19.

INVESTIGATION OF THE TRANSFORMATION OF CERTAIN ELLIPTIC FUNCTIONS.

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THE function sinam u (ϕu for shortness) may be expressed in the form

$$\phi u = u\,\Pi\left(1+\frac{u}{2mK+2m'K'i}\right) \div \Pi\left(1+\frac{u}{2mK+(2m'+1)K'i}\right)\ldots\ldots\ldots.(1)$$

where m, m' receive any integer, positive or negative, values whatever, omitting only the combination $m=0$, $m'=0$ in the numerator (Abel, *Œuvres*, t. I. p. 212, [Ed. 2, p. 343] but with modifications to adapt it to Jacobi's notation; also the positive and negative values of m, m' are not collected together as in Abel's formulæ). We deduce from this

$$\frac{\phi(u+\theta)}{\phi\theta} = \Pi\left(1+\frac{u}{2mK+2m'K'i+\theta}\right) \div \Pi\left(1+\frac{u}{2mK+(2m'+1)K'i+\theta}\right)\ldots(2).$$

Suppose now $K=aH+a'H'i$, $K'i=bH+b'H'i$, a, b, a', b' integers, and $ab'-a'b$ a positive number ν. Also let $\theta=fH+f'H'i$; f, f' integers such that $af'-a'f$, $bf'-b'f$, ν, have not all three any common factor. Consider the expression

$$v = \frac{\phi u\,\phi(u+2\omega)\ldots\phi(u+2(\nu-1)\omega)}{\phi(2\omega)\ldots\phi(2(\nu-1)\omega)}\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ldots(3),$$

from which

$$v = u\Pi\left(1+\frac{u}{2mK+2m'K'i+2r\theta}\right) \div \Pi\left(1+\frac{u}{2mK+(2m'+1)K'i+2r\theta}\right)\ldots\ldots(4)$$

where r extends from 0 to $\nu-1$ inclusively, the single combination $m=0$, $m'=0$, $r=0$ being omitted in the numerator. We may write

$$mK+m'K'i+r\theta=\mu H+\mu' H'i,$$

μ, μ' denoting any integers whatever. Also to given values of μ, μ' there corresponds only a single system of values of m, m', r. To prove this we must show that the equations

$$ma + m'b + rf = \mu,$$
$$ma' + m'b' + rf' = \mu',$$

can always be satisfied, and satisfied in a single manner only. Observing the value of ν,

$$\nu m + r(b'f - bf') = \mu b' - \mu' b;$$

then if ν and $b'f - bf'$ have no common factor, there is a single value of r less than ν, which gives an integer value for m. This being the case, $m'b$ and $m'b'$ are both integers, and therefore, since b, b' have no common factor (for such a factor would divide ν and $b'f - bf'$), m' is also an integer. If, however, ν and $b'f - bf'$ have a common factor c, so that $\nu = ab' - a'b = c\phi$, $b'f - bf' = c\phi'$; then $(af' - a'f) b' = c(\phi f' - \phi' f)$, or since no factor of c divides $af' - a'f$, c divides b', and consequently b. The equation for ν may therefore be divided by c. Hence, putting $\frac{\nu}{c} = \nu_{,}$, we may find a value of r, say $r_{,}$, less than $\nu_{,}$, which makes m an integer; and the general value of r less than ν which makes m an integer, is $r = r_{,} + s\nu_{,}$, where s is a positive integer less than c. But m being integral, bm', $b'm'$, and consequently cm' are integral; we have also

$$c\nu_1 m' + (r_{,} + s\nu_{,})(af' - a'f) = a\mu' - a'\mu;$$

and there may be found a single value of s less than c, giving an integer value for m'. Hence in every case there is a single system of values of m, m', r, corresponding to any assumed integer values whatever of μ, μ'. Hence

$$U = u\Pi\left(1 + \frac{u}{2\mu H + 2\mu' H' i}\right) \div \Pi\left(1 + \frac{u}{2\mu H + (2\mu' + 1) H' i}\right) = \phi_{,} u \quad \text{........} (5)$$

$\phi_{,} u$ being a function similar to ϕu, or sin am u, but to a different modulus, viz. such that the complete functions are H, H' instead of K, K'. We have therefore

$$\phi_{,} u = \frac{\phi u \phi(u + 2\omega) \dots \phi(u + 2(\nu - 1)\omega)}{\phi(2\omega) \dots \phi(2(\nu - 1)\omega)} \quad \text{........................} (6).$$

Expressing ω in terms of K, K', we have $\nu H = b'K - a'K'i$, $-\nu H'i = bK - aK'i$, and therefore $\nu\omega = (b'f - bf')K - (a'f - af')K'i$. Let g, g' be any two integer numbers having no common factor, which is also a factor of ν, we may always determine a, b, a', b', so that $\nu\omega = gK - g'K'i$. This will be the case if $g = b'f - bf'$, $g' = a'f - af'$. One of the quantities f, f' may be assumed equal to 0. Suppose $f' = 0$, then $g = b'f$, $g' = a'f$; whence $ag - bg' = \nu f$. Let k be the greatest common measure of g, g', so that $g = kg_{,}$, $g' = kg'_{,}$; then, since no factor of k divides ν, k must divide f, or $f = kf_{,}$, but $g_{,} = b'f$, $g'_{,} = af'$, and a', b' are integers, or f must divide $g_{,}$, g'; whence $f_{,} = 1$, or $f = k$. Also $ag_{,} - bg'_{,} = \nu$, where $g_{,}$ and $g'_{,}$ are prime to each other, so that integer values may always be found for a and b; so that in the equation (1) we have

$$\omega = \frac{gK - g'K'i}{\nu} \quad \text{..................................} (7),$$

g, g' being any integer numbers such that no common factor of g, g' also divides ν.

The above supposition, $f'=0$, is, however, only a particular one; omitting it, the conditions to be satisfied by a, b, a', b', may be written under the form

$$\left.\begin{aligned} ab'-a'b&=\nu, \\ ag-bg'&\equiv 0\ [\text{mod. } \nu], \\ a'g-b'g'&\equiv 0\ [\text{mod. } \nu], \end{aligned}\right\} \qquad (8)$$

to which we may join the equations before obtained,

$$\left.\begin{aligned} \nu H&=b'K-a'K'i, \\ -\nu H'i&=bK-aKi, \end{aligned}\right\} \qquad (9)$$

which contain the theory of the modular equation. This, however, involves some further investigations, which are not sufficiently connected with the present subject to be attempted here.