33.

ON THE REDUCTION OF $\frac{du}{\surd U}$, WHEN U IS A FUNCTION OF THE FOURTH ORDER.

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It is well known that the transformation of this differential expression into a similar one, in which the function in the denominator contains only even powers of the corresponding variable, is the first step in the process of reducing $\int \frac{du}{\sqrt{U}}$ to elliptic integrals. And, accordingly, the different modes of effecting this have been examined, more or less, by most of those who have written on the subject. The simplest supposition, that adopted by Legendre, and likewise discussed in some detail by Gudermann, is that u is a fraction, the numerator and denominator of which are linear functions of the new variable. But the theory of this transformation admits of being developed further than it has yet been done, as regards the equation which determines the modulus of the elliptic function. This may be effected most easily as follows.

Suppose

$$U = a \quad + 4bu \quad + 6cu^2 \quad + 4du^3 \quad + eu^4,$$

$$P = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Also let

$$P' = a'x'^4 + 4b'x'^2y' + 6c'x'^2y'^2 + 4d'x'y'^3 + e'y'^4$$

be what P becomes after writing

$$x = \lambda x' + \mu y',$$

$$y = \lambda_{,} x' + \mu_{,} y':$$

and let

$$U' = a' \quad + 4b'u' \quad + 6c'u'^2 \quad + 4d'u'^3 \quad + e'u'^4.$$

Suppose, moreover,

$$\begin{cases} k = \lambda\mu_{,} - \lambda_{,}\mu, \\ I = ae - 4bd + 3c^2, \\ I' = a'e' - 4b'd' + 3c'^2, \\ J = ace - ad^2 - be^2 - c^3 + 2bcd, \\ J' = a'c'e' - a'd'^2 - b'e'^2 - c'^3 + 2b'c'd'; \end{cases}$$

we have evidently

$$xdy - ydx = k\,(x'dy' - y'dx'),$$

or

$$\frac{xdy - ydx}{\sqrt{P}} = k\,\frac{x'dy' - y'dx'}{\sqrt{P'}}.$$

Hence writing

$$u = \frac{y}{x}, \quad u' = \frac{y'}{x'};$$

and therefore

$$\frac{xdy - ydx}{P^{\frac{1}{2}}} = \frac{du}{U^{\frac{1}{2}}}, \quad \frac{x'dy' - y'dx'}{P'^{\frac{1}{2}}} = \frac{du'}{U'^{\frac{1}{2}}},$$

we obtain

$$\frac{du}{\sqrt{U}} = k\,\frac{du'}{\sqrt{U'}},$$

the equation between u and u' being

$$u = \frac{\lambda + \mu\, u'}{\lambda_{,} + \mu_{,} u'}.$$

Next, to determine the relations between the coefficients of U and U'. Since P, P' are obtained from each other by linear transformations (*Math. Journal*, vol. IV. p. 208), [13, p. 94], we have between the coefficients of these functions and of the transforming equations, the relations

$$I' = k^4 I,$$
$$J' = k^6 J;$$

whence also

$$\frac{J'^2}{I'^3} = \frac{J^2}{I^3}.$$

Suppose now

$$U' = a'\,(1 + pu'^2)\,(1 + qu'^2),$$

or

$$b' = 0, \quad d' = 0, \quad 6c' = a'\,(p + q), \quad e' = a'pq;$$

whence also

$$I' = \tfrac{1}{12}\,a'^2\,(p^2 + q^2 + 14pq),$$
$$J' = \tfrac{1}{216}\,a'^3\,(p + q)\,(34pq - p^2 - q^2);$$

$$p^2 + q^2 + 14pq = 12\,\frac{k^4}{a'^2}\,I,$$

$$(p+q)(34pq - p^2 - q^2) = 216\,\frac{k^6}{a'^3}\,J\,;$$

therefore

$$\frac{(p+q)^2(34pq - p^2 - q^2)^2}{(p^2+q^2+14pq)^3} = \frac{27J^2}{I^3},$$

whence also

$$\frac{108pq\,(p-q)^4}{(p^2+q^2+14pq)^3} = 1 - \frac{27J^2}{I^3},$$

which determines the relation between p and q. Also

$$\frac{k}{\sqrt{a'}} = \left(\frac{p^2+q^2+14pq}{12I}\right)^{\frac{1}{4}},$$

so that

$$\frac{du}{\sqrt{U}} = \left(\frac{p^2+q^2+14pq}{12I}\right)^{\frac{1}{4}} \frac{du'}{\sqrt{\{(1+pu'^2)(1+qu'^2)\}}}.$$

If in particular $p = -1$, writing also $-q$ for q,

$$\frac{du}{\sqrt{U}} = \left(\frac{q^2+14q+1}{12I}\right)^{\frac{1}{4}} \frac{du'}{\sqrt{\{(1-u'^2)(1-qu'^2)\}}},$$

where

$$\frac{108q\,(1-q)^4}{(q^2+14q+1)^3} = 1 - \frac{27J^2}{I^3}.$$

Suppose, for shortness,

$$M = \tfrac{27}{4}\cdot\frac{1}{\left(1 - \dfrac{27J^2}{I^3}\right)}, \quad \text{or } \tfrac{1}{108}\left(1 - \frac{27J^2}{I^3}\right) = \frac{1}{16M},$$

then

$$(q^2+14q+1)^3 - 16Mq\,(q-1)^4 = 0,$$

i.e.

$$\left(q + \frac{1}{q} + 14\right)^3 - 16M\left(q^{\frac{1}{2}} - \frac{1}{q^{\frac{1}{2}}}\right)^4 = 0.$$

Let

$$q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \frac{4}{(\theta - 1)^{\frac{1}{2}}},$$

then

$$\theta^3 - M(\theta - 1) = 0,$$

which determines θ. And then

$$q = \frac{7 + \theta + 4\,(3+\theta)^{\frac{1}{2}}}{\theta - 1}.$$

Suppose $q=\alpha$ is one of the values of q; the equation becomes

$$\frac{(q^2+14q+1)^3}{q(q-1)^4}=\frac{(\alpha^2+14\alpha+1)^3}{\alpha(\alpha-1)^4}, \quad =\frac{(\beta^8+14\beta^4+1)^3}{\beta^4(\beta^4-1)^4}, \quad \text{if } \alpha=\beta^4.$$

Now if $$q=\left(\frac{1-\beta}{1+\beta}\right)^4,$$

then $$(q^2+14q+1)=\frac{16(\beta^8+14\beta^4+1)}{(1+\beta)^8}, \quad q-1=-\frac{8\beta(1+\beta^2)}{(1+\beta)^4},$$

which values satisfy the above equation: hence also, identically,

$$(q^2+14q+1)^3-q(q-1)^4\frac{(\beta^8+14\beta^4+1)^3}{\beta^4(\beta^4-1)^4}$$

$$=(q-\beta^4)\left(q-\frac{1}{\beta^4}\right)\left\{q-\left(\frac{1-\beta}{1+\beta}\right)^4\right\}\left\{q-\left(\frac{1+\beta}{1-\beta}\right)^4\right\}\left\{q-\left(\frac{1-\beta i}{1+\beta i}\right)^4\right\}\left\{q-\left(\frac{1+\beta i}{1-\beta i}\right)^4\right\};$$

or the values of q take the form

$$\beta^4, \quad \frac{1}{\beta^4}, \quad \left(\frac{1-\beta}{1+\beta}\right)^4, \quad \left(\frac{1+\beta}{1-\beta}\right)^4, \quad \left(\frac{1-\beta i}{1+\beta i}\right)^4, \quad \left(\frac{1+\beta i}{1-\beta i}\right)^4.$$

(Comp. *Abel. Œuv.* tom. I. p. 310 [Ed. 2, p. 459].)

The equation $$\theta^3-M\theta+M=0$$

has its three roots real if $27-4M$ is negative, and only a single real root if $27-4M$ is positive. Writing the equation under the form

$$(\theta+3)^3-9(\theta+3)^2+(27-M)(\theta+3)-(27-4M)=0,$$

we see that in the former case θ has two values greater than -3, and a single value less than -3. Writing the equation under the form

$$(\theta-1)^3+3(\theta-1)^2+(3-M)(\theta-1)+1=0, \ (3-M \text{ is negative})$$

the positive roots are both greater than 1. Hence, in this case, q has four positive values and two imaginary ones. In the second case θ has a single real value, which is greater than -3 and less than 1. Hence q has two negative values and four imaginary ones. In the former case, I^3-27J^2 is positive, and the function U has either four imaginary factors or four real ones. In the second case, I^3-27J^2 is negative, or the function U has two real and two imaginary factors.