8.

ON LAGRANGE'S THEOREM.

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The value given by Lagrange's theorem for the expansion of any function of the quantity x, determined by the equation

$$x = u + hfx \quad\quad (1),$$

admits of being expressed in rather a remarkable symbolical form. The *à priori* deduction of this, independently of any expansion, presents some difficulties; I shall therefore content myself with showing that the form in question satisfies the equations

$$\frac{d}{du} \cdot \int F'x\, fx\, dx = \frac{d}{dh} \cdot \int F'x\, dx \quad\quad (2),$$

$$Fx = Fu \text{ for } h = 0 \quad\quad (3),$$

deduced from the equation (1), and which are sufficient to determine the expansion of Fx, considered as a function of u and h in powers of h.

Consider generally the symbolical expression

$$\phi\left(h\frac{d}{dh}\right)\Xi h \quad\quad (4),$$

$\phi\left(h\frac{d}{dh}\right)$ involving in general symbols of operation relative to any of the other variables entering into Ξh. Then, if Ξh be expansible in the form

$$\Xi h = \Sigma\,(Ah^m) \quad\quad (5),$$

it is obvious that

$$\phi\left(h\frac{d}{dh}\right)\Xi h = \Sigma\,\{\phi m\,.\,(Ah^m)\} = \Sigma\,\{(\phi m\,.\,A)\,h^m\} \quad\quad (6).$$

For instance, u representing a variable contained in the function Ξh, and taking a particular form of $\phi\left(h\frac{d}{dh}\right)$,

$$\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi h = \Sigma\left(\frac{d^m A}{du^m}h^m\right) \quad\text{.......................... (7).}$$

From this it is easy to demonstrate

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi h\right\} = \frac{1}{h}\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\{h\Xi h\} \quad\text{...................... (8),}$$

$$\frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi h\right\} = \frac{1}{h}\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\{h\Xi' h\} \quad\text{...................... (9),}$$

where $\Xi' h$ denotes $\frac{d}{dh}\Xi h$, as usual. Hence also

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi' h\right\} = \frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi h\right\} \quad\text{...................... (10),}$$

of which a particular case is

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}F'u\, fu\, e^{hfu}\right\} = \frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}F'u\, e^{hfu}\right\} \quad\text{..............(11).}$$

Also, $$\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\, e^{hfu}) = Fu \text{ for } h=0 \quad\text{......................(12).}$$

Hence the form in question for Fx is

$$Fx = \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\, e^{hfu}) \quad\text{......................(13);}$$

from which, differentiating with respect to u, and writing F instead of F',

$$\frac{Fx}{1-hf'x} = \left(\frac{d}{du}\right)^{h\frac{d}{dh}}(Fu\, e^{hfu}) \quad\text{..............................(14),}$$

a well-known form of Lagrange's theorem, almost equally important with the more usual one. It is easy to deduce (13) from (14). To do this, we have only to form the equation

$$\frac{-hFx\,f'x}{1-hf'x} = -h\left(\frac{d}{du}\right)^{h\frac{d}{dh}}(Fu\,f'u\, e^{hfu}) \quad\text{........................(15),}$$

deduced from (14) by writing $Fxf'x$ for fx, and adding this to (14),

$$\begin{aligned} Fx &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}}(Fu\, e^{hfu}) - h\left(\frac{d}{du}\right)^{h\frac{d}{dh}}(Fu\,f'u\, e^{hfu}) \\ &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}\left\{\frac{d}{du}(Fu\, e^{hfu}) - h\,f'u\,Fu\, e^{hfu}\right\} \\ &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\, e^{hfu}) \quad\text{.................................(16).} \end{aligned}$$

In the case of several variables, if

$$x = u + hf(x,\ x_1 \ldots),\quad x_1 = u_1 + h_1 f_1(x,\ x_1 \ldots),\ \&c. \ldots\ldots\ldots\ldots\ (17),$$

writing for shortness

$$F,\ f,\ f_1 \ldots \text{ for } F(u,\ u_1 \ldots),\quad f(u,\ u_1 \ldots),\quad f_1(u,\ u_1 \ldots),\ \ldots$$

then the formula is

$$\frac{F(x,\ x_1 \ldots)}{\{1 - hf'(x)\}\{1 - h_1 f_1'(x_1) \ldots\}} = \left(\frac{d}{du}\right)^{h\frac{d}{dh}} \left(\frac{d}{du_1}\right)^{h_1\frac{d}{dh_1}} \ldots (Fe^{hf + h_1 f_1 + \cdots}) \ldots\ldots (18),$$

$\{$where $f'(x)$ is written to denote $\frac{d}{dx} f(x,\ x_1 \ldots)$, &c.$\}$

or the coefficient of $h^n h_1^{n_1} \ldots\ldots$ in the expansion of

$$\frac{F(x,\ x_1 \ldots\ldots)}{\{1 - hf'(x)\}\{1 - h_1 f_1'(x_1)\} \ldots\ldots} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

is

$$= \frac{1}{1\,.\,2 \ldots n\,.\,1\,.\,2 \ldots n_1} \left(\frac{d}{du}\right)^n \left(\frac{d}{du_1}\right)^{n_1} \ldots Ff^n f_1^{n_1} \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

From the formula (18), a formula may be deduced for the expansion of $F(x,\ x_1 \ldots\ldots)$, in the same way as (13) was deduced from (14), but the result is not expressible in a simple form by this method. An apparently simple form has indeed been given for this expansion by Laplace, *Mécanique Celeste*, [Ed. 1, 1798] tom. I. p. 176; but the expression there given for the general term, requires first that certain differentiations should be performed, and then that certain changes should be made in the result, quantities $z,\ z' \ldots\ldots$, are to be changed into $z^n,\ z_1^{n_1} \ldots\ldots$; in other words, the general term is not really expressed by known symbols of operation only. The formula (18) is probably known, but I have not met with it anywhere.