34.

NOTE ON THE MAXIMA AND MINIMA OF FUNCTIONS OF THREE VARIABLES.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 74, 75.]

If A, B, C, F, G, H, be any real quantities, such that

$$BC + CA + AB - F^2 - G^2 - H^2,$$

and

$$(A+B+C)(ABC - AF^2 - BG^2 - CH^2 + 2FGH)$$

are positive; the six quantities

$$BC - F^2,\ CA - G^2,\ AB - H^2,\ AK,\ BK,\ CK,$$

(where $K = ABC - AF^2 - BG^2 - CH^2 + 2FGH$) are all of them positive. It is unnecessary to point out the connection of this property with the theory of maxima and minima.

To demonstrate this, writing as usual

$$BC - F^2 = A',\quad GH - AF = F',$$
$$CA - G^2 = B',\quad HF - BG = G',$$
$$AB - H^2 = C',\quad FG - CH = H',$$

and K as above: then if $A'', B'', C'', F'', G'', H'', K'$ be formed from A', B', C', F', G', H', as these and K are from A, B, C, F, G, H, we have the well-known formulæ

$$A'' = KA,\quad F'' = KF,\quad K' = K^2.$$
$$B'' = KB,\quad G'' = KG,$$
$$C'' = KC,\quad H'' = KH,$$

It is required to show that if $A'+B'+C'$ and $A''+B''+C''$ are positive, A', B', C', A'', B'', C'' are so likewise.

Consider the cubic equation

$$(A'-k)(B'-k)(C'-k)-(A'-k)F'^2-(B'-k)G'^2-(C'-k)H'^2+2F'G'H'=0,$$

the roots of which are all real. By the formulæ just given this may be written

$$k^3-k^2(A'+B'+C')+k(A''+B''+C'')-K^2=0;$$

and the terms of this equation are alternately positive and negative; i.e. the roots are all positive. Hence the roots of the limiting equation

$$(B'-k)(C'-k)-F'^2=0$$

are positive, i.e. $B'+C'$ and $B'C'$ are positive: but from the second condition B', C' are of the same sign: consequently they are of the same sign with $B'+C'$, or positive. Also $A''=B'C'-F'^2$ is positive. Similarly, considering the other limiting equations, A', B', C', A'', B'', C'' are all of them positive.

In connection with the above I may notice the following theorem. The roots of the equation

$$(A-ka)(B-kb)(C-ck)-(A-ka)(F-kf)^2-(B-kb)(G-kg)^2-(C-kc)(H-kh)^2 \\ +2(F-kf)(G-kg)(H-kh)=0,$$

are all of them real, if either of the functions

$$Ax^2+By^2+Cz^2+2Fyz+2Gxz+2Hxy,$$
$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy,$$

preserve constantly the same sign. The above form parts of a general system of properties of functions of the second order.