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NOTES ON THE ABELIAN INTEGRALS.—JACOBI'S SYSTEM OF DIFFERENTIAL EQUATIONS.

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THE theory of elliptic functions depends, it is well known, on the differential equation $\frac{dx}{\sqrt{(fx)}} + \frac{dy}{\sqrt{(fy)}} = 0$, ($fx$ denoting a rational and integral function of the fourth order), the integral of which was discovered by Euler, though first regularly derived from the differential equation by Lagrange. The theory of the Abelian integrals depends in like manner, as is proved by Jacobi, in the memoir "Considerationes generales de transcendentibus Abelianis" (*Crelle*, t. IX. [1832] p. 394) to depend, upon the system of equations

$$\Sigma \frac{dx}{\sqrt{(fx)}} = 0, \quad \Sigma \frac{xdx}{\sqrt{(fx)}} = 0, \ldots\ldots \Sigma \frac{x^{n-2}\, dx}{\sqrt{(fx)}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where fx is a rational and integral function of the order $2n-1$ or $2n$, and the sums Σ contain n terms.

The integration of this system of equations is of course virtually comprehended in Abel's theorem; the problem was to obtain $(n-1)$ integrals each of them containing a single independent arbitrary constant. One such integral was first obtained by Richelot (*Crelle*, t. XXIII. [1842] p. 354), "Ueber die Integration eines merkwürdigen Systems Differentialgleichungen," by a method founded on that of Lagrange for the solution of Euler's equation; and a second integral very ingeniously deduced from it. A complete system of integrals in the required form is afterwards obtained, not by direct integration, but by means of Abel's theorem: there is this objection to them, however, that any one of them contains two roots of the equation $fx = 0$. The next paper on the subject is one by Jacobi, "Demonstratio Nova theorematis Abeliani" (*Crelle*, t. XXIV. [1842] p. 28), in which a complete system of equations is deduced by direct integration, each of which contains only a single root of the equation $fx=0$. But in Richelot's second memoir "Einige neue Integralgleichungen des Jacobischen Systems Differentialgleichungen" (*Crelle*, t. XXV. [1843] p. 97), the equations are obtained by direct integration in a form not involving any of the roots of this equation; the method employed in obtaining them being in a great measure founded upon the memoir just quoted of Jacobi's. The following is the process of integration.

Denoting the variables by $x_1, x_2 \ldots x_n$, and writing

$$F\alpha = (\alpha - x_1)(\alpha - x_2)\ldots(\alpha - x_n),$$

so that

$$F'x_1 = (x_1 - x_2)\ldots(x_1 - x_n),$$
$$\&c.$$

then the system of differential equations is satisfied by assuming that $x_1, x_2 \ldots x_n$ are functions of a new variable t, determined by the equations

$$\frac{dx_1}{dt} = \frac{\sqrt{(fx_1)}}{F'x_1}, \ \&c.$$

(In fact these equations give $\Sigma \dfrac{dx}{\sqrt{(fx)}} = dt\, \Sigma \dfrac{1}{F'x} = 0$, &c.)

From these we deduce, by differentiation,

$$\frac{d^2x_1}{dt^2} = \tfrac{1}{2}\frac{d}{dx_1}\,\frac{fx_1}{(F'x_1)^2} + \frac{\sqrt{(fx_1)}}{F'x_1}\Sigma'\frac{\sqrt{(fx)}}{(x_1 - x)F'x}$$

(where Σ' refers to all the roots except x_1) and a set of analogous equations for $x_2, x_3 \ldots x_n$.

Dividing this by $\alpha - x_1$, where α is arbitrary, and reducing by

$$\frac{1}{(\alpha - x_1)(x_1 - x)} = \frac{1}{2(\alpha - x)(\alpha - x_1)}\left(1 - \frac{x + x_1 - 2\alpha}{x_1 - x}\right),$$

we have

$$\frac{1}{\alpha - x_1}\,\frac{d^2x_1}{dt^2} = \frac{1}{2(\alpha - x_1)}\,\frac{d}{dx_1}\,\frac{fx_1}{(F'x_1)^2}$$
$$+\tfrac{1}{2}\frac{\sqrt{(fx_1)}}{(\alpha - x_1)F'x_1}\Sigma'\frac{\sqrt{(fx)}}{(\alpha - x)F'x} - \tfrac{1}{2}\Sigma'\frac{\sqrt{(fx)}\sqrt{(fx_1)}}{F'x\,F'x_1}\,\frac{(x_1 + x - 2\alpha)}{(\alpha - x)(\alpha - x_1)(x_1 - x)},$$

that is

$$\frac{1}{\alpha - x_1}\,\frac{d^2x_1}{dt^2} = \frac{1}{2(\alpha - x_1)}\,\frac{d}{dx_1}\,\frac{fx_1}{(F'x_1)^2}$$
$$+\tfrac{1}{2}\frac{\sqrt{(fx_1)}}{(\alpha - x_1)F'x_1}\Sigma\frac{\sqrt{(fx)}}{(\alpha - x)F'x} - \tfrac{1}{2}\frac{fx_1}{(\alpha - x_1)^2(F'x_1)^2}$$
$$-\tfrac{1}{2}\Sigma'\frac{\sqrt{(fx)}\sqrt{(fx_1)}}{F'x\,F'x_1}\,\frac{(x_1 + x - 2\alpha)}{(\alpha - x)(\alpha - x_1)(x_1 - x)};$$

and taking the sum of all the equations of this form, the last term disappears on account of the factor $x_1 - x$ in the denominator, and the result is

$$\Sigma \frac{1}{\alpha - x}\frac{d^2x}{dt^2} = \tfrac{1}{2}\Sigma \frac{1}{\alpha - x}\frac{d}{dx}\frac{fx}{(F'x)^2} + \tfrac{1}{2}\left\{\Sigma \frac{\sqrt{(fx)}}{(\alpha - x)F'x}\right\}^2 - \tfrac{1}{2}\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2}.$$

This being premised, assume

$$y = \sqrt{(F\alpha)},$$

which, by differentiation, gives

$$\frac{dy}{dt} = -\tfrac{1}{2}y\Sigma \frac{1}{\alpha - x}\frac{dx}{dt} = -\tfrac{1}{2}y\Sigma \frac{\sqrt{(fx)}}{(\alpha - x)F'x},$$

and thence

$$\frac{d^2y}{dt^2} = -\tfrac{1}{2}\frac{dy}{dt}\Sigma \frac{1}{\alpha - x}\frac{dx}{dt} + \tfrac{1}{2}y\Sigma \frac{1}{(\alpha - x)^2}\left(\frac{dx}{dt}\right)^2 - \tfrac{1}{2}y\Sigma \frac{1}{(\alpha - x)}\frac{d^2x}{dt^2},$$

that is
$$\frac{d^2y}{dt^2} = \tfrac{1}{4}y\left(\Sigma \frac{\sqrt{(fx)}}{(\alpha - x)F'x}\right)^2 - \tfrac{1}{2}y\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2} - \tfrac{1}{2}y\Sigma \frac{1}{\alpha - x}\frac{d^2x}{dt^2}.$$

Substituting the preceding value of

$$\Sigma \frac{1}{\alpha - x}\frac{d^2x}{dt^2},$$

we have
$$\frac{d^2y}{dt^2} = -\tfrac{1}{4}y\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2} - \tfrac{1}{4}y\Sigma \frac{1}{\alpha - x}\frac{d}{dx}\frac{fx}{(F'x)^2};$$

that is
$$4\frac{d^2y}{dt^2} + y\left\{\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2} + \Sigma \frac{1}{(\alpha - x)}\frac{d}{dx}\frac{fx}{(F'x)^2}\right\} = 0.$$

Now the fractional part of $\frac{f\alpha}{(F\alpha)^2}$ is equal to

$$\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2} + \Sigma \frac{1}{\alpha - x}\frac{d}{dx}\frac{fx}{(F'x)^2}.$$

Also if L be the coefficient of x^{2n} in $f\alpha$, the integral part is simply equal to L, (since $(F\alpha)^2$ is a function of the order $2n$, in which the coefficient of α^{2n} is unity). Hence the coefficient of y in the last equation is simply

$$\frac{f\alpha}{(F\alpha)^2} - L, \quad = \frac{f\alpha}{y^4} - L;$$

or we have

$$4\frac{d^2y}{dt^2} + y\left(\frac{f\alpha}{y^4} - L\right) = 0,$$

viz. multiplying by the factor $2\dfrac{dy}{dt}$, and integrating,

$$4\left(\frac{dy}{dt}\right)^2 - \frac{f\alpha}{y^2} - Ly^2 = C.$$

Hence replacing y and $\dfrac{dy}{dt}$ by their values

$$\sqrt{(F\alpha)} \text{ and } -\tfrac{1}{2}\sqrt{(F\alpha)}\,\Sigma\frac{\sqrt{(fx)}}{(\alpha - x)\,F'x},$$

we have

$$F\alpha\left\{\Sigma\frac{\sqrt{(fx)}}{(\alpha-x)\,F'x}\right\}^2 - \frac{f\alpha}{F\alpha} - LF\alpha = C,$$

for one of the integrals of the proposed system of equations: and since α is arbitrary, the complete system is obtained by giving any $(n-1)$ particular values to α, and changing the value of the constant of integration C; or by expanding the first side of the equation in terms of α, and equating the different coefficients to arbitrary constants. The *à posteriori* demonstration that all the results so obtained are equivalent to $(n-1)$ independent equations would probably be of considerable interest.