6.

ON THE MOTION OF ROTATION OF A SOLID BODY.

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In the fifth volume of Liouville's Journal, in a paper "Des lois géometriques qui régissent les déplacemens d'un système solide," M. Olinde Rodrigues has given some very elegant formulæ for determining the position of two sets of rectangular axes with respect to each other, employing rational functions of three quantities only. The principal object of the present paper is to apply these to the problem of the rotation of a solid body; but I shall first demonstrate the formulæ in question, and some others connected with the same subject which may be useful on other occasions.

Let Ax, Ay, Az; $Ax_{,}$, $Ay_{,}$, $Az_{,}$, be any two sets of rectangular axes passing through the point A: $x, y, z, x_{,}, y_{,}, z_{,}$ being taken for the points where these lines intersect the spherical surface described round the centre A with radius unity. Join $xx_{,}$, $yy_{,}$, $zz_{,}$, by arcs of great circles, and through the central points of these describe great circles cutting them at right angles: these are easily seen to intersect in a certain point P. Let $Px=f$, $Py=g$, $Pz=h$; then also $Px_{,}=f$, $Py_{,}=g$, $Pz_{,}=h$: and $\angle xPx_{,}=\angle yPy_{,}=\angle zPz_{,}, =\theta$ suppose, θ being measured from xP towards yP, yP towards zP, or zP towards xP. The cosines of f, g, h, are of course connected by the equation

$$\cos^2 f+\cos^2 g+\cos^2 h=1.$$

Let α, β, γ; α', β', γ'; α'', β'', γ'', represent the cosines of $x_{,}x$, $y_{,}x$, $z_{,}x$; $x_{,}y$, $y_{,}y$, $z_{,}y$; $x_{,}z$, $y_{,}z$, $z_{,}z$: these quantities are to be determined as functions of f, g, h, θ.

Suppose for a moment,

$$\angle yPz=\mathrm{x},\qquad \angle zPx=\mathrm{y},\qquad \angle xPy=\mathrm{z};$$

then
$$\begin{aligned}
\alpha &= \cos^2 f \quad + \sin^2 f \cos\theta,\\
\alpha' &= \cos f \cos g + \sin f \sin g \cos(\mathrm{z}-\theta),\\
\alpha'' &= \cos f \cos h + \sin f \sin h \cos(\mathrm{y}+\theta),\\
\beta &= \cos g \cos f + \sin g \sin f \cos(\mathrm{z}+\theta),\\
\beta' &= \cos^2 g \quad + \sin^2 g \cos\theta,\\
\beta'' &= \cos g \cos h + \sin g \sin h \cos(\mathrm{x}-\theta),\\
\gamma &= \cos h \cos f + \sin h \sin f \cos(\mathrm{y}-\theta),\\
\gamma' &= \cos h \cos g + \sin h \sin g \cos(\mathrm{x}+\theta),\\
\gamma'' &= \cos^2 h \quad + \sin^2 h \cos\theta.
\end{aligned}$$

Also
$$\begin{aligned}
\sin g \sin h \cos \mathrm{x} &= -\cos g \cos h,\\
\sin h \sin f \cos \mathrm{y} &= -\cos h \cos f,\\
\sin f \sin g \cos \mathrm{z} &= -\cos f \cos g,
\end{aligned}$$

and
$$\begin{aligned}
\sin g \sin h \sin \mathrm{x} &= \cos f,\\
\sin h \sin f \sin \mathrm{y} &= \cos g,\\
\sin f \sin g \sin \mathrm{z} &= \cos h.
\end{aligned}$$

Substituting,
$$\begin{aligned}
\alpha &= \cos^2 f \quad + \sin^2 f \cos\theta,\\
\alpha' &= \cos f \cos g (1-\cos\theta) + \cos h \sin\theta,\\
\alpha'' &= \cos f \cos h (1-\cos\theta) - \cos g \sin\theta,\\
\beta &= \cos g \cos f (1-\cos\theta) - \cos h \sin\theta,\\
\beta' &= \cos^2 g \quad + \sin^2 g \cos\theta,\\
\beta'' &= \cos g \cos h (1-\cos\theta) + \cos f \sin\theta,\\
\gamma &= \cos h \cos f (1-\cos\theta) + \cos g \sin\theta,\\
\gamma' &= \cos h \cos g (1-\cos\theta) - \cos f \sin\theta,\\
\gamma'' &= \cos^2 h \quad + \sin^2 h \cos\theta.
\end{aligned}$$

Assume $\lambda = \tan\frac{1}{2}\theta \cos f$, $\mu = \tan\frac{1}{2}\theta\cos g$, $\nu = \tan\frac{1}{2}\theta \cos h$, and $\sec^2\frac{1}{2}\theta = 1+\lambda^2+\mu^2+\nu^2 = \kappa$;

then
$$\begin{aligned}
&\kappa\alpha = 1+\lambda^2-\mu^2-\nu^2, && \kappa\alpha' = 2(\lambda\mu+\nu), && \kappa\alpha'' = 2(\nu\lambda-\mu),\\
&\kappa\beta = 2(\lambda\mu-\nu), && \kappa\beta' = 1+\mu^2-\nu^2-\lambda^2, && \kappa\beta'' = 2(\mu\nu+\lambda),\\
&\kappa\gamma = 2(\nu\lambda+\mu), && \kappa\gamma' = 2(\mu\nu-\lambda), && \kappa\gamma'' = 1+\nu^2-\lambda^2-\mu^2;
\end{aligned}$$

which are the formulæ required, differing only from those in Liouville, by having λ, μ, ν, instead of $\frac{1}{2}m$, $\frac{1}{2}n$, $\frac{1}{2}p$; and $\alpha, \alpha', \alpha''$; β, β', β''; $\gamma, \gamma', \gamma''$, instead of a, b, c; a', b', c'; a'', b'', c''. It is to be remarked, that β', β'', β; $\gamma'', \gamma, \gamma'$, are deduced from $\alpha, \alpha', \alpha''$, by writing μ, ν, λ; ν, λ, μ, for λ, μ, ν.

Let $1+\alpha+\beta'+\gamma'' = \upsilon$; then $\kappa\upsilon = 4$, and we have

$$\begin{aligned}
&\lambda\upsilon = \beta''-\gamma', && \mu\upsilon = \gamma-\alpha'', && \nu\upsilon = \alpha'-\beta,\\
&\lambda^2\upsilon = 1+\alpha-\beta'-\gamma'', && \mu^2\upsilon = 1-\alpha+\beta'-\gamma'', && \nu^2\upsilon = 1-\alpha-\beta'-\gamma''.
\end{aligned}$$

Suppose that Ax, Ay, Az, are referred to axes $A\mathrm{X}$, $A\mathrm{Y}$, $A\mathrm{Z}$, by the quantities l, m, n, k, analogous to λ, μ, ν, κ, these latter axes being referred to $Ax_{,}$, $Ay_{,}$, $Az_{,}$, by the quantities $l_{,}$, $m_{,}$, $n_{,}$, $k_{,}$.

Let a, b, c; a', b', c'; a'', b'', c''; $a_{,}$, $b_{,}$, $c_{,}$; $a_{,}'$, $b_{,}'$, $c_{,}'$; $a_{,}''$, $b_{,}''$, $c_{,}''$, denote the quantities analogous to α, β, γ; α', β', γ'; α'', β'', γ''. Then we have, by spherical trigonometry, the formulæ

$$\alpha\ = a\,a_{,} + b\,a_{,}' + c\,a_{,}'', \qquad \beta\ = a\,b_{,} + b\,b_{,}' + c\,b_{,}'', \qquad \gamma\ = a\,c_{,} + b\,c_{,}' + c\,c_{,}'';$$

$$\alpha' = a'a_{,} + b'a_{,}' + c'a_{,}'', \qquad \beta' = a'b_{,} + b'b_{,}' + c'b_{,}'', \qquad \gamma' = a'c_{,} + b'c_{,}' + c'c_{,}'';$$

$$\alpha'' = a''a_{,} + b''a_{,}' + c''a_{,}'', \qquad \beta'' = a''b_{,} + b''b_{,}' + c''b_{,}'', \qquad \gamma'' = a''c_{,} + b''c_{,}' + c''c_{,}''.$$

Then expressing a, b, c; a', b', c'; a'', b'', c''; $a_{,}$, $b_{,}$, $c_{,}$; $a_{,}'$, $b_{,}'$, $c_{,}'$; $a_{,}''$, $b_{,}''$, $c_{,}''$, in terms of l, m, n; $l_{,}$, $m_{,}$, $n_{,}$, after some reductions we arrive at

$$kk_{,}\nu \qquad\qquad = 4\,(1 - ll_{,} - mm_{,} - nn_{,})^2, \ \ = 4\Pi^2 \text{ suppose},$$

$$kk_{,}\,(\beta'' - \gamma\ ) = 4\,(l\ + l_{,}\ + n_{,}m - nm_{,})\,\Pi,$$

$$kk_{,}\,(\gamma\ -\ \alpha'\ ) = 4\,(m + m_{,} + l_{,}m\ - lm_{,}\ )\,\Pi,$$

$$kk_{,}\,(\alpha'\ -\ \beta'') = 4\,(n\ + n_{,}\ + m_{,}n - mn_{,})\,\Pi\,;$$

and hence

$$\Pi\ \ = 1 - ll_{,} - mm_{,} - nn_{,}, \qquad \Pi\lambda = l + l_{,} + n_{,}m - nm_{,},$$

$$\Pi\mu = m + m_{,} + l_{,}m - lm_{,}, \qquad \Pi\nu = n + n_{,} + m_{,}n - mn_{,},$$

which are formulæ of considerable elegance for exhibiting the combined effect of successive displacements of the axes. The following analogous ones are readily obtained:

$$P\ \ = 1 + \lambda l + \mu m + \nu n, \qquad Pl_{,} = \lambda - l - \nu m + \mu n,$$

$$Pm_{,} = \mu - m - \lambda n + \nu l, \qquad Pn_{,} = \nu - n - \mu l + \lambda m:$$

and again,

$$P_{,}\ \ = 1 + \lambda l_{,} + \mu m_{,} + \nu n_{,}, \qquad P_{,}l = \lambda - l_{,} + \nu m_{,} - \mu n_{,},$$

$$P_{,}m = \mu - m_{,} + \lambda n_{,} - \nu l_{,}, \qquad P_{,}n = \nu - n_{,} + \mu l_{,} - \lambda m_{,}.$$

These formulæ will be found useful in the integration of the equations of rotation of a solid body.

Next it is required to express the quantities p, q, r, in terms of λ, μ, ν, where as usual

$$p = \gamma\frac{d\beta}{dt} + \gamma'\frac{d\beta'}{dt} + \gamma''\frac{d\beta''}{dt},$$

$$q = \alpha\frac{d\gamma}{dt} + \alpha'\frac{d\gamma'}{dt} + \alpha''\frac{d\gamma''}{dt},$$

$$r = \beta\frac{d\alpha}{dt} + \beta'\frac{d\alpha'}{dt} + \beta''\frac{d\alpha''}{dt}.$$

Differentiating the values of $\beta\kappa$, $\beta'\kappa$, $\beta''\kappa$, multiplying by γ, γ', γ'', and adding,

$$\kappa p = 2\lambda'\,(\gamma\mu - \gamma'\lambda + \gamma'') + 2\mu'\,(\gamma\lambda - \gamma'\mu + \gamma''\nu) + 2\nu'\,(-\gamma - \gamma'\nu + \gamma''\mu),$$

where λ', μ', ν', denote $\frac{d\lambda}{dt}, \frac{d\mu}{dt}, \frac{d\nu}{dt}$. Reducing, we have

$$\kappa p = 2(\lambda' + \nu\mu' - \nu'\mu):$$

from which it is easy to derive the system

$$\begin{aligned}\kappa p &= 2(\lambda' + \nu\mu' - \nu'\mu),\\ \kappa q &= 2(-\nu\lambda' + \mu' + \nu'\lambda),\\ \kappa r &= 2(\mu\lambda' - \lambda\mu' + \nu');\end{aligned}$$

or, determining λ', μ', ν', from these equations, the equivalent system

$$\begin{aligned}2\lambda' &= (1+\lambda^2)\,p + (\lambda\mu - \nu)\,q + (\nu\lambda + \mu)\,r,\\ 2\mu' &= (\lambda\mu + \nu)\,p + (1+\mu^2)\,q + (\mu\nu - \lambda)\,r,\\ 2\nu' &= (\nu\lambda - \mu)\,p + (\mu\nu + \lambda)\,q + (1+\nu^2)\,r.\end{aligned}$$

The following equation also is immediately obtained,

$$\kappa' = \kappa(\lambda p + \mu q + \nu r).$$

The subsequent part of the problem requires the knowledge of the differential coefficients of p, q, r, with respect to $\lambda, \mu, \nu; \lambda', \mu', \nu'$. It will be sufficient to write down the six

$$\kappa\frac{dp}{d\lambda'} = 2, \qquad \kappa\frac{dp}{d\lambda} + 2p\lambda = 0,$$

$$\kappa\frac{dq}{d\lambda'} = -2\nu, \qquad \kappa\frac{dq}{d\lambda} + 2q\lambda = 2\nu',$$

$$\kappa\frac{dr}{d\lambda'} = 2\mu, \qquad \kappa\frac{dr}{d\lambda} + 2r\lambda = -2\mu',$$

from which the others are immediately obtained.

Suppose now a solid body acted on by any forces, and revolving round a fixed point. The equations of motion are

$$\frac{d}{dt}\frac{dT}{d\lambda'} - \frac{dT}{d\lambda} = \frac{dV}{d\lambda},$$

$$\frac{d}{dt}\frac{dT}{d\mu'} - \frac{dT}{d\mu} = \frac{dV}{d\mu},$$

$$\frac{d}{dt}\frac{dT}{d\nu'} - \frac{dT}{d\nu} = \frac{dV}{d\nu};$$

where $T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2); \quad V = \Sigma\,[\int(Xdx + Ydy + Zdz)]\,dm;$

or if $Xdx + Ydy + Zdz$ is not an exact differential, $\frac{dV}{d\lambda}, \frac{dV}{d\mu}, \frac{dV}{d\nu}$, are independent symbols standing for

$$\Sigma\left(X\frac{dx}{d\lambda} + Y\frac{dy}{d\lambda} + Z\frac{dz}{d\lambda}\right)dm, \ldots\ldots$$

see *Mécanique Analytique*, Avertissement, t. I. p. V. [Ed. 3, p. VII.]: only in this latter case V stands for the disturbing function, the principal forces vanishing.

Now, considering the first of the above equations

$$\frac{dT}{d\lambda'} = Ap\frac{dp}{d\lambda} + Bq\frac{dq}{d\lambda} + Cr\frac{dr}{d\lambda}, \quad = \frac{2}{\kappa}(Ap - \nu Bq + \mu Cr);$$

whence, writing p', q', r', κ', for $\frac{dp}{dt}, \frac{dq}{dt}, \frac{dr}{dt}, \frac{d\kappa}{dt}$,

$$\frac{d}{dt}\frac{dT}{d\lambda'} = \frac{2}{\kappa}(Ap' - \nu Bq' + \mu Cr') - \frac{2}{\kappa}Bq\nu' + \frac{2}{\kappa}Cr\mu' - \frac{2\kappa'}{\kappa^2}(Ap - \nu Bq + \mu Cr).$$

Also $$\frac{dT}{d\lambda} = Ap\frac{dp}{d\lambda} + Bq\frac{dq}{d\lambda} + Cr\frac{dr}{d\lambda}, \quad = -\frac{2\lambda}{\kappa}(Ap^2 + Bq^2 + Cr^2) + \frac{2}{\kappa}Bq\nu' - \frac{2}{\kappa}Cr\mu';$$

and hence $\frac{1}{2}\left(\frac{d}{dt}\frac{dT}{d\lambda'} - \frac{dT}{d\lambda}\right)$

$$= \frac{1}{\kappa}(Ap' - \nu Bq' + \mu Cr') - \frac{2}{\kappa}Bq\nu' + \frac{2}{\kappa}Cr\mu' + \frac{\lambda}{\kappa}(Ap^2 + Bq^2 + Cr^2) - \frac{\kappa'}{\kappa^2}(Ap - \nu Bq + \mu Cr).$$

Substituting for $\lambda', \mu', \nu', \kappa'$, after all reductions,

$$\frac{1}{2}\left(\frac{d}{dt}\frac{dT}{d\lambda'} - \frac{dT}{d\lambda}\right) = \frac{1}{\kappa}\left[\{Ap' + (C-B)qr\} - \nu\{Bq' + (A-C)rp\} + \mu\{Cr + (B-A)pq\}\right];$$

and, forming the analogous quantities in μ, ν, and substituting in the equations of motion, these become

$$\{Ap' + (C-B)qr\} - \nu\{Bq' + (A-C)rp\} + \mu\{Cr' + (B-A)pq\} = \tfrac{1}{2}\kappa\frac{dV}{d\mu},$$

$$\nu\{Ap' + (C-B)qr\} + \{Bq' + (A-C)rp\} - \lambda\{Cr' + (B-A)pq\} = \tfrac{1}{2}\kappa\frac{dV}{d\mu},$$

$$\mu\{Ap' + (C-B)qr\} + \lambda\{Bq' + (A-C)rp\} + \{Cr' + (B-A)pq\} = \tfrac{1}{2}\kappa\frac{dV}{d\nu};$$

or eliminating, and replacing p', q', r', by $\frac{dp}{dt}, \frac{dq}{dt}, \frac{dr}{dt}$, we obtain

$$A\frac{dp}{dt} + (C-B)qr = \tfrac{1}{2}\left\{(1+\lambda^2)\frac{dV}{d\lambda} + (\lambda\mu + \nu)\frac{dV}{d\mu} + (\nu\lambda - \mu)\frac{dV}{d\nu}\right\},$$

$$B\frac{dq}{dt} + (A-C)rp = \tfrac{1}{2}\left\{(\lambda\mu - \nu)\frac{dV}{d\lambda} + (1+\mu^2)\frac{dV}{d\mu} + (\mu\nu + \lambda)\frac{dV}{d\nu}\right\},$$

$$C\frac{dr}{dt} + (B-A)pq = \tfrac{1}{2}\left\{(\nu\lambda + \mu)\frac{dV}{d\lambda} + (\mu\nu - \lambda)\frac{dV}{d\mu} + (1+\nu^2)\frac{dV}{d\nu}\right\};$$

to which are to be joined

$$\kappa p = 2\left(\frac{d\lambda}{dt} + \nu\frac{d\mu}{dt} - \mu\frac{d\nu}{dt}\right),$$

$$\kappa q = 2\left(-\nu\frac{d\lambda}{dt} + \frac{d\mu}{dt} + \lambda\frac{d\nu}{dt}\right),$$

$$\kappa r = 2\left(\mu\frac{d\lambda}{dt} - \lambda\frac{d\mu}{dt} + \frac{d\nu}{dt}\right);$$

where it will be recollected

$$\kappa = 1 + \lambda^2 + \mu^2 + \nu^2;$$

and on the integration of these six equations depends the complete determination of the motion.

If we neglect the terms depending on V, the first three equations may be integrated in the form

$$p^2 = p_1{}^2 - \frac{C-B}{A}\phi, \qquad q^2 = q_1{}^2 - \frac{A-C}{B}\phi, \qquad r_2 + r_1{}^2 - \frac{B-A}{C}\phi,$$

$$2t = \int \frac{d\phi}{\sqrt{\left\{\left(p_1{}^2 - \frac{C-B}{A}\phi\right)\left(q_1{}^2 - \frac{A-C}{B}\phi\right)\left(r_1{}^2 - \frac{B-A}{C}\phi\right)\right\}}};$$

and considering p, q, r as functions of ϕ, given by these equations, the three latter ones take the form

$$\frac{\kappa}{4qr} = \frac{d\lambda}{d\phi} + \nu\frac{d\mu}{d\phi} - \mu\frac{d\nu}{d\phi},$$

$$\frac{\kappa}{4rp} = -\nu\frac{d\lambda}{d\phi} + \frac{d\mu}{d\phi} + \lambda\frac{d\nu}{d\phi},$$

$$\frac{\kappa}{4pq} = \mu\frac{d\lambda}{d\phi} - \lambda\frac{d\mu}{d\phi} + \frac{d\nu}{d\phi};$$

of which, as is well known, the equations following, equivalent to two independent equations, are integrals,

$$\kappa g = Ap\,(1 + \lambda^2 - \mu^2 - \nu^2) + 2Bq\,(\lambda\mu - \nu) + 2Cr\,(\nu\lambda + \mu),$$

$$\kappa g' = 2Ap\,(\lambda\mu + \nu) + Bq\,(1 + \mu^2 - \lambda^2 - \nu^2) + 2Cr\,(\mu\nu - \lambda),$$

$$\kappa g'' = 2Ap\,(\nu\lambda - \mu) + 2Bq\,(\mu\nu + \lambda) + Cr\,(1 + \nu^2 - \lambda^2 - \mu^2);$$

where g, g', g'', are arbitrary constants satisfying

$$g^2 + g'^2 + g''^2 = A^2p_1{}^2 + B^2q_1{}^2 + C^2r_1{}^2.$$

To obtain another integral, it is apparently necessary, as in the ordinary theory, to revert to the consideration of the invariable plane. Suppose $g' = 0$, $g'' = 0$, then

$$g'' = \sqrt{(A^2p_1{}^2 + B^2q_1{}^2 + C^2r_1{}^2)}, \ = k \text{ suppose}.$$

We easily obtain, where λ_0, μ_0, ν_0, κ_0 are written for λ, μ, ν, κ, to denote this particular supposition,

$$\kappa_0 Ap = 2(\nu_0\lambda_0 - \mu_0)k,$$
$$\kappa_0 Bq = 2(\mu_0\nu_0 + \lambda_0)k,$$
$$\kappa_0 Cr = (1 + \nu_0^2 - \lambda_0^2 - \mu_0^2)k;$$

whence, and from $\kappa_0 = 1 + \lambda_0^2 + \mu_0^2 + \nu_0^2$, $\kappa_0 Cr = (2 + 2\nu_0^2 - \kappa_0)k$, we obtain

$$\kappa_0 = \frac{(2+2\nu_0^2)k}{k+Cr}, \qquad \nu_0\lambda_0 - \mu_0 = \frac{(1+\nu_0^2)Ap}{k+Cr}, \qquad \mu_0\nu_0 + \lambda_0 = \frac{(1+\nu_0^2)Bq}{k+Cr}.$$

Hence, writing $h = Ap_1^2 + Bq_1^2 + Cr_1^2$, the equation

$$\frac{d\nu_0}{d\phi} = \frac{1}{4pqr}\{(\nu_0\lambda_0 - \mu_0)p + (\mu_0\nu_0 + \lambda_0)q + (1+\nu^2)r\}$$

reduces itself to $$\frac{4}{1+\nu_0^2}\frac{d\nu_0}{d\phi} = \frac{h+kr}{(k+Cr)pqr},$$

or, integrating, $$4\tan^{-1}\nu_0 = \int\frac{(h+kr)\,d\phi}{(k+Cr)pqr}.$$

The integral takes rather a simpler form if p, q, ϕ be considered functions of r, and becomes

$$2\tan^{-1}\nu_0 = \int\frac{h+kr}{k+Cr}\,\frac{C\sqrt{(AB)}\,dr}{\sqrt{[\{k^2 - Bh + (B-C)Cr^2\}\{Ah - k^2 + (C-A)Cr^2\}]}};$$

and then, ν_0 being determined, λ_0, μ_0 are given by the equations

$$\lambda_0 = \frac{\nu_0 Ap + Bq}{k+Cr}, \qquad \mu_0 = \frac{\nu_0 Bq - Ap}{k+Cr}.$$

Hence l, m, n, denoting arbitrary constants, the general values of λ, μ, ν, are given by the equations

$$P_0 = 1 - l\lambda_0 - m\mu_0 - n\nu_0,$$
$$P_0\lambda = l + \lambda_0 + m\nu_0 - n\mu_0,$$
$$P_0\mu = m + \mu_0 + n\lambda_0 - l\nu_0,$$
$$P_0\nu = n + \nu_0 + l\mu_0 - m\lambda_0.$$

In a following paper I propose to develope the formulæ for the variations of the arbitrary constants p_1, q_1, r_1, l, m, n, when the terms involving V are taken into account.

Note. It may be as well to verify independently the analytical conclusion immediately deducible from the preceding formulæ, viz. if λ, μ, ν, be given by the differential equations,

$$\kappa p = \frac{d\lambda}{dt} + \nu \frac{d\mu}{dt} - \mu \frac{d\nu}{dt},$$

$$\kappa q = -\nu \frac{d\lambda}{dt} + \frac{d\mu}{dt} + \lambda \frac{d\nu}{dt},$$

$$\kappa r = \mu \frac{d\lambda}{dt} - \lambda \frac{d\mu}{dt} + \frac{d\nu}{dt},$$

where $\kappa = 1 + \lambda^2 + \mu^2 + \nu^2$, and p, q, r, are any functions of t. Then if λ_0, μ_0, ν_0, be particular values of λ, μ, ν, and l, m, n, arbitrary constants, the general integrals are given by the system

$$P_0 = 1 - l\lambda_0 - m\mu_0 - n\nu_0,$$

$$P_0\lambda = l + \lambda_0 + m\nu_0 - n\mu_0,$$

$$P_0\mu = m + \mu_0 + n\lambda_0 - l\nu_0,$$

$$P_0\nu = n + \nu_0 + l\mu_0 - m\lambda_0.$$

Assuming these equations, we deduce the equivalent system,

$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, l = \lambda - \lambda_0 + \nu_0\mu - \nu\mu_0,$$

$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, m = \mu - \mu_0 + \lambda_0\nu - \lambda\nu_0,$$

$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, n = \nu - \nu_0 + \mu_0\lambda - \mu\lambda_0.$$

Differentiate the first of these and eliminate l, the result takes the form $0 =$

$$-(\mu_0^2 + \nu_0^2)(\lambda' + \nu\mu' - \nu'\mu) - (\nu_0 - \lambda_0\mu_0)(-\nu\lambda' + \mu' + \lambda\nu') + (\mu_0 + \lambda_0\nu_0)(\mu\lambda' - \lambda\mu' + \nu') + \kappa_0\lambda',$$

$$+(\mu^2 + \nu^2)(\lambda_0' + \nu_0\mu_0' - \nu_0'\mu_0) + (\nu - \lambda\mu)(-\nu_0\lambda_0' + \mu_0' + \lambda_0\nu_0') - (\mu + \lambda\nu)(\mu_0\lambda_0' - \lambda_0\mu_0' + \nu_0') - \kappa\lambda_0',$$

where λ', &c. denote $\frac{d\lambda}{dt}$, &c. and $\kappa_0 = 1 + \lambda_0^2 + \mu_0^2 + \nu_0^2$.

Reducing by the differential equations in λ, μ, ν; λ_0, μ_0, ν_0, this becomes

$$\kappa_0 \{\lambda' + \tfrac{1}{2}p(\mu^2 + \nu^2) + \tfrac{1}{2}q(\nu - \lambda\mu) - \tfrac{1}{2}r(\mu + \lambda\nu)\}$$

$$-\kappa\{\lambda_0' + \tfrac{1}{2}p(\mu_0^2 + \nu_0^2) + \tfrac{1}{2}q(\nu_0 - \lambda_0\mu_0) - \tfrac{1}{2}r(\mu_0 + \lambda\nu_0)\} = 0;$$

or substituting for λ', λ_0', we have the identical equation

$$\tfrac{1}{2}p(\kappa_0\kappa - \kappa\kappa_0) = 0:$$

and similarly may the remaining equations be verified.