOGENEOUS FUNCTIONS OF THE SECOND ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 47—50.]

THE theory of the simultaneous transformation by linear substitutions of two homogeneous functions of the second order has been developed by Jacobi in the memoir "De binis quibuslibet functionibus &c., *Crelle*, t. XII. [1834], p. 1; but the simplest method of treating the problem is the one derived from Mr Boole's Theory of Linear Transformations, combined with the remark in his "Notes on Linear Transformations," in the *Cambridge Mathematical Journal*, vol. IV. [1845], p. 167. As I shall have occasion to refer to the results of this theory in the second part of my paper "On the Attraction of Ellipsoids," in the present number of the *Journal* [75], I take this opportunity of developing the formula in question; considering for greater convenience the case of three variables only.

Suppose that by a linear transformation,

$$x = \alpha\ x_1 + \beta\ y_1 + \gamma\ z_1,$$

$$y = \alpha'\ x_1 + \beta'\ y_1 + \gamma'\ z_1,$$

$$z = \alpha'' x_1 + \beta'' y_1 + \gamma'' z_1,$$

we have identically,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = a_1x_1^2 + b_1y_1^2 + c_1z_1^2 + 2f_1y_1z_1 + 2g_1z_1x_1 + 2h_1x_1y_1,$$

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = A_1x_1^2 + B_1y_1^2 + C_1z_1^2 + 2F_1y_1z_1 + 2G_1z_1x_1 + 2H_1x_1y_1.$$

Of course, whatever be the values of a, b, c, f, g, h, the same transformation gives

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + 2\mathrm{f}yz + 2\mathrm{g}zx + 2\mathrm{h}xy = \mathrm{a}_1x_1^2 + \mathrm{b}_1y_1^2 + \mathrm{c}_1z_1^2 + 2\mathrm{f}_1y_1z_1 + 2\mathrm{g}_1z_1x_1 + 2\mathrm{h}_1x_1y_1,$$

www.rcin.org.pl

provided that we have

$$\left\{\begin{array}{l} \mathrm{a}_1 = \mathrm{a}\alpha^2 + \mathrm{b}\alpha'^2 + \mathrm{c}\alpha''^2 + 2\mathrm{f}\alpha'\alpha'' + 2\mathrm{g}\alpha''\alpha + 2\mathrm{h}\alpha\alpha', \\ \mathrm{b}_1 = \mathrm{a}\beta^2 + \mathrm{b}\beta'^2 + \mathrm{c}\beta''^2 + 2\mathrm{f}\beta'\beta'' + 2\mathrm{g}\beta''\beta + 2\mathrm{h}\beta\beta', \\ \mathrm{c}_1 = \mathrm{a}\gamma^2 + \mathrm{b}\gamma'^2 + \mathrm{c}\gamma''^2 + 2\mathrm{f}\gamma'\gamma'' + 2\mathrm{g}\gamma''\gamma + 2\mathrm{h}\gamma\gamma', \\ \mathrm{f}_1 = \mathrm{a}\beta\gamma + \mathrm{b}\beta'\gamma' + \mathrm{c}\beta''\gamma'' + \mathrm{f}(\beta'\gamma'' + \beta''\gamma') + \mathrm{g}(\beta''\gamma + \beta\gamma'') + \mathrm{h}(\beta\gamma' + \beta'\gamma), \\ \mathrm{g}_1 = \mathrm{a}\gamma\alpha + \mathrm{b}\gamma'\alpha' + \mathrm{c}\gamma''\alpha'' + \mathrm{f}(\gamma'\alpha'' + \gamma''\alpha') + \mathrm{g}(\gamma''\alpha + \gamma\alpha'') + \mathrm{h}(\gamma\alpha' + \gamma'\alpha), \\ \mathrm{h}_1 = \mathrm{a}\alpha\beta + \mathrm{b}\alpha'\beta' + \mathrm{c}\alpha''\beta'' + \mathrm{f}(\alpha'\beta'' + \alpha''\beta') + \mathrm{g}(\alpha''\beta + \alpha\beta'') + \mathrm{h}(\alpha\beta' + \alpha'\beta). \end{array}\right.$$

Representing for a moment the equations between the pairs of functions of the second order by

$$u = u_1, \quad U = U_1, \quad v = v_1,$$

we have, whatever be the value of λ,

$$\lambda u + U + v = \lambda u_1 + U_1 + v_1.$$

Hence, if $\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \Pi$; then

$$\begin{vmatrix} \lambda a_1 + A_1 + \mathrm{a}_1, & \lambda h_1 + H_1 + \mathrm{h}_1, & \lambda g_1 + G_1 + \mathrm{g}_1 \\ \lambda h_1 + H_1 + \mathrm{h}_1, & \lambda b_1 + B_1 + \mathrm{b}_1, & \lambda f_1 + F_1 + \mathrm{f}_1 \\ \lambda g_1 + G_1 + \mathrm{g}_1, & \lambda f_1 + F_1 + \mathrm{f}_1, & \lambda c_1 + C_1 + \mathrm{c}_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} \lambda a + A + \mathrm{a}, & \lambda h + H + \mathrm{h}, & \lambda g + G + \mathrm{g} \\ \lambda h + H + \mathrm{h}, & \lambda b + B + \mathrm{b}, & \lambda f + F + \mathrm{f} \\ \lambda g + G + \mathrm{g}, & \lambda f + F + \mathrm{f}, & \lambda c + C + \mathrm{c} \end{vmatrix}$$

Hence, since a, b, c, f, g, h, are arbitrary,

$$\begin{vmatrix} \lambda a_1 + A_1, & \lambda h_1 + H_1, & \lambda g_1 + G_1 \\ \lambda h_1 + H_1, & \lambda b_1 + B_1, & \lambda f_1 + F_1 \\ \lambda g_1 + G_1, & \lambda f_1 + F_1, & \lambda c_1 + C_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} \lambda a + A, & \lambda h + H, & \lambda g + G \\ \lambda h + H, & \lambda b + B, & \lambda f + F \\ \lambda g + G, & \lambda f + F, & \lambda c + C \end{vmatrix}$$

which determine the relations which must subsist between the coefficients of the functions of the second order. We derive

$$\begin{vmatrix} a_1, & h_1, & g_1 \\ h_1, & b_1, & f_1 \\ g_1, & f_1, & c_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

and by comparing the coefficients of a, &c., if we write for shortness,

$$\mathfrak{A} = \begin{vmatrix} 1, & ., & . \\ ., & \lambda b + B, & \lambda f + F \\ ., & \lambda f + F, & \lambda c + C \end{vmatrix} \text{ \&c., then}$$

www.rcin.org.pl

$$\mathfrak{A}_1\alpha^2 + \mathfrak{B}_1\beta^2 + \mathfrak{C}_1\gamma^2 + 2\mathfrak{F}_1\beta\gamma + 2\mathfrak{G}_1\gamma\alpha + 2\mathfrak{H}_1\alpha\beta = \Pi^2\mathfrak{A},$$
$$\mathfrak{A}_1\alpha'^2 + \mathfrak{B}_1\beta'^2 + \mathfrak{C}_1\gamma'^2 + 2\mathfrak{F}_1\beta'\gamma' + 2\mathfrak{G}_1\gamma'\alpha' + 2\mathfrak{H}_1\alpha'\beta' = \Pi^2\mathfrak{B},$$
$$\mathfrak{A}_1\alpha''^2 + \mathfrak{B}_1\beta''^2 + \mathfrak{C}_1\gamma''^2 + 2\mathfrak{F}_1\beta''\gamma'' + 2\mathfrak{G}_1\gamma''\alpha'' + 2\mathfrak{H}_1\alpha''\beta'' = \Pi^2\mathfrak{C},$$
$$\mathfrak{A}_1\alpha'\alpha'' + \mathfrak{B}_1\beta'\beta'' + \mathfrak{C}_1\gamma'\gamma'' + \mathfrak{F}_1(\beta'\gamma'' + \beta''\gamma') + \mathfrak{G}_1(\gamma'\alpha'' + \gamma''\alpha') + \mathfrak{H}_1(\alpha'\beta'' + \alpha''\beta') = \Pi^2\mathfrak{F},$$
$$\mathfrak{A}_1\alpha''\alpha + \mathfrak{B}_1\beta''\beta + \mathfrak{C}_1\gamma''\gamma + \mathfrak{F}_1(\beta''\gamma + \beta\gamma'') + \mathfrak{G}_1(\gamma''\alpha + \gamma\alpha'') + \mathfrak{H}_1(\alpha''\beta + \alpha\beta'') = \Pi^2\mathfrak{G},$$
$$\mathfrak{A}_1\alpha\alpha' + \mathfrak{B}_1\beta\beta' + \mathfrak{C}_1\gamma\gamma' + \mathfrak{F}_1(\beta\gamma' + \beta'\gamma) + \mathfrak{G}_1(\gamma\alpha' + \gamma'\alpha) + \mathfrak{H}_1(\alpha\beta' + \alpha'\beta) = \Pi^2\mathfrak{H},$$

each of which virtually contains three equations on account of the indeterminate quantity λ. A somewhat more elegant form may be given to these equations; thus the first of them is

$$\begin{vmatrix} & \alpha, & \beta, & \gamma, \\ \alpha, & \lambda a_1 + A_1, & \lambda h_1 + H_1, & \lambda g_1 + G_1 \\ \beta, & \lambda h_1 + H_1, & \lambda b_1 + B_1, & \lambda f_1 + F_1 \\ \gamma, & \lambda g_1 + G_1, & \lambda f_1 + F_1, & \lambda c_1 + C_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} 1, & . & . \\ ., & \lambda b + B, & \lambda f + F \\ ., & \lambda f + F, & \lambda c + C \end{vmatrix}$$

from which the form of the whole system is sufficiently obvious. The actual values of the coefficients α, β, &c. can only be obtained in the particular case where $f_1 = g_1 = h_1 = F_1 = G_1 = H_1 = 0$. If we suppose besides (which is no additional loss of generality) that $a_1 = b_1 = c_1 = 1$, then the whole system of formulæ becomes

$$(A_1+\lambda)(B_1+\lambda)(C_1+\lambda) = \Pi^2 \begin{vmatrix} \lambda a + A, & \lambda h + H, & \lambda g + G \\ \lambda h + H, & \lambda b + B, & \lambda f + F \\ \lambda g + G, & \lambda f + F, & \lambda c + C \end{vmatrix};$$

$$1 = \Pi^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \text{ or } \Pi^2 = \kappa^{-1} \text{ suppose; and then}$$

$$(B_1+\lambda)(C_1+\lambda)\alpha^2 + (C_1+\lambda)(A_1+\lambda)\beta^2 + (A_1+\lambda)(B_1+\lambda)\gamma^2 = \frac{1}{\kappa}\mathfrak{A},$$
$$(B_1+\lambda)(C_1+\lambda)\alpha'^2 + (C_1+\lambda)(A_1+\lambda)\beta'^2 + (A_1+\lambda)(B_1+\lambda)\gamma'^2 = \frac{1}{\kappa}\mathfrak{B},$$
$$(B_1+\lambda)(C_1+\lambda)\alpha''^2 + (C_1+\lambda)(A_1+\lambda)\beta''^2 + (A_1+\lambda)(B_1+\lambda)\gamma''^2 = \frac{1}{\kappa}\mathfrak{C},$$
$$(B_1+\lambda)(C_1+\lambda)\alpha'\alpha'' + (C_1+\lambda)(A_1+\lambda)\beta'\beta'' + (A_1+\lambda)(B_1+\lambda)\gamma'\gamma'' = \frac{1}{\kappa}\mathfrak{F},$$
$$(B_1+\lambda)(C_1+\lambda)\alpha''\alpha + (C_1+\lambda)(A_1+\lambda)\beta''\beta + (A_1+\lambda)(B_1+\lambda)\gamma''\gamma = \frac{1}{\kappa}\mathfrak{G},$$
$$(B_1+\lambda)(C_1+\lambda)\alpha\alpha' + (C_1+\lambda)(A_1+\lambda)\beta\beta' + (A_1+\lambda)(B_1+\lambda)\gamma\gamma' = \frac{1}{\kappa}\mathfrak{H},$$

www.rcin.org.pl

where, writing down the expanded values of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$,

$$(\lambda b + B)(\lambda c + C) - (\lambda f + F)^2 = \mathfrak{A},$$

$$(\lambda c + C)(\lambda a + A) - (\lambda g + G)^2 = \mathfrak{B},$$

$$(\lambda a + A)(\lambda b + B) - (\lambda h + H)^2 = \mathfrak{C},$$

$$(\lambda g + G)(\lambda h + H) - (\lambda a + A)(\lambda f + F) = \mathfrak{F},$$

$$(\lambda h + H)(\lambda f + F) - (\lambda b + B)(\lambda g + G) = \mathfrak{G},$$

$$(\lambda f + F)(\lambda g + G) - (\lambda c + C)(\lambda h + H) = \mathfrak{H}.$$

By writing successively $\lambda = -A_1$, $\lambda = -B_1$, $\lambda = -C_1$, we see in the first place that A_1, B_1, C_1 are the roots of the same cubic equation, and we obtain next the values of α^2, β^2, γ^2, &c in terms of these quantities A_1, B_1, C_1, and of the coefficients a, b, &c., A, B, &c. It is easy to see how the above formulæ would have been modified if a_1, b_1, c_1, instead of being equal to unity, had one or more of them been equal to unity with a negative sign. It is obvious that every step of the preceding process is equally applicable whatever be the number of variables.

www.rcin.org.pl