62.

ON AN INTEGRAL TRANSFORMATION.

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THE following transformation, given for elliptic functions by Gudermann (*Crelle*, t. XXIII. [1842], p. 330) is useful for some other integrals.

If
$$y = \frac{dbc - dba - dca + abc - (bc - ad) z}{(bc - ad) + (d - b - c + a) z},$$
then, putting
$$K = (bc - ad) + (d - b - c + a) z,$$
we have, supposing $a < b < c < d$, so that $(b - a)$, $(c - a)$, $(d - b)$, $(d - c)$ are positive,
$$K(y - a) = (b - a)(c - a)(d - z),$$
$$K(y - b) = (b - a)(d - b)(c - z),$$
$$K(y - c) = (c - a)(d - c)(b - z),$$
$$K(y - d) = (d - b)(d - c)(a - z),$$
$$K^2\, dy \quad = -(b - a)(c - a)(d - b)(d - c)\, dz.$$

In particular, if $\alpha + \beta + \gamma + \delta = -2$,
$$(y - a)^\alpha (y - b)^\beta (y - c)^\gamma (y - d)^\delta\, dy = -M(z - a)^\delta (z - b)^\gamma (z - c)^\beta (z - d)^\alpha\, dz,$$
where
$$M = (b - a)^{\alpha+\beta+1} (c - a)^{\alpha+\gamma+1} (d - b)^{\beta+\delta+1} (d - c)^{\gamma+\delta+1}.$$

Thus, if $\alpha = \beta = \gamma = \delta = -\frac{1}{2}$,
$$\frac{dy}{\{-(y - a)(y - b)(y - c)(y - d)\}^{\frac{1}{2}}} = \frac{-dz}{\{-(z - a)(z - b)(z - c)(z - d)\}^{\frac{1}{2}}}.$$

In any case when $y = a$, $y = b$, the corresponding values of z are $z = d$, $z = c$; the last formula becomes by this means
$$\int_a^b \frac{dy}{\{-(y - a)(y - b)(y - c)(y - d)\}^{\frac{1}{2}}} = \int_c^d \frac{dy}{\{-(z - a)(z - b)(z - c)(z - d)\}^{\frac{1}{2}}}.$$