5.

ON THE INTERSECTION OF CURVES.

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THE following theorem is quoted in a note of Chasles' Aperçu Historique &c., *Memoires de Bruxelles*, tom. XI. p. 149, where M. Chasles employs it in the demonstration of Pascal's theorem: "If a curve of the third order pass through eight of the points of intersection of two curves of the third order, it passes through the ninth point of intersection." The application in question is so elegant, that it deserves to be generally known. Consider a hexagon inscribed in a conic section. The aggregate of three alternate sides may be looked upon as forming a curve of the third order, and that of the remaining sides, a second curve of the same order. These two intersect in nine points, viz. the six angular points of the hexagon, and the three points which are the intersections of pairs of opposite sides. Suppose a curve of the third order passing through eight of these points, viz. the aggregate of the conic section passing through the angular points of the hexagon, and of the line joining two of the three intersections of pairs of opposite sides. This passes through the ninth point, by the theorem of Chasles, i.e. the three intersections of pairs of opposite sides lie in the same straight line, (since obviously the third intersection does *not* lie in the conic section); which is Pascal's theorem.

The demonstration of the above property of curves of the third order is one of extreme simplicity. Let $U=0$, $V=0$, be the equations of two curves of the third order, the curve of the same order which passes through eight of their points of intersection (which may be considered as eight perfectly arbitrary points), and a ninth arbitrary point, will be perfectly determinate. Let U_0, V_0, be the values of U, V, when the coordinates of this last point are written in place of x, y. Then $UV_0 - U_0V = 0$, satisfies the above conditions, or it is the equation to the curve required; but it is an equation which is satisfied by all the nine points of intersection of the two curves, i.e. any curve that passes through eight of these points of intersection, passes also through the ninth.

Consider generally two curves, $U_m = 0$, $V_n = 0$, of the orders m and n respectively, and a curve of the r^{th} order (r not less than m or n) passing through the mn points of intersection. The equation to such a curve will be of the form

$$U = u_{r-m} U_m + v_{r-n} V_n = 0,$$

u_{r-m}, v_{r-n}, denoting two polynomes of the orders $r - m$, $r - n$, with all their coefficients complete. It would at first sight appear that the curve $U = 0$ might be made to pass through as many as $\{1 + 2 \ldots + (r - m + 1)\} + \{1 + 2 \ldots (r - n + 1)\} - 1$, arbitrary points, i.e.

$$\tfrac{1}{2}(r - m + 1)(r - m + 2) + \tfrac{1}{2}(r - n + 1)(r - n + 2) - 1;$$

or, what is the same thing,

$$\tfrac{1}{2}r(r + 3) - mn + \tfrac{1}{2}(r - m - n + 1)(r - m - n + 2)$$

arbitrary points, such being apparently the number of disposable constants. This is in fact the case as long as r is not greater than $m + n - 1$; but when r exceeds this, there arise, between the polynomes which multiply the disposable coefficients, certain linear relations which cause them to group themselves into a smaller number of disposable quantities. Thus, if r be not less than $m + n$, forming different polynomes of the form $x^\alpha y^\beta V_n$ [$\alpha + \beta =$ or $< m$], and multiplying by the coefficients of $x^\alpha y^\beta$ in U_m and adding, we obtain a sum $U_m V_n$, which might have been obtained by taking the different polynomes of the form $x^\gamma y^\delta U_m$ [$\gamma + \delta =$ or $< n$], multiplying by the coefficients of $x^\gamma y^\delta$ in V_n, and adding: or we have a linear relation between the different polynomes of the forms $x^\alpha y^\beta V_n$, and $x^\gamma y^\delta U_m$. In the case where r is not less than $m + n + 1$, there are two more such relations, viz. those obtained in the same way from the different polynomes $x^\alpha y^\beta . x V_n$, $x^\gamma y^\delta . x U_m$, and $x^\alpha y^\beta . y V_n$, $x^\gamma y^\delta . y U_m$, &c.; and in general, whatever be the excess of r above $m + n - 1$, the number of these linear relations is

$$1 + 2 \ldots (r - m - n + 1) = \tfrac{1}{2}(r - m - n + 1)(r - m - n + 2).$$

Hence, if r be not less than $m + n$, the number of points through which a curve of the r^{th} order may be made to pass, in addition to the mn points which are the intersections of $U_m = 0$, $V_n = 0$, is simply $\tfrac{1}{2}r(r + 3) - mn$. In the case of $r = m + n - 1$, or $r = m + n - 2$, the two formulæ coincide. Hence we may enunciate the theorem

"A curve of the r^{th} order, passing through the mn points of intersection of two curves of the m^{th} and n^{th} orders respectively, may be made to pass through $\tfrac{1}{2}r(r + 3) - mn + \tfrac{1}{2}(m + n - r - 1)(m + n - r - 2)$ arbitrary points, if r be not greater than $m + n - 3$: if r be greater than this value, it may be made to pass through $\tfrac{1}{2}r(r + 3) - mn$ points only."

Suppose r not greater than $m + n - 3$, and a curve of the r^{th} order made to pass through

$$\tfrac{1}{2}r(r + 3) - mn + \tfrac{1}{2}(m + n - r - 1)(m + n - r - 2)$$

arbitrary points, and

$$mn - \tfrac{1}{2}(m + n - r - 1)(m + n - r - 2)$$

of the mn points of intersection above. Such a curve passes through $\frac{1}{2}r(r+3)$ given points, and though the $mn - \frac{1}{2}(m+n-r-1)(m+n-r-2)$ latter points are not perfectly arbitrary, there appears to be no reason why the relation between the positions of these points should be such as to prevent the curve from being *completely determined* by these conditions. But if it be so, then the curve must pass through the remaining $\frac{1}{2}(m+n-r-1)(m+n-r-2)$ points of intersection, or we have the theorem

"If a curve of the r^{th} order (r not less than m or n, not greater than $m+n-3$) pass through

$$mn - \tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of the points of intersection of two curves of the m^{th} and n^{th} orders respectively, it passes through the remaining

$$\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

points of intersection."