7.

ON A CLASS OF DIFFERENTIAL EQUATIONS, AND ON THE LINES OF CURVATURE OF AN ELLIPSOID.

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CONSIDER the primitive equation

$$fx + gy + hz + \ldots\ldots = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

between n variables x, y, z, the constants f, g, h being connected by the equation

$$H(f, g, h\ldots\ldots) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

H denoting a homogeneous function. Suppose that $f, g, h\ldots\ldots$ are determined by the conditions

$$\begin{array}{l} fx_1 \ \ + gy_1 \ \ + hz_1 \ \ \ldots\ldots = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3), \\ \ \vdots \qquad\ \ \vdots \qquad\ \ \vdots \\ fx_{n-2} + gy_{n-2} + hz_{n-2} \ldots\ldots = 0. \end{array}$$

Then writing

$$X = \left| \begin{array}{lll} y &, z &, \ldots \\ y_1 &, z_1 &, \ldots \\ \vdots & \ \vdots & \\ y_{n-2}, & z_{n-2}, & \ldots \end{array} \right| \quad\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

with analogous expressions for $y, z\ldots\ldots$; the equations (3) give $f, g, h,\ldots\ldots$ proportional to $x, y, z,\ldots\ldots$ or eliminating $f, g, h\ldots\ldots$ by the equation (2),

$$H(X, Y, Z\ldots\ldots) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Conversely the equation (5), which contains, in appearance, $n(n-2)$ arbitrary constants, is equivalent to the system (1), (2). And if H be a rational integral function of the order r, the first side of the equation (5) is the product of r factors, each of them of the form given by the system (1), (2).

Now from the equation (1), we have the system

$$\begin{aligned} &fx \quad + gy \quad + hz \quad \ldots\ldots = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),\\ &fdx \quad + gdy \quad + hdx \quad \ldots\ldots = 0,\\ &\vdots \qquad\quad \vdots \qquad\quad \vdots\\ &fd^{n-2}x + gd^{n-2}y + hd^{n-2}z \ldots\ldots = 0, \end{aligned}$$

or writing

$$X = \begin{vmatrix} y & , z & , \ldots \\ dy & , dz & , \ldots \\ \vdots & \vdots & \\ d^{n-2}y, & d^{n-2}z, & \ldots \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

with analogous expressions for Y, Z......; then from the equations (6), f, g, h...... are proportional to X, Y, Z......: or, eliminating by (2),

$$H(X, Y, Z\ldots\ldots) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Conversely the integral of the equation (8) of the order $(n-2)$ is given either by the system of equations (1), (2), in which it is evident that the number of arbitrary constants is reduced to $(n-2)$; or, by the equation (5), which contains in appearance $n(n-2)$ arbitrary constants, but which we have seen is equivalent in reality to the system (1), (2).

Thus, with three variables, the integral of

$$H(ydz - zdy,\ zdx - xdz,\ xdy - ydx) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

may be expressed in the form

$$H(yz_1 - y_1z,\quad zx_1 - z_1x,\quad xy_1 - x_1y) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

and also in the form

$$fx + gy + hz = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where

$$H(f, g, h) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

The above principles afford an elegant mode of integrating the differential equation for the lines of curvature of an ellipsoid. The equation in question is

$$(b^2 - c^2)\, xdydz + (c^2 - a^2)\, ydzdx + (a^2 - b^2)\, zdxdy = 0 \ldots\ldots\ldots\ldots\ldots(13),$$

where x, y, z are connected by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14);$$

writing

$$\frac{x^2}{a^2} = u, \quad \frac{y^2}{b^2} = v, \quad \frac{z^2}{c^2} = w \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

these become

$$(b^2 - c^2)\, udvdw + (c^2 - a^2)\, vdwdu + (a^2 - b^2)\, wdudv = 0 \ldots\ldots\ldots\ldots\ldots(16),$$

$$u + v + w = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Multiplying by

$$-\{(vdu-udv)(wdv-vdw)(udw-wdu)\}^{-1},$$

the first of these becomes

$$\frac{-a^2du}{(vdu-udv)(udw-wdu)}+\frac{-b^2dv}{(wdv-vdw)(vdu-udv)}+\frac{-c^2dw}{(udw-wdu)(wdv-vdw)}=0 \ldots \quad (18);$$

but writing (17) and its derived equations under the form

$$u+\ (v+\ w)=1 \ \ldots\ldots\ldots\ldots \quad (19),$$

$$du+(dv+dw)=0,$$

we deduce

$$-du\,(v+w)+u\,(dv+dw)=-du \ldots\ldots\ldots\ldots (20),$$

i.e.

$$-du=-(vdu-udv)+(udw-wdu) \ldots\ldots\ldots\ldots (21),$$

and similarly

$$-dv=-(wdv-vdw)+(vdu-udv),$$

$$-dw=-(udw-wdu)+wdv-vdw).$$

Substituting,

$$\frac{b^2-c^2}{wdv-vdw}+\frac{c^2-a^2}{udw-wdu}+\frac{a^2-b^2}{vdu-udv}=0 \ \ldots\ldots\ldots\ldots (22);$$

the integral of which may be written in the form

$$\frac{b^2-c^2}{wv_1-vw_1}+\frac{c^2-a^2}{uw_1-wu_1}+\frac{a^2-b^2}{vu_1-v_1u}=0 \ \ldots\ldots\ldots\ldots (23),$$

where, on account of (17),

$$u_1+v_1+w_1=1 \ \ldots\ldots\ldots\ldots (24);$$

and also in the form

$$fu+gv+hw=0 \ \ldots\ldots\ldots\ldots (25),$$

where f, g, h are connected by

$$\frac{b^2-c^2}{f}+\frac{c^2-a^2}{g}+\frac{a^2-b^2}{g}=0 \ldots\ldots\ldots\ldots (26);$$

this last equation is satisfied identically by

$$f=\frac{b^2-c^2}{B^2-C^2}, \qquad g=\frac{c^2-a^2}{C^2-A^2}, \qquad h=\frac{a^2-b^2}{A^2-B^2} \ \ldots\ldots\ldots\ldots (27).$$

Restoring x, y, z, x_1, y_1, z_1 for u, v, w, u_1, v_1, w_1, the equations to a line of curvature passing through a given point x_1, y_1, z_1, on the ellipsoid, are the equation (14) and

$$\frac{(b^2-c^2)}{a^2\,(y_1^2z^2-y^2z_1^2)}+\frac{(c^2-a^2)}{b^2\,(z_1^2x^2-z^2x_1^2)}+\frac{(a^2-b^2)}{c^2\,(x_1^2y^2-x^2y_1^2)}=0 \ \ldots\ldots\ldots\ldots (28),$$

or again, under a known form, they are the equation (14) and

$$\frac{(b^2-c^2)}{B^2-C^2}\,\frac{x^2}{a^2}+\frac{c^2-a^2}{C^2-A^2}\,\frac{y^2}{b^2}+\frac{a^2-b^2}{A^2-B^2}\,\frac{z^2}{c^2}=0 \ \ldots\ldots\ldots\ldots (29).$$

From the equations (14), (29) it is easy to prove the well-known form

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1 \quad\ldots\ldots\ldots\ldots(30);$$

in fact, multiplying (29) by m, and adding to (14), we have the equation (30), if the equations

$$\frac{1}{a^2}+m\,\frac{b^2-c^2}{B^2-C^2}\,\frac{1}{a^2}=\frac{1}{a^2+\theta}, \quad\ldots\ldots\ldots\ldots(31),$$

$$\frac{1}{b^2}+m\,\frac{c^2-a^2}{C^2-A^2}\,\frac{1}{b^2}=\frac{1}{b^2+\theta},$$

$$\frac{1}{c^2}+m\,\frac{a^2-b^2}{A^2-B^2}\,\frac{1}{c^2}=\frac{1}{(c^2+\theta)},$$

are satisfied.

But on reduction, these take the form

$$(B^2-C^2)\,\theta+(b^2-c^2)\,m\theta+ma^2\,(b^2-c^2)=0, \quad\ldots\ldots\ldots\ldots(32),$$

$$(C^2-A^2)\,\theta+(c^2-a^2)\,m\theta+mb^2\,(c^2-a^2)=0,$$

$$(A^2-B^2)\,\theta+(a^2-b^2)\,m\theta+mc^2\,(a^2-b^2)=0,$$

and since, by adding, an identical equation is obtained, m and θ may be determined to satisfy these equations. The values of θ, m are

$$\theta=\frac{(a^2-b^2)\,(b^2-c^2)\,(c^2-a^2)}{a^2\,(B^2-C^2)+b^2\,(C^2-A^2)+c^2\,(A^2-B^2)} \quad\ldots\ldots\ldots\ldots(33),$$

$$m=\frac{b^2c^2\,(B^2-C^2)+c^2a^2\,(C^2-A^2)+a^2b^2\,(A^2-B^2)}{(a^2-b^2)\,(b^2-c^2)\,(c^2-a^2)} \quad\ldots\ldots\ldots\ldots(34).$$