14.

ON LINEAR TRANSFORMATIONS.

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In continuing my researches on the present subject, I have been led to a new manner of considering the question, which, at the same time that it is much more general, has the advantage of applying directly to the only case which one can possibly hope to develope with any degree of completeness, that of functions of two variables. In fact the question may be proposed, "To find all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables." By Derivative I understand a function deduced in any manner whatever from the given functions, and I give the name of Hyperdeterminant Derivative, or simply of Hyperdeterminant, to those derivatives which have the property just enunciated. These derivatives may easily be expressed explicitly, by means of the known method of the separation of symbols. We thus obtain the most general expression of a hyperdeterminant. But there remains a question to be resolved, which appears to present very great difficulties, that of determining the *independent* derivatives, and the relation between these and the remaining ones. I have only succeeded in treating a very particular case of this question, which shows however in what way the general problem is to be attacked.

Imagine p series each of m variables

$$x_1,\quad y_1, \ldots \text{\&c.}\quad x_2,\quad y_2, \ldots \text{\&c.}\quad x_p,\quad y_p, \ldots \text{\&c.},$$

where p is at least as great as m.

Similarly p' series each of m' variables

$$x_1',\quad y_1', \ldots \text{\&c.}\quad x_2',\quad y_2', \ldots, \text{\&c.}\quad \ldots x'_{p'},\quad y'_{p'}, \ldots \text{\&c.},$$

$p^{\backprime}$ at least as great as $m^{\backprime}$, and so on. Let the analogous variables $\dot{x}$, $\dot{y}$ ... be connected with these by the equations

$$\begin{aligned} x &= \lambda \dot{x} + \mu \dot{y} + \ldots, \\ y &= \lambda' \dot{x} + \mu' \dot{y} + \ldots, \\ &\vdots \\ x^{\backprime} &= \lambda^{\backprime} \dot{x}^{\backprime} + \mu^{\backprime} \dot{y}^{\backprime} + \ldots, \\ y^{\backprime} &= \lambda'^{\backprime} \dot{x}^{\backprime} + \mu'^{\backprime} \dot{y}^{\backprime} + \ldots, \\ &\vdots \end{aligned}$$

where $x, y, \ldots$ stand for $x_1, y_1, \ldots$ or $x_2, y_2, \ldots$ or $x_p, y_p, \ldots$; $x^{\backprime}, y^{\backprime}, \ldots$ stand for $x_1^{\backprime}, y_1^{\backprime}, \ldots$ or $x_2^{\backprime}, y_2^{\backprime}, \ldots$ or $x^{\backprime}_{p^{\backprime}}, y^{\backprime}_{p^{\backprime}}$, &c.... The coefficients $\lambda, \mu, \ldots, \lambda', \mu', \ldots$ &c.; $\lambda^{\backprime}, \mu^{\backprime}, \ldots, \lambda'^{\backprime}, \mu'^{\backprime}, \ldots$ remain the same in all these systems. Suppose next,

$$\xi = \delta_x, \quad \eta = \delta_y,$$

i.e.

$$\xi_1 = \delta_{x_1}, \quad \eta_1 = \delta_{y_1}, \ldots, \xi_1 = \delta_{x_1}, \ldots$$

(where $\delta_x, \delta_y \ldots$ are the symbols of differentiation relative to x, y, &c.). Then evidently

$$\begin{aligned} \dot{\xi} &= \lambda \xi + \lambda' \eta + \ldots, \\ \dot{\eta} &= \mu \xi + \mu' \eta + \ldots, \\ &\vdots \end{aligned}$$

with similar equations for $\dot{\xi}^{\backprime}, \dot{\eta}^{\backprime}, \ldots$ Suppose

$$\|\Omega\| = \left\| \begin{matrix} \xi_1, & \xi_2, \ldots \xi_p \\ \eta_1, & \eta_2, \ldots \eta_p \\ \vdots & \end{matrix} \right\|, \quad \|\Omega^{\backprime}\| = \left\| \begin{matrix} \xi_1^{\backprime}, & \xi_2^{\backprime}, \ldots \xi^{\backprime}_{p^{\backprime}} \\ \eta_1^{\backprime}, & \eta_2^{\backprime}, \ldots \eta^{\backprime}_{p^{\backprime}} \\ \vdots & \end{matrix} \right\|,$$

that is to say $\|\Omega\|$ is the series of determinants formed by choosing any m vertical columns to compose a determinant, and similarly $\|\Omega^{\backprime}\|$, &c. Suppose, besides,

$$E = \left| \begin{matrix} \lambda, & \mu, \ldots \\ \lambda', & \mu', \ldots \\ \vdots & \end{matrix} \right|, \quad E^{\backprime} = \left| \begin{matrix} \lambda^{\backprime}, & \mu^{\backprime}, \ldots \\ \lambda'^{\backprime}, & \mu'^{\backprime}, \ldots \\ \vdots & \end{matrix} \right|.$$

Then, by the known properties of determinants,

$$\|\dot{\Omega}\| = E \, \|\Omega\|, \quad \|\dot{\Omega}^{\backprime}\| = E^{\backprime} \, \|\Omega^{\backprime}\| \text{ \&c.},$$

i.e. the terms on the one side are respectively equal to the terms on the other. Hence if

$$\Box = F(\|\Omega\|^f, \|\Omega^{\backprime}\|^{f^{\backprime}}, \ldots),$$

i.e. $\Box$ a rational and integral function, homogeneous of the order f in the quantities of the series $\|\Omega\|$, homogeneous of the order $f^{\backprime}$ in the quantities of the series $\|\Omega^{\backprime}\|$, &c., we have immediately

$$\dot{\Box} = E^f E^{\backprime f^{\backprime}} \ldots \Box\,;$$

or if U be any function whatever of the variables $x, y \ldots$ which is transformed by the linear substitutions above into $\acute{U}$, then

$$\Box \acute{U} = E^f E'^{f'} \ldots \Box U;$$

or the function $\Box U$

is by the above definition a hyperdeterminant derivative. The symbol $\Box$ may be called "symbol of hyperdeterminant derivation," or simply "hyperdeterminant symbol."

Let $A, B, \ldots$ represent the different quantities of the series $\|\Omega\|$,—A', B',... those of the series $\|\Omega'\|$, &c. ..., then $\Box$ may be reduced to a single term, and we may write

$$\Box = A^\alpha B^\beta \ldots A'^{\alpha'} B'^{\beta'} \ldots$$

Also U may be supposed of the form

$$U = \Theta\Phi \ldots$$

where Θ, Φ are functions of the variables of one of the sets $x, y, \ldots$, of one of the sets $x', y', \ldots$, &c., thus Θ is of the form

$$F(x_1, y_1, \ldots x_1', y_1', \ldots),$$

and so on. The functions Θ, $\Phi \ldots$ may be the same or different. It may be supposed after the differentiations that several of the sets $x, y, \ldots$ or of the sets $x', y', \ldots$ become identical: in such cases it will always be assumed that the functions $\Theta, \ldots$ into which these sets of variables enter, are similar; so that they become absolutely identical, when the variables they contain are made so. Thus the general expression of a hyperdeterminant is

$$\Box U = A^\alpha B^\beta \ldots A'^{\alpha'} B'^{\beta'} \ldots \Theta\Phi \ldots$$

in which, after the differentiations, any number of the sets of variables are made equal. For instance, if all the sets $x, y \ldots$ and all the sets $x', y' \ldots$ are made equal, the hyperdeterminant refers to a single function $F(x, y \ldots x', y' \ldots)$. In any other case it refers not to a single function but to several.

What precedes, is the general theory: it might perhaps have been made clearer by confining it to a particular case: and by doing this from the beginning it will be seen that it presents no real difficulties. Passing at present to some developments, to do this, I neglect entirely the sets $x', y' \ldots$ and I assume that the number m of variables in each of the sets $x, y \ldots$ reduces itself to two; so that I consider functions of two variables x, y only. The functions Θ, Φ, &c. reduce themselves to functions $V_1, V_2 \ldots V_p$ of the variables x_1, y_1, or $x_2, y_2 \ldots$ or x_p, y_p. Writing also

$$\xi_1\eta_2 - \xi_2\eta_1 = \overline{12}, \text{ \&c.}$$

the symbols $A, B \ldots$ reduce themselves to $\overline{12}$, $\overline{13} \ldots$ Hence for functions of two variables, there results the following still tolerably general form

$$\Box U = \overline{12}^\alpha\, \overline{13}^\beta\, \overline{14}^\gamma \ldots \overline{23}^{\beta'}\, \overline{24}^{\gamma'} \ldots \overline{34}^{\gamma''} \ldots V_1 V_2 V_3 V_4 \ldots$$

The functions V_1, $V_2 \ldots$ may be the same or different: but they will be supposed the same whenever the corresponding variables are made equal. This equality will be denoted by writing, for instance,

$$\square VV'VV \ldots$$

to represent the value assumed by

$$\square V_1 V_2 V_3 V_4 \ldots$$

when after the differentiations

$$x_1,\ y_1 = x_3,\ y_3 = x_4,\ y_4 = x,\ y\ ;$$
$$x_2,\ y_2 = x',\ y'\ ;$$
&c.

It is easy to determine the general term of $\square U$. To do this, writing for shortness

$$\alpha + \beta + \gamma \ \ldots = f_1,$$
$$\alpha + \beta' + \gamma' \ldots = f_2,$$
$$\beta + \beta' + \gamma'' \ldots = f_3,$$
&c.

$$N = (-)^{r+s+t\cdots+r'+s'\cdots t'\cdots} \frac{[\alpha]^r}{[r]^r} \frac{[\beta]^s}{[s]^s} \frac{[\gamma]^t}{[t]^t} \cdots \frac{[\beta']^{s'}}{[r']^{r'}} \frac{[\gamma']^{t'}}{[s']^{t'}} \cdots \frac{[\gamma'']^{t''}}{[t'']^{t''}} \cdots$$

$$\xi^{f-r}\eta^r V \text{ or } \delta_x{}^{f-r}\delta_y{}^r V = \overset{f}{V}{}^{,r} \text{ or } V^{,r},$$

the general term is

$$N \overset{f_1}{V_1}{}^{,r+s+t\cdots} \overset{f_2}{V_2}{}^{,\alpha-r+s'+t'\cdots} \overset{f_3}{V_3}{}^{,\beta-s+\beta'-s'+t'\cdots}$$

where r, s, $t, \ldots s'$, $t', \ldots t'', \ldots$ extend from 0 to α, β, $\gamma \ldots \beta'$, $\gamma', \ldots \gamma'' \ldots$ respectively. It would be easy to change this general term in a way similar to that which will be employed presently for the particular case of $\square V_1 V_2 V_3$.

If several of the functions become identical, and for these some of the letters f are equivalent, it is clear that the derivative $\square U$ refers to a certain number of functions V_1, $V_2 \ldots$ the same or different, of the variables x, y; x', y'; ... and besides that this derivative is homogeneous, of the degrees θ_1, $\theta_1', \ldots$ with respect to the differential coefficients of the orders f_1, $f_1', \ldots$ &c. of V_1, (consequently homogeneous of the order $\theta_1 + \theta_1' + \ldots$ with respect to these differential coefficients collectively), homogeneous and of the degrees θ_2, $\theta_2', \ldots$ with respect to the differential coefficients of the orders f_2, $f_2' \ldots$ of V_2, (consequently of the order $\theta_2 + \theta_2' \ldots$ with respect to these collectively), and so on. The degree with respect to all the functions is of course $\theta_1 + \theta_1' \ldots + \theta_2 + \theta_2' + \ldots, = p$ suppose. In general, only a single function will be considered, and it will be assumed that $\square U$ only contains the differential coefficients of the f^{th} order. In this case, the derivative is said to be of the degree p and of the order f. The most convenient classification is by degrees, rather than by orders.

Commencing with the simplest case, that of functions of the second order (and writing V, W instead of V_1, V_2), we have

$$\square VW = \overline{12}^{\alpha}\, VW,$$

(where ξ_1, η_1 apply to V and ξ_2, η_2 to W). This will be constantly represented in the sequel by the notation

$$\overline{12}^{\alpha}\, VW = B_\alpha(V,\ W).$$

Hence, writing $\qquad \delta_x{}^{\alpha} V = V^{,0}, \quad \delta_x{}^{\alpha-1}\delta_y V = V^{,1} \ldots,$

we have $\qquad B_\alpha(V,\ W) = V^{,0}W^{,\alpha} - \dfrac{[\alpha]^1}{[1]^1} V^{,1}W^{,\alpha-1} + \ldots;$

and in particular, according as α is odd or even,

$$B_\alpha(V,\ V) = 0,$$

$$\tfrac{1}{2}B_\alpha(V,\ V) = V^{,0}V^{,\alpha} - \frac{\alpha}{1} V^{,1}V^{,\alpha-1} + \ldots,$$

continued to the term which contains $V^{,\frac{1}{2}\alpha}V^{,\frac{1}{2}\alpha}$, the coefficient of this last term being divided by two.

Thus, for the functions $\frac{1}{2}(ax^2 + 2bxy + cy^2)$, $\frac{1}{24}(ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4)$, &c., if α be made equal to 2, 4, &c. respectively, we have the *constant* derivatives

$$ac - b^2,$$

$$ae - 4bd + 3c^2,$$

$$ag - 6bf + 15ce - 10d^2,$$

$$ai - 8bh + 28cg - 56df + 35e^2,$$

$$\vdots$$

which have all of them the property of remaining unaltered, *à un facteur près*, when the variables are transformed by means of $x = \lambda\dot{x} + \mu\dot{y}$, $y = \lambda'\dot{x} + \mu'\dot{y}$. Thus, for instance, if these equations give

$$ax^2 + 2bxy + cy^2 = \dot{a}\dot{x}^2 + 2\dot{b}\dot{x}\dot{y} + \dot{c}\dot{y}^2,$$

then $\qquad \dot{a}\dot{c} - \dot{b}^2 = (\lambda\mu' - \lambda'\mu)^2 \cdot (ac - b^2),$

and so on. This is the general property, which we call to mind for the case of these constant derivatives.

The above functions may be transformed by means of the identical equation

$$B_a(V,\ W) = \overline{12}^{\,a-k} B_k(V,\ W),$$

to make use of which, it is only necessary to remark the general formula

$$\xi_1{}^\lambda \eta_1{}^\mu \xi_2{}^\rho \eta_2{}^\sigma B_k(V,\ W) = B_k(\xi^\lambda \eta^\mu V,\ \xi^\rho \eta^\sigma W).$$

Thus, if $k=1$, we obtain for the above series, the new forms

$$\begin{aligned}
&ac - b^2,\\
&(ae - bd) - 3\,(bd - c^2),\\
&(ag - bf) - 5\,(bf - ce) + 10\,(ce - d^2),\\
&(ai - bh) - 7\,(bh - cg) + 21\,(cg - df) - 35\,(df - e^2),\\
&\&c.,
\end{aligned}$$

the law of which is evident. This shows also that these functions may be linearly expressed by means of the series of determinants

$$\left\| \begin{matrix} a, & b \\ b, & c \end{matrix} \right\| \ \left\| \begin{matrix} a, & b, & c \\ b, & c, & d \end{matrix} \right\| \ \&c.$$

We may also immediately deduce from them the derivatives B which relate to two functions. For example, for functions of the sixth order this is

$$ag' + a'g - 6\,(bf' + b'f) + 15\,(ce' + c'e) - 20dd',$$

which has an obvious connection with

$$ag - 6bf + 15ce - 10d^2;$$

and the same is the case for functions of any order.

The following theorem is easily verified; but I am unacquainted with the general theory to which it belongs.

"If U, V are any functions of the second order, and $W = \lambda U + \mu V$; then

$$B_2'\,[B_2(W,\ W),\quad B_2(W,\ W)] = 0$$

(where B_2' relates to λ, μ) is the same that would be obtained by the elimination of x, y between $U=0$, $V=0$." (See Note[1].)

In fact this becomes

$$4\,(ac - b^2)\,(a'c' - b'^2) - (ac' + a'c - 2bb')^2 = 0,$$

which is one of the forms under which the result of the elimination of the variables from two quadratic equations may be written. This is a result for which I am indebted to Mr Boole.

[1] Not given with the present paper.

Passing to the third degree, we may consider in particular the derivatives

$$\square UVW = \overline{23}^{\alpha}\, \overline{31}^{\alpha}\, \overline{12}^{\alpha}\, UVW = C_{\alpha}(U,\ V,\ W):$$

writing for shortness

$$A_r = \frac{[\alpha]^r}{[r]^r}, \quad \delta_x^{\,2\alpha-r}\delta_y^{\,r} U = U^{\cdot r},$$

we have the general term

$$C_{\alpha}(U,\ V,\ W) = \Sigma\,\{(-)^{r+s+t} A_r A_s A_t U^{\cdot\alpha+t-s} V^{\cdot\alpha+r-t} W^{\cdot\alpha+s-r}\},$$

where r, s, t extend from 0 to α. By changing the suffixes r, s the following more convenient formula

$$C_{\alpha}(U,\ V,\ W) = \Sigma\Sigma\,\{(-)^{\sigma+\rho} U^{\cdot\rho} V^{\cdot\sigma} W^{\cdot 3\alpha-\rho-\sigma} \Sigma\,[(-)^t A_{\rho-t} A_{\sigma+t-\alpha} A_t]\},$$

where t extends from 0 to 2α: ρ, σ, and $3\alpha-\rho-\sigma$ must be positive and not greater than 2α.

In particular, according as α is odd or even,

$$C_{\alpha}(U,\ U,\ U) = 0,$$

$$C_{\alpha}(U,\ U,\ U) = 6\Sigma\Sigma\,\{(-)^{\rho+\sigma} U^{\cdot\rho} U^{\cdot\sigma} U^{\cdot 3\alpha-\rho-\sigma} \Sigma\,[(-)^t A_{\rho-t} A_{\sigma+t-\alpha} A_t]\},$$

omitting therein those values of ρ, σ for which $\rho > \sigma$ or $\sigma > 3\alpha-\rho-\sigma$, and dividing by two the terms in which $\rho = \sigma$ or $\sigma = 3\alpha-\rho-\sigma$, and by six the term for which

$$\rho = \sigma = 3\alpha - \rho - \sigma,\ = \alpha.$$

In particular, for functions of the fourth or eighth orders we have the constant derivatives

$$ace - ad^2 - b^2e - c^3 + 2bcd\,;$$

$$aei - 4ibd - 4afh + 3ag^2 + 3ic^2 + 12beh - 8chd - 8bgf - 22ceg + 24cf^2 + 24d^2g - 36def + 15e^3;$$

the first of which is a simple determinant. Thus we have been led to the functions $ae - 4bd + 3c^2$ and $ace - ad^2 - eb^2 - c^3 + 2bcd$, which occur in my "Note sur quelques formules &c." (*Crelle*, vol. XXIX. [1845] [15]), and in the forms which M. Eisenstein has given for the solutions of equations of the first four degrees.

Let U be a function of the order 4α: the derivative C may be expressed by means of the derivatives B.

For, consider the function

$$B_{4\alpha}[U,\ B_{2\alpha}(V,\ W)];$$

paying attention to the signification of B, this may be written

$$\overline{1\theta}^{4\alpha}\, \overline{23}^{2\alpha}\, UVW,$$

where the symbols ξ_θ, η_θ refer to the two systems $x_2, y_2 : x_3, y_3$. Thus it is easily seen that we may write

$$\xi_\theta = \xi_2 + \xi_3, \quad \eta_\theta = \eta_2 + \eta_3, \text{ or } \overline{1\theta} = \overline{12} + \overline{13} = \overline{12} - \overline{31},$$

whence the function becomes

$$(\overline{12} - \overline{31})^{4\alpha}\, \overline{23}^{2\alpha}\, UVW,$$

of which all the terms vanish except

$$\frac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}}\, \overline{12}^{2\alpha}\, \overline{23}^{2\alpha}\, \overline{31}^{2\alpha}\, UVW.$$

Hence putting

$$K = \frac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}} = \frac{2^{4\alpha}\, 1 \,.\, 3 \ldots (4\alpha - 1)}{2 \,.\, 4 \ldots 4\alpha},$$

we have $$B_{4\alpha}[U,\ B_{2\alpha}(V,\ W)] = KC_\alpha(U,\ V,\ W),$$

or in particular

$$B_{4\alpha}[U,\ B_{2\alpha}(U,\ U)] = KC_\alpha(U,\ U,\ U).$$

Thus for example, neglecting a numerical factor,

$$(ax^2 + 2bxy + cy^2)(cx^2 + 2dxy + ey^2) - (bx^2 + 2cxy + dy^2)^2$$

$$= (ac - b^2)\, x^4 + 2\,(ad - bc)\, x^3y + (ae + 2bd - 3c^2)\, x^2y^2 + 2\,(be - cd)\, xy^3 + (ce - d^2)\, y^4,$$

and then

$$e\,(ac - b^2) - 4d\,\tfrac{2}{4}\,(ad - bc) + 6c\,\tfrac{1}{6}\,(ae + 2bd - 3c^2) - 4b\,\tfrac{2}{4}\,(be - cd) + a\,(ce - d^2)$$

$$= 3\,(ace - ad^2 - b^2e - c^3 + 2bcd).$$

We have likewise the singular equation

$$B_{2\alpha}(V,\ W) = K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\,\frac{d}{da_{4\alpha-1}} \ldots + y^{4\alpha}\frac{d}{da_0}\right) C_\alpha(U,\ V,\ W)$$

where $$U = \frac{1}{[4\alpha]^{4\alpha}}\left(a_0x^{4\alpha} - \frac{[4\alpha]^1}{1}\, a_1x^{4\alpha-1}y \ldots + a_{4\alpha}y^{4\alpha}\right), \text{ \&c.}$$

If however $U = V = W$, we must write

$$B_{2\alpha}(U,\ U) = \tfrac{1}{3}K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\,\frac{d}{da_{4\alpha-1}} \ldots + y^{4\alpha}\frac{d}{da_0}\right) C_\alpha(U,\ U,\ U),$$

the reason of which is easily seen. This subject will be resumed in the sequel.

The functions C may be transformed in the same way as the functions B have been. In fact

$$C_\alpha(U,\ V,\ W) = \overline{12}^{\,\alpha-k}\ \overline{23}^{\,\alpha-k}\ \overline{31}^{\,\alpha-k}\ C_k(U,\ V,\ W);$$

if in particular $k=1$, then

$$C_1(U,\ V,\ W) = \begin{vmatrix} U^{,0} & U^{,1} & U^{,2} \\ V^{,0} & V^{,1} & V^{,2} \\ W^{,0} & W^{,1} & W^{,2} \end{vmatrix},\quad U^{,0} \text{ for } \overset{2}{U}{}^{,0}, \text{ \&c.}$$

but in general

$$\xi_1^{\rho'}\eta_1^{\rho}\xi_2^{\sigma'}\eta_2^{\sigma}\xi_3^{\tau'}\eta_3^{\tau} C_1(U,\ V,\ W), \text{ where } \rho+\rho'=\sigma+\sigma'=\tau+\tau'=2\alpha-2,$$

$$= C_1(\overset{2\alpha-2}{U}{}^{,\rho}\ \overset{2\alpha-2}{V}{}^{,\sigma}\ \overset{2\alpha-2}{W}{}^{,\tau}) = \begin{vmatrix} U^{,\rho} & U^{,\rho+1} & U^{,\rho+2} \\ V^{,\sigma} & V^{,\sigma+1} & V^{,\rho+2} \\ W^{,\tau} & W^{,\tau+1} & W^{,\rho+2} \end{vmatrix},\quad U^{,\rho} \text{ for } \overset{2\alpha}{U}{}^{,\rho}, \text{ \&c.}$$

whence $C_\alpha(U,\ V,\ W) = \Sigma\Sigma\left\{(-)^{\rho+\sigma} \begin{vmatrix} U^{,\rho} & V^{,\sigma-1} & W^{,3\alpha-\rho-\sigma-2} \\ U^{,\rho+1} & V^{,\sigma} & W^{,3\alpha-\rho-\sigma-1} \\ U^{,\rho+2} & V^{,\sigma+1} & W^{,3\alpha-\rho-\sigma} \end{vmatrix} \Sigma\left[(-)^t A_t' A'_{\rho-t} A'_{\sigma-\alpha+t}\right]\right\}$,

where $A_t' = \dfrac{[\alpha-1]^t}{[t]^t}$; t extends from 0 to $\alpha-1$; ρ, $\sigma-1$, and $3\alpha-\rho-\sigma-2$ may have each of them any positive values not greater than $2\alpha-2$.

In particular

$$C_\alpha(U,\ U,\ U) = 6\Sigma\Sigma\left\{(-)^{\rho+\sigma} \begin{vmatrix} U^{,\rho} & U^{,\sigma-1} & U^{,3\alpha-\rho-\sigma-2} \\ U^{,\rho+1} & U^{,\sigma} & U^{,3\alpha-\rho-\sigma-1} \\ U^{,\rho+2} & U^{,\sigma+1} & U^{,3\alpha-\rho-\sigma} \end{vmatrix} \Sigma\left[(-)^t A'_t A'_{\rho-t} A'_{\sigma-\alpha-t}\right]\right\},$$

where ρ, σ need only have such values that $\rho < \sigma-1$, $\sigma-1 < 3\alpha-\rho-\sigma-2$.

In particular the derivative $aei - \ldots + 15e^3$ may be transformed into

$$\begin{vmatrix} a, & d, & g \\ b, & e, & h \\ c, & f, & i \end{vmatrix} - 3 \begin{vmatrix} a, & e, & f \\ b, & f, & g \\ c, & g, & h \end{vmatrix} - 3 \begin{vmatrix} b, & c, & g \\ c, & d, & h \\ d, & e, & i \end{vmatrix} + 6 \begin{vmatrix} b, & d, & f \\ c, & e, & g \\ d, & f, & h \end{vmatrix} - 15 \begin{vmatrix} c, & d, & e \\ d, & e, & f \\ e, & f, & g \end{vmatrix}$$

in which form it is obviously a linear function of the determinants

$$\left\|\begin{matrix} a, & b, & c, & d, & e, & f, & g \\ b, & c, & d, & e, & f, & g, & h \\ c, & d, & e, & f, & g, & h, & i \end{matrix}\right\|,$$

which is true generally.

Omitting for the present the theory of derivatives of the form

$$\square UVW = \overline{23}^{\alpha}\, \overline{31}^{\beta}\, \overline{12}^{\gamma}\, UVW,$$

we pass on to the derivatives of the fourth degree, considering those forms in which all the differential coefficients are of the same order. We may write

$$\square UVWX = (\overline{12}\,.\,\overline{34})^{\alpha}\,(\overline{13}\,.\,\overline{42})^{\beta}\,(\overline{14}\,.\,\overline{23})^{\gamma}\, UVWX = D_{\alpha,\beta,\gamma}(U,\ V,\ W,\ X) = D_{\alpha,\beta,\gamma};$$

or if for shortness

$$\overline{12}\,.\,\overline{34} = \mathfrak{A},\quad \overline{13}\,.\,\overline{42} = \mathfrak{B},\quad \overline{14}\,.\,\overline{23} = \mathfrak{C},$$

we have
$$D_{\alpha,\beta,\gamma} = \mathfrak{A}^{\alpha}\mathfrak{B}^{\beta}\mathfrak{C}^{\gamma}\,.\,UVWX.$$

Suppose $U = V = W = X$, and consider the derivatives which correspond to the same value f of $\alpha+\beta+\gamma$. The question is to determine how many of these are independent, and to express the remaining ones in terms of these. Since the functions become equal after the differentiations, we are at liberty before the differentiations to interchange the symbolic numbers 1, 2, 3, 4 in any manner whatever. We have thus

$$D_{\alpha,\beta,\gamma} = D_{\beta,\gamma,\alpha} = D_{\gamma,\alpha,\beta} = (-)^f D_{\alpha,\gamma,\beta} = (-)^f D_{\gamma,\beta,\alpha} = (-)^f D_{\beta,\alpha,\gamma};$$

but the identical equation

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0,$$

multiplied by $\mathfrak{A}^a\mathfrak{B}^b\mathfrak{C}^c$ and applied to the product $UVWX$, gives

$$D_{a+1,b,c} + D_{a,b+1,c} + D_{a,b,c+1} = 0;$$

whence if $a+b+c = f-1$, we have a set of equations between the derivatives $D_{\alpha,\beta,\gamma}$ for which $\alpha+\beta+\gamma = f$. Reducing these by the conditions first found, suppose Θf is the number of divisions of an integer f into three parts, zero admissible, but permutations of the same three parts rejected. The number of derivatives is Θf, and the number of relations between them is $\Theta(f-1)$. Hence $\Theta f - \Theta(f-1)$ of these derivatives are independent: only when f is even, one of these is $D_{f,0,0}$, i.e. $\overline{12}^f\,\overline{34}^f\,.\,UVWX$, i.e. $\overline{12}^f UV\,.\,\overline{34}^f WX$, or $B_f(U,\ V)\,B_f(X,\ W)$, i.e. $[B_f(U,\ U)]^2$; rejecting this, the number of independent derivatives, when f is even, is $\Theta f - \Theta(f-1) - 1$. Let $E\left(\dfrac{a}{b}\right)$ be the greatest integer contained in the fraction $\dfrac{a}{b}$; the number required may be shown to be

$$E\frac{f}{6} \text{ or } E\frac{f+3}{6},$$

according as f is even or odd. Giving to f the six forms

$$6g, \quad 6g+1, \quad 6g+2, \quad 6g+3, \quad 6g+4, \quad 6g+5,$$

the corresponding numbers of the independent derivatives are

$$g, \quad g, \quad g, \quad g+1, \quad g, \quad g+1;$$

thus there is a single derivative for the orders 3, 5, 6, 7, 8, 10, ... two for the orders 9, 11, 12, 13, 14, 16, ... &c.

When f is even, the terms $D_{f-3,3,0}$, $D_{f-6,6,0}\ldots$, and when f is odd, the terms $D_{f-1,1,0}$, $D_{f-4,4,0}$, $D_{f-7,7,0}$, &c. may be taken for independent derivatives: by stopping immediately before that in which the second suffix exceeds the first, the right number of terms is always obtained. Thus, when $f=9$ the independent derivatives are D_{810}, D_{540}, and we have the system of equations

$$\begin{aligned}
D_{900}+D_{810}+D_{801}&=0, & D_{621}+D_{531}+D_{522}&=0,\\
D_{810}+D_{720}+D_{711}&=0, & D_{540}+D_{450}+D_{441}&=0,\\
D_{720}+D_{630}+D_{621}&=0, & D_{531}+D_{441}+D_{432}&=0,\\
D_{711}+D_{621}+D_{612}&=0, & D_{522}+D_{432}+D_{423}&=0,\\
D_{630}+D_{531}+D_{540}&=0, & D_{432}+D_{342}+D_{333}&=0,
\end{aligned}$$

which are to be reduced by

$$D_{900}=-D_{900}=0, \qquad D_{801}=-D_{810}, \text{ \&c.}$$

It is easy to form the table

$D_{200}=\quad B_2{}^2,$	$D_{500}=0,$	$D_{700}=0,$
$D_{110}=-\frac{1}{2}B_2{}^2,$	$D_{410},$	$D_{610},$
	$D_{320}=-D_{410},$	$D_{520}=-D_{610},$
$D_{300}=0,$	$D_{311}=0,$	$D_{511}=0,$
$D_{210},$	$D_{221}=0,$	$D_{430}=\quad D_{610},$
$D_{111}=0,$		$D_{421}=0,$
	$D_{600}=\quad B_6{}^2,$	$D_{331}=0,$
$D_{400}=\quad B_4{}^2,$	$D_{510}=\quad -\frac{1}{2}B_6{}^2,$	$D_{222}=0,$
$D_{310}=-\frac{1}{2}B_4{}^2,$	$D_{420}=-\frac{2}{3}D_{330}+\frac{1}{6}B_6{}^2,$	
$D_{220}=\quad \frac{1}{2}B_4{}^2,$	$D_{411}=\quad \frac{2}{3}D_{330}+\frac{1}{3}B_6{}^2,$	
$D_{211}=0,$	$D_{330},$	
	$D_{321}=-\frac{1}{3}D_{330}-\frac{1}{6}B_6{}^2,$	
	$D_{222}=\quad \frac{2}{3}D_{330}+\frac{1}{3}B_6{}^2,$	

$$\begin{array}{ll}
D_{800} = \quad B_8^2, & D_{900} = 0, \\
D_{710} = \quad -\tfrac{1}{2} B_8^2, & D_{810}, \\
D_{620} = -\tfrac{2}{3} D_{530} + \tfrac{1}{6} B_8^2, & D_{720} = - D_{810}, \\
D_{611} = \quad \tfrac{2}{3} D_{530} + \tfrac{1}{3} B_8^2, & D_{711} = 0, \\
D_{530}, & D_{630} = \tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540}, \\
D_{521} = -\tfrac{1}{3} D_{530} - \tfrac{1}{12} B_8^2, & D_{621} = \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540}, \\
D_{440} = -\tfrac{16}{15} D_{530} - \tfrac{1}{30} B_8^2, & D_{540}, \\
D_{431} = \quad \tfrac{1}{15} D_{530} - \tfrac{1}{30} B_8^2, & D_{531} = -\tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540}, \\
D_{422} = \quad \tfrac{4}{15} D_{530} + \tfrac{2}{15} B_8^2, & D_{522} = 0, \\
D_{332} = -\tfrac{2}{15} D_{530} - \tfrac{1}{15} B_8^2, & D_{441} = 0, \\
 & D_{432} = \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540}, \\
 & D_{333} = 0.
\end{array}$$

Whatever be the value, all the tables except the three first commence thus, according as f is even or odd,

$$\begin{array}{ll}
D_{f,0,0} = B_f^2, & \text{or } D_{f,0,0} = 0, \\
D_{f-1,1,0} = -\tfrac{1}{2} B_f^2, & D_{f-1,1,0}, \\
D_{f-2,2,0} = -\tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{6} B_f^2, & D_{f-2,2,0} = - D_{f-1,1,0}, \\
D_{f-2,1,1} = \tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{3} B_f^2, & D_{f-2,1,1} = 0, \\
D_{f-3,3,0} & \vdots \\
\vdots &
\end{array}$$

but beyond this I am not acquainted with the law.

To give some formulæ for the transformation of these derivatives; we have, for example,

$$D_{f-1,1,0} = (\overline{12}\,.\,\overline{34})\,\overline{13}^{f-1}\,.\,\overline{42}\; UUUU = \overline{13}\,.\,\overline{42}\, B_{f-1}(U,\ U)\, B_{f-1}(U,\ U).$$

But $\qquad \overline{13}\,.\,\overline{42} = \xi_1\eta_2\eta_3\xi_4 - \xi_1\xi_2\eta_3\eta_4 - \eta_1\eta_2\xi_3\xi_4 + \eta_1\xi_2\xi_3\eta_4,$

and $\qquad \xi_1\eta_2\eta_3\xi_4\, B_{f-1}(U,\ U)\, B_{f-1}(U,\ U) = B_{f-1}(\xi U,\ \eta U)\, B_{f-1}(\eta U,\ \xi U)$

$$= B_{f-1}(U^{\cdot 0}\, U^{\cdot 1})\, B_{f-1}(U^{\cdot 1}\, U^{\cdot 0}), \text{ \&c.}$$

(where $U^{\cdot 0}$, $U^{\cdot 1}$ stand for $\overset{1}{U}{}^{\cdot 0}\, \overset{1}{U}{}^{\cdot 1}$, &c.); or

$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{\cdot 0}\, U^{\cdot 0})\, B_{f-1}(U^{\cdot 1}\, U^{\cdot 1}) - B_{f-1}(U^{\cdot 0}\, U^{1})\, B_{f-1}(U^{\cdot 1}\, U^{\cdot 0})\},$$

which reduces itself to

$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{\cdot 0}\, U^{\cdot 1})\}^2,$$

$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{\cdot 0}\, U^{\cdot 0})\, B_{f-1}(U^{\cdot 1}\, U^{\cdot 1}) - [B_{f-1}(U^{\cdot 0} U^{\cdot 1})]^2\},$$

according as f is even or odd.

For example, for the orders 3, 5, 7, 9, we have

$$D_{210} = -2\{4(ac-b^2)(bd-c^2)-(ad-bc)^2\},$$

$$D_{410} = -2\{4(ae-4bd+3c^2)(bf-4ce+3d^2)-(af-3be+2cd)^2\},$$

$$D_{610} = -2\{4(ag-6bf+15ce-10d^2)(bh-6cg+15df-10e^2)-(ah-5bg+9cf-5de)^2\},$$

$$D_{810} = -2\{4(ai-8bh+28cg-56df+35e^2)(bj-8ci+28dh-56eg+35f^2) - (aj-7bi+20ch-28dg+14ef)^2\}.$$

The derivatives D will be presently calculated in a completely expanded form up to the ninth order. We have, therefore, still to find the derivatives of the sixth and eighth orders, and a second derivative of the ninth order. For the sixth order, the simplest method is to make use of D_{222}, which is easily seen to be equal to

$$24\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

For the two others we have the general formulæ

$$D_{f-2,2,0} = 2\{B_{f-2}(U^{,0}U^{,0})\,B_{f-2}(U^{,2}U^{,2}) - 4B_{f-2}(U^{,0}U^{,1})\,B_{f-2}(U^{,2}U^{,1}) + B_{f-2}(U^{,0}U^{,2})\,B_{f-2}(U^{\,2}U^{,0}) + 2[B_{f-2}(U^{,1}U^{,1})]^2\},$$

where $U^{,0}$, $U^{,1}$, $U^{,2}$ have been written for $\overset{2}{U}{}^{,0}$, $\overset{2}{U}{}^{,1}$, $\overset{2}{U}{}^{,2}$; a formula which is demonstrated in precisely the same way as that for $D_{f-1,1,0}$.

$$D_{f-3,3,0} = -2\{\ B_{f-3}(U^{,0}U^{,3})\,B_{f-3}(U^{,3}U^{,0}) - 6B_{f-3}(U^{,0}U^{,1})\,B_{f-3}(U^{,3}U^{,2}) + 6B_{f-3}(U^{,0}U^{,2})\,B_{f-3}(U^{,3}U^{,1}) + 9B_{f-3}(U^{,1}U^{,1})\,B_{f-3}(U^{,2}U^{,2}) - 9B_{f-3}(U^{,1}U^{,2})\,B_{f-3}(U^{,2}U^{,1}) - B_{f-3}(U^{,0}U^{,3})\,B_{f-3}(U^{,3}U^{,0})\},$$

(in which $U^{,0}$, &c. stand for $\overset{0}{U}{}^{,0}$, &c.). In particular

$$D_{620} = 2\{4(ag-6bf+15ce-10d^2)(ci-6dh+15eg-10f^2) - 4(ah-5bg+9cf-5de)\times(bi+5ch+9dg-5ef) + (ai-6bh+16cg-26df+15e^2)^2 + 8(bh-6cg+15df-10e^2)^2\},$$

$$D_{630} = -2\{4(ag-6bf+15ce-10d^2)(dj-6ei+15fh-10g^2) - 6(ah-5bg+9cf-5de)\times(cj-5di+9eh-5fg) + 6(ai-6bh+16cg-26df+15e^2)(bj-6ci+16dh-26eg+15f^2) + 36(bh-6cg+15df-10e^2)(ci-6dh+15eg-10f^2) - 9(bi-5ch+9dg-5ef)^2 - (aj-6bi+15ch-19dg+9ef)^2\}.$$

Hence we have all the elements necessary for the calculation of the following table of the independent constant derivatives of the fourth degree, up to the ninth order. [I have arranged the terms alphabetically and in tabular form as in my Memoirs on Quantics, and have corrected some inaccuracies];

$D_{210} = 2 \times$

a^2d^2	$+1$
$abcd$	-6
ac^3	$+4$
b^3d	$+4$
b^2c^2	-3
	± 9

$D_{410} = 2 \times$

a^2f^2	$+1$
$abef$	-10
$acdf$	$+4$
ace^2	$+16$
ad^2e	-12
b^2df	$+16$
b^2e^2	$+9$
bc^2f	-12
$bcde$	-76
bd^3	$+48$
c^3e	$+48$
c^2d^2	-32
	± 142

$D_{222} = 24 \times$

$aceg$	$+1$
acf^2	-1
ad^2g	-1
$adef$	$+2$
ae^3	-1
b^2eg	-1
b^2f^2	$+1$
$bcdg$	$+2$
$bcef$	-2
bd^2f	-2
bde^2	$+2$
c^3g	-1
c^2df	$+2$
c^2e^2	$+1$
cd^2e	-3
d^4	$+1$
	± 12

$D_{610} = 2 \times$

a^2h^2	$+1$
$abgh$	-14
$acfh$	$+18$
acg^2	$+24$
$adeh$	-10
$adfg$	-60
ae^2g	$+40$
b^2fh	$+24$
b^2g^2	$+25$
$bceh$	-60
$bcfg$	-234
bd^2h	$+40$
$bdeg$	$+50$
bdf^2	$+360$
be^2f	-240
c^2eg	$+360$
c^2f^2	$+81$
cd^2g	-240
$cdef$	-990
ce^3	$+600$
d^3f	$+600$
d^2e^2	$+375$
	± 2223

$D_{620} = 2 \times$

a^2i^2	$+1$
$abhi$	-16
$acgi$	$+36$
ach^2	$+20$
$adfi$	-52
$adgh$	-60
ae^2i	$+30$
$aefh$	$+20$
aeg^2	$+60$
af^2g	-40
b^2gi	$+20$
b^2h^2	$+44$
$bcfi$	-60
$bcgh$	-388
$bdei$	$+20$
$bdfh$	$+696$
bdg^2	$+180$
be^2h	-340
$befg$	-460
bf^3	$+240$
c^2ei	$+60$
c^2fh	$+180$
c^2g^2	$+544$
cd^2i	-40
$cdeh$	-460
$cdfg$	-2596
ce^2g	$+2340$
cef^2	-420
d^3h	$+240$
d^2eg	-420
d^2f^2	$+12876$
de^2f	-3280
e^4	$+1025$
	± 8632

	$D_{810} = 2 \times$	$D_{621} = 4 \times$
a^2j^2	$+1$	......
$abij$	-18	......
$achj$	$+40$	$+2$
aci^2	$+32$	-2
$adgj$	-56	-7
$adhi$	-112	$+7$
$aefj$	$+28$	$+5$
$aegi$	$+224$	$+22$
aeh^2	......	-27
af^2i	-140	-25
$afgh$	......	$+45$
ag^3	......	-20
b^2hj	$+32$	-2
b^2i^2	$+49$	$+2$
$bcgj$	-112	$+7$
$bchi$	-536	-7
$bdfj$	$+224$	$+22$
$bdgi$	$+392$	-74
bdh^2	$+896$	$+52$
be^2j	-140	-25
$befi$	-196	$+73$
$begh$	-1792	-23
bf^2h	$+1120$	-70
bfg^2	......	$+45$
c^2fj	......	-27
c^2gi	$+896$	$+52$
c^2h^2	$+400$	-25
$cdej$	......	$+45$
$cdfi$	-1792	-23
$cdgh$	-4256	-22
ce^2i	$+1120$	-70
$cefh$	$+560$	$+127$
ceg^2	$+6272$	-32
cf^2g	-3920	-25
d^3j	......	-20
d^2fh	$+6272$	-32
d^2g^2	$+784$	$+47$
d^2ei	......	$+45$
de^2h	-3920	-25
$defg$	-13328	-85
df^3	$+7840$	$+50$
e^3g	$+7840$	$+50$
e^2f^2	-4704	-30
	± 35022	± 698

$$D_{330} = \tfrac{3}{2} D_{222} + \tfrac{1}{2} B_6^2,$$
$$D_{530} = -\tfrac{3}{2} D_{620} + \tfrac{1}{4} B_8^2,$$
$$D_{540} = 2 D_{621} + D_{810},$$

equations which give the functions D_{330}, D_{530}, D_{540}, by means of which the others have been expressed.

We may now proceed to demonstrate an important property of the derivatives of the fourth degree, analogous to the one which exists for the third degree. Let U, V, W, X be functions of any order f: then, investigating the value of the expression

$$B_{2f-2\alpha}[B_\alpha(U, V), \; B_\alpha(W, X)],$$

this reduces itself in the first place to

$$\overline{\theta\phi}^{2f-2\alpha}\,\overline{12}^\alpha\,\overline{34}^\alpha\,UVWX,$$

where ξ_θ, η_θ refer to U and V, and ξ_ϕ, η_ϕ to W and X: this comes to writing $\xi_\theta = \xi_1 + \xi_2$, $\eta_\theta = \eta_1 + \eta_2$, and $\xi_\phi = \xi_3 + \xi_4$, $\eta_\phi = \eta_3 + \eta_4$; whence

$$\overline{\theta\phi} = \overline{13} + \overline{14} + \overline{23} + \overline{24},$$

or the function in question is

$$(13 + 14 + 23 + 24)^{2f-2\alpha}\,\overline{12}^\alpha\,\overline{34}^\alpha\,UVWX.$$

But all the terms of this where the sum of the indices of ξ_1, η_1 or ξ_2, η_2 or ξ_3, η_3 or ξ_4, η_4, exceed f, vanish: whence it is only necessary to consider those of the form

$$K_r(\overline{13}\,.\,\overline{42})^r\,(\overline{14}\,.\,\overline{23})^{f-\alpha-r}\,(\overline{12}\,.\,\overline{34})^\alpha\,UVWX,$$

where K_r denotes the numerical coefficient

$$\frac{(-)^r\,[2f-2\alpha]^{2f-2\alpha}}{[r]^r\,[r]^r\,[f-\alpha-r]^{f-\alpha-r}\,[f-\alpha-r]^{f-\alpha-r}},$$

or
$$B_{2f-2\alpha}[B_\alpha(U, V), \; B_\alpha(W, X)] = \Sigma\{K_r D_{\alpha, r, f-\alpha-r}(U, V, W, X)\}.$$

In particular, if $U = V = W = X$, writing also B_α for $B_\alpha(U, U)$,

$$B_{2f-2\alpha}(B_\alpha, B_\alpha) = \Sigma\,(K_r\,D_{\alpha, r, f-\alpha-r}).$$

If α is odd, this becomes

$$0 = \Sigma\,(K_r\,D_{\alpha, r, f-\alpha-r}),$$

an equation which must be satisfied identically by the relations that exist between the quantities D: if, on the contrary, α is even, we see that there are as many independent functions of the form

$$B_{2f-2\alpha}(B_\alpha, B_\alpha)$$

as there are of the form D; and that these two systems may be linearly expressed, either by means of the other. Thus, for the orders 3, 5, 7, the derivatives D are respectively equal, neglecting a numerical factor, to

$$B_6(U^2, U^2), \;\; B_{10}(U^2, U^2), \;\; B_{14}(U^2, U^2);$$

for the sixth order they may be linearly expressed by means of

$$B_{12}(U^2, U^2), \;\; B_6^2,$$

and so on. All that remains to complete the theory of the fourth degree is to find the general solution of this system of equations, as also of the system connecting the derivatives D.

Passing on to a more general property; let $U_1, U_2, \ldots U_p$ be functions of the orders $f_1, f_2 \ldots f_p$; and suppose

$$\Theta(U_2 \ldots U_p), \;= \square U_2 \ldots U_p,$$

a function of the degree f_1 in the variables: suppose that $\Theta(U_2 \ldots U_p)$ contains the differential coefficients of the order r_2 for U_2, r_3 for U_3, &c., so that $f_1 = (f_2 - r_2) + \ldots (f_p - r_p)$. Consider the expression

$$B_{f_1}\{U_1, \Theta(U_2 \ldots U_p)\},$$

which reduces itself in the first place to

$$(\overline{12} + \overline{13} \ldots + \overline{1p})^{f_1} \square U_1 U_2 \ldots U_p,$$

then to $$K(\overline{12}^{f_2 - r_2}\, \overline{13}^{f_3 - r_3} \ldots \overline{1p}^{f_p - r_p} \square U_1 U_2 \ldots U_p;$$

where for shortness

$$K = \frac{[f_1]^{f_1}}{[f_2 - r_2]^{f_2 - r_2} \ldots [f_p - r_p]^{f_p - r_p}}.$$

For if one of the indices were smaller another would be greater, for instance that of $\overline{12}$: and the symbols ξ_2, η_2 in $\overline{12}^{f_2 - r_2 - \lambda} \square$ would rise to an order higher than f_2, or the term would vanish. Hence, writing

$$\square' = \overline{12}^{f_2 - r_2}\, \overline{13}^{f_3 - r_3} \ldots \overline{1p}^{f_p - r_p},$$

and $$\Theta'(U_1, U_2 \ldots, U_p) = \square' U_1 U_2 \ldots U_p,$$

we have $$B_{f_1}\{U_1, \Theta(U_2, \ldots U_p)\} = K\Theta'(U_1, U_2 \ldots U_p);$$

i.e. the first side is a constant derivative of $U_1, U_2 \ldots U_p$.

Suppose $$U_1 = \frac{1}{[f_1]^{f_1}}(a_0 x^{f_1} + \ldots),$$

$$\Theta(U_2, \ldots U_p) = \frac{1}{[f_1]^{f_1}}(A_0 x^{f_1} + \ldots),$$

then $$K\Theta'(U_1 \ldots U_p) = a_0 A_{f_1} - \frac{f_1}{1} a_1 A_{f_1 - 1} + \ldots;$$

i.e. $$A_{f_1} = K\frac{d}{da_0}\Theta'(U_1 \ldots U_p), \quad \frac{f_1}{1} A_{f_1 - 1} = K\frac{d}{da_1}\Theta'(U_1, U_2 \ldots U_p) \ldots,$$

or finally, $$\Theta(U_2, \ldots U_p) = \frac{K}{[f_1]^{f_1}}\left(x^{f_1}\frac{d}{da_{f_1}} - x^{f_1 - 1} y \frac{d}{da_{f_1 - 1}} + \ldots\right)\Theta'(U_1, \ldots U_p),$$

an equation which holds good (changing, however, the numerical factor,) when several of the functions $U_1 \ldots U_p$ become identical. Hence the theorem: if U be a function given by

$$U = \frac{1}{[f]^f}(a_0 x^f + a_1 x^{f-1} y + \ldots),$$

and Θ be any constant derivative whatever of U, then

$$\left(x^f \frac{d}{da_f} - x^{f-1}y\frac{d}{da_{f-1}} + \dots\right)\Theta$$

is a derivative of U, and its value, neglecting a numerical factor, may be found by omitting in the symbol $\square$, which corresponds to the derivative Θ, the factors which contain any one, no matter which, of the symbolic numbers.

If, for example,

$$-\tfrac{1}{2}D_{210} = \Theta = 6abcd - 4ac^3 - 4bd^3 + 3b^2c^2 - a^2d^2,$$

or

$$\square = \overline{12}^3 \,.\, \overline{34}^2 \,.\, \overline{13} \,.\, \overline{42}\,;$$

then

$$\left(x^3\frac{d}{dd} - x^2y\frac{d}{dc} + xy^2\frac{d}{db} - y^3\frac{d}{da}\right)\Theta$$

reduces itself, omitting a numerical factor, to

$$\overline{12}^2\ \overline{13}\ UUU = -\tfrac{1}{2}B_1\{U,\ B_1(U,\ U)\}.$$

This may be compared with some formulæ of M. Eisenstein's (*Crelle*, vol. XXVII. [1844, pp. 105, 106]; adopting his notation, we have

$$\Phi = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

$$F = \tfrac{1}{36}B_2(\Phi,\ \Phi) = (ac - b^2)\,x^2 + (ad - bc)\,xy + (bd - c^2)\,y^2,$$

$$\Phi_1 = -\tfrac{1}{2}\left(x^3\frac{d}{da} - x^2y\frac{d}{dc} + xy^2\frac{d}{db} - y^3\frac{d}{da}\right)D,$$

where D is the same as Θ. Hence to the system of formulæ which he has given, we may add the two following:

$$\Phi_1 = \tfrac{1}{3}\left(\frac{d\Phi}{dx}\ \frac{dF}{dy} - \frac{d\Phi}{dy}\ \frac{dF}{dx}\right),$$

$$\Phi_1 = -\tfrac{1}{216}\left\{\frac{d^3\Phi}{dx^3}\ \frac{d^2\Phi}{dx^2}\ \frac{d\Phi}{dy} - \frac{d^3\phi}{dx^2\,dy}\left(2\frac{d^2\Phi}{dx\,dy}\ \frac{d\Phi}{dy} + \frac{d^2\Phi}{dy^2}\ \frac{d\Phi}{dx}\right)\right.$$
$$\left. + \frac{d^3\Phi}{dx\,dy^2}\left(2\frac{d^2\Phi}{dx\,dy}\ \frac{d\Phi}{dx} + \frac{d^2\Phi}{dx^2}\ \frac{d\Phi}{dy}\right) - \frac{d^3\Phi}{dy^3}\ \frac{d^2\Phi}{dx^2}\ \frac{d\Phi}{dx}\right\},$$

the first of which explains most simply the origin of the function Φ_1.

It will be sufficient to indicate the reductions which may be applied to derivatives of the form

$$C_{\alpha,\,\beta,\,\gamma}(U,\ V,\ W) = \overline{23}^{\alpha}\,.\,\overline{31}^{\beta}\,.\,\overline{12}^{\gamma}\ UVW,$$

where U, V, W are *homogeneous* functions. In fact, if

$$\xi x + \eta y = \Xi,$$

the above becomes, neglecting a numerical factor,

$$(\Xi_1\,.\,\overline{23})^{\alpha}\,.\,(\Xi_2\,.\,\overline{31})^{\beta}\,.\,(\Xi_3\,.\,\overline{12})^{\gamma}\ UVW,$$

where the symbols ξ, η are supposed not to affect the x, y which enter into the expressions Ξ. But we have identically

$$\Xi_1\,\overline{23}+\Xi_2\,\overline{31}+\Xi_3\,\overline{12}=0,$$

an equation which gives rise to reductions similar to those which have been found for the derivatives $D_{\alpha,\beta,\gamma}$, but which require to be performed with care, in order to avoid inacccuracies with respect to the numerical factors. It may, however, be at once inferred, that the number of independent derivatives $C_{\alpha,\beta,\gamma}$ is the same with that of the independent derivatives $D_{\alpha,\beta,\gamma}$ for the same value of $\alpha+\beta+\gamma$.

From similar reasonings to those by which $B\{U,\ B(U,\ U)\}$ has been found, the following general theorem may be inferred.

"The derivative of any number of the derivatives of one or more functions, or even of any number of functions of these derivatives, is itself a derivative of the original functions."

For the complete reduction of these double derivatives, it would be sufficient, theoretically, to be able to reduce to the smallest number possible, the derivatives of any given degree whatever. This has been done for the derivatives of the third degree $C_{\alpha,\beta,\gamma}$, and for those of the fourth degree, in which all the differentiations rise to the same order $(D_{\alpha,\beta,\gamma})$: it seems, however, very difficult to extend these methods even to the next simplest cases,—extensive researches in the theory of the division of numbers would probably be necessary. Important results might be obtained by connecting the theory of hyperdeterminants with that of elimination, but I have not yet arrived at anything satisfactory upon this subject. I shall conclude with the remark, that it is very easy to find a series, or rather a series of series of hyperdeterminants of all degrees, viz. the determinants

$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},\quad \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}\ \&\text{c.}$$

$$\begin{vmatrix} . & a, & 2b, & c \\ . & b, & 2c, & d \\ a, & 2b, & c, & . \\ b, & 2c, & d, & . \end{vmatrix},\quad \begin{vmatrix} . & a, & 4b, & 6c, & 4d, & e \\ . & b, & 4c, & 6d, & 4e, & f \\ . & c, & 4d, & 6e, & 4f, & g \\ a, & 4b, & 6c, & 4d, & e, & . \\ b, & 4c, & 6d, & 4e, & f, & . \\ c, & 4d, & 6e, & 4f, & g, & . \end{vmatrix}\ \&\text{c.}\quad \begin{vmatrix} . & . & a, & 3b, & 3c, & d \\ . & . & b, & 3c, & 3d, & e \\ . & a, & 3b, & 3c, & d, & . \\ . & b, & 3c, & 3d, & e, & . \\ a, & 3b, & 3c, & d, & . & . \\ b, & 3c, & 3d, & e, & . & . \end{vmatrix}\ \&\text{c.}$$

[I have inserted in these determinants the numerical coefficients which were by mistake omitted.]

However, these functions are not all independent; e.g. the last may be linearly expressed by the square of the second and the cube of $(ae-4bd+3c^2)$; nor do I know the symbolical form of these hyperdeterminant determinants.