11.

CHAPTERS IN THE ANALYTICAL GEOMETRY OF (n) DIMENSIONS.

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CHAP. 1. *On some preliminary formulæ.*

I TAKE for granted all the ordinary formulæ relating to determinants. It will be convenient, however, to write down a few, relating to a certain system of determinants, which are of considerable importance in that which follows: they are all of them either known, or immediately deducible from known formulæ.

Consider the series of terms

$$\begin{matrix} x_1, & x_2 \ldots\ldots x_n \\ A_1, & A_2 \ldots\ldots A_n \\ \vdots & \vdots \\ K_1, & K_2 \ldots\ldots K_n \end{matrix} \qquad \ldots\ldots (1).$$

the number of the quantities $A \ldots\ldots K$ being equal to q $(q < n)$. Suppose $q+1$ vertical rows selected, and the quantities contained in them formed into a determinant, this may be done in $\dfrac{n(n-1)\ldots\ldots(q+2)}{1 \cdot 2 \ldots\ldots n-q-1}$ different ways. The system of determinants so obtained will be represented by the notation

$$\left\| \begin{matrix} x_1, & x_2 \ldots\ldots x_n \\ A_1, & A_2 \ldots\ldots A_n \\ \vdots & \vdots \\ K_1, & K_2 \ldots\ldots K_n \end{matrix} \right\| \qquad \ldots\ldots (2),$$

and the system of equations, obtained by equating each of these determinants to zero, by the notation

$$\left\| \begin{matrix} x_1, & x_2 \ldots\ldots x_n \\ A_1, & A_2 \ldots\ldots A_n \\ \vdots & \vdots \\ K_1 & K_2 \ldots\ldots K_n \end{matrix} \right\| = 0 \qquad \ldots\ldots (3).$$

The $\dfrac{n(n-1)\ldots\ldots(q+2)}{1.2\ldots\ldots(n-q+1)}$ equations represented by this formula reduce themselves to $(n-q)$ independent equations. Imagine these expressed by

$$(1)=0,\quad (2)=0\ldots\ldots(n-q)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

any one of the determinants of (2) is reducible to the form

$$\Theta_1(1)+\Theta_2(2)\ldots+\Theta_{n-q}(n-q)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where $\Theta_1, \Theta_2\ldots\Theta_{n-q}$ are coefficients independent of $x_1, x_2\ldots x_n$. The equations (3) may be replaced by

$$\left\|\begin{matrix} \lambda_1x_1+\lambda_2x_2+\ldots\lambda_nx_n, & \mu_1x_1+\ldots, & \ldots\tau_1x_1+\ldots \\ \lambda_1A_1+\lambda_2A_2+\ldots\lambda_nx_n, & \mu_1A_1+\ldots, & \ldots\tau_1A_1+\ldots \\ \vdots & & \\ \lambda_1K_1+\lambda_2K_2+\ldots\lambda_nK_n, & \mu_1K_1+\ldots, & \tau_1K_1+\ldots \end{matrix}\right\|=0\ldots\ldots\ldots\ldots(6),$$

and conversely from (6) we may deduce (3), unless

$$\left|\begin{matrix} \lambda_1, & \lambda_2, & \ldots\lambda_n \\ \mu_1, & \mu_2, & \ldots\mu_n \\ \vdots & & \\ \tau_1, & \tau_2, & \ldots\tau_n \end{matrix}\right|=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

(The number of the quantities $\lambda, \mu\ldots\tau$ is of course equal to n.) The equations (3) may also be expressed in the form

$$\left\|\begin{matrix} x_1 & , & x_2, & \ldots & x_n \\ \lambda_1A_1+\ldots\omega_1K_1, & & \lambda_1A_2+\ldots\omega_1K_2, & \ldots & \lambda_1A_n\ldots+\omega_1K_n \\ \vdots & & \vdots & & \\ \lambda_qA_1+\ldots\omega_qK_1, & & \lambda_qA_2+\ldots\omega_qK_2, & \ldots & \lambda_qA_n\ldots+\omega_qK_n \end{matrix}\right\|\ldots\ldots\ldots\ldots(8),$$

the number of the quantities $\lambda, \mu\ldots\omega$ being q.

And conversely (3) is deducible from (8), unless

$$\left|\begin{matrix} \lambda_1, \ldots\omega_1 \\ \vdots \\ \lambda_q, \ldots\omega_q \end{matrix}\right|=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

CHAP. 2. *On the determination of linear equations in $x_1, x_2,\ldots x_n$ which are satisfied by the values of these quantities derived from given systems of linear equations.*

It is required to find linear equations in $x_1,\ldots x_n$ which are satisfied by the values of these quantities derived—1. from the equations $\mathfrak{A}'=0, \mathfrak{B}'=0\ldots\mathfrak{G}'=0$; 2. from the equations $\mathfrak{A}''=0, \mathfrak{B}''=0\ldots\mathfrak{H}''=0$; 3. from $\mathfrak{A}'''=0, \mathfrak{B}'''=0\ldots\mathfrak{K}'''=0$, &c. &c., where

$$\mathfrak{A}'=A_1x_1+A_2x_2\ldots+A_nx_n,\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$\mathfrak{B}'=B_1x_1+B_2x_2\ldots+B_nx_n,$$
$$\vdots$$

and similarly $\mathfrak{A}'', \mathfrak{B}'',\ldots, \mathfrak{A}''', \mathfrak{B}''',\ldots$, &c. are linear functions of the coordinates $x_1, x_2,\ldots x_n$.

Also r', $r''\dots$ representing the number of equations in the systems (1), (2) ... and k the number of these given systems,

$$(n-r')+(n-r'')+\dots \not> n-1 \text{ or } (k-1)n+1 \not> r'+r''+\dots$$

Assume

$$0=\lambda'\mathfrak{A}'+\mu'\mathfrak{B}'+\dots,$$

$$\lambda'\mathfrak{A}'+\mu'\mathfrak{B}'+\dots=\lambda''\mathfrak{A}''+\mu''\mathfrak{B}''+\dots=\lambda'''\mathfrak{A}'''+\mu'''\mathfrak{B}'''+\dots=\&c. \;\dots\dots\; (2),$$

the latter equations denoting the equations obtained by equating to zero the terms involving x_1, those involving x_2, &c.... separately. Suppose, in addition to these, a set of linear equations in λ', $\mu'\dots\lambda''$, $\mu''\dots$ so that, with the preceding ones, there is a sufficient number of equations for the elimination of these quantities. Then, performing the elimination, we thus obtain equations $\Psi=0$, where Ψ is a function of x_1, $x_2\dots$ which vanishes for the values of these quantities derived from the equations (1) or (2) ... &c. The series of equations $\Psi=0$ may be expressed in the form

$$\left\|\begin{matrix} \mathfrak{A}', & \mathfrak{B}', \dots & \mathfrak{C}', & & & & & & \\ A_1', & B_1', \dots & G_1', & A_1'', \dots & O''_1, & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ A_n', & B_n', \dots & G_n', & A_n'', \dots & O_n'', & & & & \\ & & & A_1'', \dots & O_1'', & A_1''', \dots & R_1''', & & \\ & & & \vdots & \vdots & \vdots & \vdots & & \\ & & & A_n'', \dots & O_n'', & A_n''', \dots & R_n''', & & \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}\right\| = 0 \;\dots\dots\; (3).$$

CHAP. 3. *On reciprocal equations.*

Consider a system of equations

$$\begin{aligned} &A_1x_1+A_2x_2\dots+A_nx_n=0, \quad \dots\dots\dots\dots\dots\dots\dots\dots(1),\\ &\vdots\\ &K_1x_1+K_2x_2\dots+K_nx_n=0, \end{aligned}$$

(r in number).

The reciprocal system with respect to a given function (U) of the second order in x_1, $x_2\dots x_n$, is said to be

$$\left\|\begin{matrix} d_{x_1}U, & d_{x_2}U, \dots & d_{x_n}U \\ A_1, & A_2, \dots & A_n \\ \vdots & \vdots & \vdots \\ K_1, & K_2, \dots & K_n \end{matrix}\right\| = 0 \;\dots\dots\dots\dots\dots\dots\dots(2),$$

($n-r$ in number).

It must first be shown that the reciprocal system to (2) is the system (1), or that the systems (1), (2) are reciprocals of each other.

Consider, in general, the system of equations

$$\alpha_1 d_{x_1} U + \alpha_2 d_{x_2} U \ldots + \alpha_n d_{x_n} U = 0 \quad \ldots\ldots\ldots\ldots (3).$$
$$\vdots$$
$$\lambda_1 d_{x_1} U + \lambda_2 d_{x_2} U \ldots + \lambda_n d_{x_n} U = 0.$$

Suppose $\quad 2U = \Sigma (\alpha^2) x_\alpha^2 + 2\Sigma (\alpha\beta)\, x_\alpha x_\beta$, so that $d_{x_s} U = \Sigma (s\alpha)\, x_\alpha$ ....(4), (5).

The equations (3) may be written

$$x_1 \{\alpha_1 (1^2) + \alpha_2 (12) \ldots + \alpha_n (1n)\} + \ldots + x_n \{\alpha_1 (n1) + \alpha_2 (n2) \ldots + \alpha_n (n^2)\} = 0 \quad \ldots\ldots\ldots (6),$$

&c.

and forming the reciprocals of these, also replacing $d_{x_1} U$, $d_{x_2} U \ldots$ by their values, we have

$$\left\| \begin{matrix} x_1 (1^2) + x_2 (12) + \ldots x_n (1n), \ldots x_1 (n1) + x_2 (n2) \ldots + x_n (n^2) \\ \alpha_1 (1^2) + \alpha_2 (12) + \ldots \alpha_n (1n), \ldots \alpha_1 (n1) + \alpha_2 (n2) \ldots + \alpha_n (n^2) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \lambda_1 (1^2) + \lambda_2 (12) + \ldots \lambda_n (1n), \ldots \lambda_1 (n1) + \lambda_2 (n2) \ldots + \lambda_n (n^2) \end{matrix} \right\| = 0 \ldots\ldots (7).$$

From these, assuming

$$\left| \begin{matrix} (1^2), & (12), \ldots (1n) \\ (21), & (2^2), \ldots (2n) \\ \vdots & \\ (n1), & (n2), \ldots (n^2) \end{matrix} \right| \neq 0 \quad \ldots\ldots\ldots\ldots (8)$$

we obtain, for the reciprocal system of (3),

$$\left\| \begin{matrix} x_1, & x_2, \ldots x_n \\ \alpha_1, & \alpha_2, \ldots \alpha_n \\ \vdots & \\ \lambda_1, & \lambda_2, \ldots \lambda_n \end{matrix} \right\| = 0 \quad \ldots\ldots\ldots\ldots (9).$$

Now, suppose the equations (3) represent the system (2); their number in this case must be $n - r$. Also if θ represent any one of the quantities $\alpha, \beta \ldots \lambda$, we have

$$A_1\theta_1 + A_2\theta_2 \ldots + A_n\theta_n = 0 \quad \ldots\ldots\ldots\ldots (10),$$
$$\vdots$$
$$K_1\theta_1 + K_2\theta_2 \ldots + K_n\theta_n = 0.$$

By means of these equations, the system (9) may be reduced to the form

$$\left\| \begin{matrix} A_1 x_1 + A_2 x_2 \ldots + A_n x_n, & \ldots K_1 x_1 + K_2 x_2 \ldots + K_n x_n, & x_{r+1}, & x_{r+2}, \ldots x_n \\ 0 & , \ldots \quad 0 & , \alpha_{r+1}, & \alpha_{r+2}, \ldots \alpha_n \\ \vdots & & \vdots & \\ 0 & , \ldots \quad 0 & , \lambda_{r+1}, & \lambda_{r+2}, \ldots \lambda_n \end{matrix} \right\| = 0 \ldots (11),$$

which are satisfied by the equations (1). Hence the reciprocal system to (2) is (1), or (1), (2) are reciprocals to each other.

THEOREM. Consider the equations

$$(\mathfrak{A}' = 0,\ \mathfrak{B}' = 0 \dots \mathfrak{G}' = 0) \dots\dots (12),$$
$$(\mathfrak{A}'' = 0,\ \mathfrak{B}'' = 0 \dots \mathfrak{P}'' = 0),$$
$$(\mathfrak{A}''' = 0,\ \mathfrak{B}''' = 0 \dots \mathfrak{K}''' = 0),$$
&c.

of Chap. 2. The equations

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, \dots & d_{x_n}U \\ A_1', & A_2', \dots & A_n' \\ \vdots & & \vdots \\ G_1', & G_2', \dots & G_n' \end{matrix} \right\| = 0, \quad \left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, \dots & d_{x_n}U \\ A_1'', & A_2'', \dots & A_n'' \\ \vdots & & \\ O_1'', & O_2'', \dots & O_n'' \end{matrix} \right\| = 0, \dots (13),$$
&c.

which are the reciprocals of these systems, represent taken conjointly the reciprocal of the system of equations (3) of the same chapter.

Let this system, which contains $n - \{(n-r) + (n-r') + \dots\}$ equations, be represented by

$$\alpha_1 x_1 + \alpha_2 x_2 \dots + \alpha_n x_n = 0 \dots\dots (14),$$
$$\beta_1 x_1 + \beta_2 x_2 \dots + \beta_n x_n = 0.$$
$$\vdots$$
$$\zeta_1 x_1 + \zeta_2 x_2 \dots + \zeta_n x_n = 0.$$

The reciprocal system is

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, \dots & d_{x_n}U \\ \alpha_1, & \alpha_2, \dots & \alpha_n \\ \vdots & & \\ \zeta_1, & \zeta_2, \dots & \zeta_n \end{matrix} \right\| = 0 \dots\dots (15),$$

containing $(n-r) + (n-r') +$ &c. ... equations.

Also, by the formulæ in Chap. 2,

$$\alpha_1 x_1 + \dots + \alpha_n x_n = \lambda_1' \mathfrak{A}' + \mu_1' \mathfrak{B}' + \dots \sigma_1' \mathfrak{G}' \text{ } (\lambda, \mu \dots \sigma, \ r' \text{ in number}).$$
$$\beta_1 x_1 + \dots + \beta_n x_n = \lambda_2' \mathfrak{A}' + \mu_2' \mathfrak{B}' + \dots \sigma_2' \mathfrak{G}'$$
$$\vdots$$
$$\zeta_1 x_1 \quad \dots + \zeta_n x_n = \lambda_\theta' \mathfrak{A}' + \mu_\theta' \mathfrak{B}' + \dots \sigma_\theta' \mathfrak{G} \dots\dots (16),$$

writing $\theta = n - \{(n-r) + (n-r') + \dots\}$.

Also, assuming any arbitrary quantities $\eta_1, \ \eta_2 \dots \eta_n \dots \phi_1, \ \phi_2 \dots \phi_n$ (the number of sets being $(r' - \theta)$,) such that

$$\eta_1 x_1 \dots + \eta_n x_n = \lambda_{\theta+1}' \mathfrak{A}' + \mu_{\theta+1} \mathfrak{B}' + \dots \sigma_{\theta+1} \mathfrak{G}' \dots\dots (17),$$
$$\vdots$$
$$\phi_1 x_1 \dots + \phi_n x_n = \lambda_{r'}' \mathfrak{A}' + \mu_{r'}' \mathfrak{B}' + \dots \sigma_{r'}' \mathfrak{G}'.$$

From the equations (15) we deduce the $(n-r)$ equations

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, & \dots & d_{x_n}U \\ \eta_1, & \eta_2, & \dots & \eta_n \\ \vdots & & & \\ \phi_1, & \phi_2, & \dots & \phi_n \end{matrix} \right\| = 0 \quad \text{.......................... (18).}$$

Hence, writing

$$\eta = \lambda_1' A + \mu_1' B + \dots \sigma_1' G \quad \text{............................ (19).}$$
$$\vdots$$
$$\phi = \lambda_r' A + \mu_r' B + \dots \sigma_r' G,$$

and reducing, by the formula (8) of Chap. 1, we have

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, & \dots & d_{x_n}U \\ A_1', & A_2', & \dots & A_n' \\ \vdots & & & \\ G_1', & G_2', & \dots & G_n' \end{matrix} \right\| = 0 \quad \text{.......................... (20);}$$

and similarly may the remaining formulæ of (13) be deduced from the equation (15). Hence the required theorem is demonstrated, a theorem which may be more clearly stated as follows :—

The reciprocals of several systems of equations form together the reciprocal of the equation which is satisfied by the values of the variables which satisfy each of the original systems of equations. (The theorem requires that the number of all the reciprocal equations shall be less than the number of variables.)

Conversely, consider several systems of equations, the whole number of the equations being less than the number of variables. These systems, taken conjointly, have for their reciprocal, the equation which is satisfied by the values satisfying the reciprocal system of each of the given systems.

CHAP. 4. *On some properties of functions of the second order.*

Suppose, as before, U denotes the general function of the second order, or

$$2U = \Sigma\,(\alpha^2)\, x_\alpha{}^2 + 2\Sigma\,(\alpha\beta)\, x_\alpha x_\beta \quad \text{..........................(21).}$$

Also let V denote a function of the second order of the form

$$V = H\left(\left\| \begin{matrix} x_1, & x_2, & \dots & x_n \\ \alpha_1, & \alpha_2, & \dots & \alpha_n \\ \vdots & & & \\ \rho_1, & \rho_2, & \dots & \rho_n \end{matrix} \right\| \right) \quad \text{............................ (22),}$$

(H being the symbol of a homogeneous function of the second order, and the number r of the quantities $\alpha, \beta \dots \rho$, being less than $n-1$). [Observe that $\alpha_1, \beta_1, \dots \rho_1, \dots \alpha_n, \beta_n, \dots \rho_n$, or say the suffixed quantities $\alpha, \beta, \dots \rho$ (r in number) are used to denote coefficients: α, β, without suffixes, are any two numbers in the series of suffixes 1, 2, 3, ... n.] Then $2U - 2kV$, k arbitrary, is of the form

$$\Sigma\,[\alpha^2]\, x_\alpha^2 + 2\Sigma\,[\alpha\beta]\, x_\alpha x_\beta \quad \text{.................................(23).}$$

Suppose $X_1, X_2, \ldots X_n$ determined by the equations

$$\begin{aligned}
&[1^2]\,X_1 + [12]\,X_2 \ldots + [1n]\,X_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),\\
&[21]\,X_1 + [2^2]\,X_2 \ldots + [2n]\,X_n = 0,\\
&\vdots\\
&[n1]\,X_1 + [n2]\,X_2 \ldots + [n^2]\,X_n = 0;
\end{aligned}$$

equations that involve the condition that k satisfies an equation of the order $n-r$, as will be presently proved.

Then shall $x_1 = X_1 \ldots x_n = X_n$ satisfy the system of equations, which is the reciprocal of

$$\left\|\begin{matrix} x_1, & x_2, \ldots x_n \\ \alpha_1, & \alpha_2, \ldots \alpha_n \\ \vdots & \\ \rho_1, & \rho_2, \ldots \rho_n \end{matrix}\right\| = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

To prove these properties, in the first place we must find the form of V. Consider the quantities $\xi_A, \xi_B, \ldots \xi_L$, $(n-r)$ in number, of the form

$$\begin{aligned}
\xi_A &= A_1\,x_1 + A_2\,x_2 \ldots + A_n\,x_n, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),\\
\xi_B &= B_1\,x_1 + B_2\,x_2 \ldots + B_n\,x_n,\\
&\vdots\\
\xi_L &= L_1\,x_1 + L_2\,x_2 \ldots + L_n\,x_n;
\end{aligned}$$

where, if Θ represent any of the quantities $A, B \ldots L$,

$$\begin{aligned}
\alpha_1\Theta_1 + \alpha_2\Theta_2 \ldots + \alpha_n\Theta_n &= 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),\\
\beta_1\Theta_1 + \beta_2\Theta_2 \ldots + \beta_n\Theta_n &= 0,\\
&\vdots\\
\rho_1\Theta_1 + \rho_2\Theta_2 \ldots + \rho_n\Theta_n &= 0,
\end{aligned}$$

$$2V = (A^2)\xi_A{}^2 + (B^2)\,\xi_B{}^2 + \ldots + 2\,(AB)\,\xi_A\xi_B + \ldots = \Sigma\,(A^2)\,\xi_A{}^2 + 2\Sigma\,(AB)\,\xi_A\xi_B.$$

Hence, if $$2V = \Sigma\,\{\alpha^2\}\,x_\alpha^2 + 2\Sigma\,\{\alpha\beta\}\,x_\alpha x_\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

we have for the coefficients of this form

$$\{1^2\} = \Sigma\,(A^2)\,A_1{}^2 + 2\Sigma\,(AB)\,A_1B_1, \quad \{12\} = \Sigma\,(A^2)\,A_1A_2 + \Sigma\,(AB)\,(A_1B_2 + A_2B_1),$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$

and consequently the coefficients of $2U - 2kV$ are

$$[1^2] = (1^2) - k\,\{1^2\}, \qquad [12] = (12) - k\,\{12\}.$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$

Hence, θ representing any of the quantities $\alpha, \beta \ldots \rho$,

$$\begin{aligned}
&\theta_1\,\{1^2\} + \theta_2\,\{12\} \ldots + \theta_n\,\{1n\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (29),\\
&\vdots\\
&\theta_1\{n1\} + \theta_2\,\{n2\} \ldots + \theta_n\,\{n^2\} = 0;
\end{aligned}$$

whence also $\theta_1[1^2] + \dots \theta_n[1n] = \theta_1(1^2) + \dots \theta_n(1n),$

$$\vdots$$

$$\theta_1[n1] + \dots \theta_n[n^2] = \theta_1(n1) + \dots \theta_n(n^2).$$

Hence, the equations for determining $X_1, \dots X_n$ may be reduced to

$$X_1[\alpha_1(1^2) + \dots \alpha_n(1n)] + X_2[\alpha_1(21) \dots + \alpha_n(2n)] \dots + X_n[\alpha_1(n1) \dots + \alpha_n(n^2)] = 0 \dots (30),$$

$$X_1[\beta_1(1^2) + \dots \beta_n(1n)] + X_2[\beta_1(21) \dots + \beta_n(2n)] \dots + X_n[\beta_1(n1) \dots + \beta_n(n^2)] = 0,$$

$$\vdots$$

$$X_1[\rho_1(1^2) + \dots \rho_n(1n)] + X_2[\rho_1(21) \dots + \rho_n(2n)] \dots + X_n[\rho_n(n1) \dots + \rho_n(n^2)] = 0.$$

$$X_1[r+1, 1] + \quad X_2[r+1, 2] \dots + \quad X_n[r+1, n] = 0,$$

$$\vdots$$

$$X_1[n, 1] + \quad X_2[n, 2] \dots + \quad X_n[n^2] = 0.$$

Eliminating $X_1 \dots X_n$, since the first r equations do not contain k, the equation in this quantity is of the order $n - r$.

Next form the reciprocals of the equations (25). These are

$$\begin{Vmatrix} d_{x_1}U, & d_{x_2}U, \dots d_{x_n}U \\ A_1, & A_2, \dots A_n \\ \vdots & \\ L_1, & L_2, \dots L_n \end{Vmatrix} = 0 \dots\dots\dots (31).$$

From which we may deduce

$$\begin{Vmatrix} \alpha_1 d_{x_1}U \dots + \alpha_n d_{x_n}U, & \beta_1 d_{x_1}U \dots + \beta_n d_x U, & \dots \rho_1 d_{x_1}U \dots \rho_n d_{x_n}U, & d_{x_{r+1}}U, \dots d_{x_n}U \\ 0, & 0, & \dots\ 0, & A_{r+1}, \dots A_n \\ \vdots & & & \\ 0, & 0, & \dots\ 0, & L_{r+1}, \dots L_n \end{Vmatrix} = 0 \dots (32),$$

which are evidently satisfied by $x_1 = X_1,\ x_2 = X_2 \dots x_n = X_n$.

In the case of four variables, the above investigation demonstrates the following properties of surfaces of the second order.

I. If a cone intersect a surface of the second order, three different cones may be drawn through the curve of intersection, and the vertices of these lie in the plane which is the polar reciprocal of the vertex of the intersecting cone.

II. If two planes intersect a surface of the second order through the curve of intersection, two cones may be drawn, and the vertices of these lie in the line which is the polar reciprocal of the line of intersection of the two planes.

Both these theorems are undoubtedly known, though I am not able to refer for them to any given place.