4.

ON CERTAIN EXPANSIONS, IN SERIES OF MULTIPLE SINES AND COSINES.

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In the following paper we shall suppose ϵ the base of the hyperbolic system of logarithms; e a constant, such that its modulus, and also the modulus of $\frac{1}{e}\{1-\sqrt{1-e^2}\}$, are each of them less than unity; $\chi\{\epsilon^{u\sqrt{(-1)}}\}$ a function of u, which, as u increases from 0 to π, passes continuously from the former of these values to the latter, without becoming a maximum in the interval, $f\{\epsilon^{u\sqrt{(-1)}}\}$ any function of u which remains finite and continuous for values of u included between the above limits. Hence, writing

$$\chi\{\epsilon^{u\sqrt{(-1)}}\}=m \quad\text{.......................................(1)},$$

and considering the quantity

$$\frac{\sqrt{1-e^2}\,f\{\epsilon^{u\sqrt{(-1)}}\}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)} \quad\text{...........................(2)},$$

as a function of m, for values of m or u included between the limits 0 and π, we have

$$\frac{\sqrt{1-e^2}\,f\{\epsilon^{u\sqrt{(-1)}}\}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\,\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)}=\frac{2}{\pi}\Sigma_{-\infty}^{\infty}\cos rm\int_0^\pi\frac{\sqrt{1-e^2}\,f\{\epsilon^{u\sqrt{(-1)}}\}\cos rm\,dm}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\,\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)}\text{...(3)},$$

(Poisson, *Mec.* tom. I. p. 650); which may also be written

$$\frac{\sqrt{1-e^2}\,f\{\epsilon^{u\sqrt{(-1)}}\}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\,\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)}=\frac{2}{\pi}\Sigma_{-\infty}^{\infty}\cos rm\int_0^\pi\frac{\sqrt{1-e^2}\,f\{\epsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\epsilon^{u\sqrt{(-1)}}\}\,du}{1-e\cos u}\text{...(4)};$$

and if the first side of the equation be generally expansible in a series of multiple cosines of m, instead of being so in particular cases only, its expanded value will always be the one given by the second side of the preceding equation.

Now, between the limits 0 and π, the function

$$f\{\epsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\epsilon^{u\sqrt{(-1)}}\}$$

will always be expansible in a series of multiple cosines of u; and if by any algebraical process the function $f\rho\cos r\chi\rho$ can be expanded in the form

$$f\rho\cos r\chi\rho = \Sigma_{-\infty}^{\infty}\alpha_s\rho^s, \quad (\alpha_s = \alpha_{-s}) \quad \ldots\ldots(5);$$

we have, in a convergent series,

$$f\{\epsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\epsilon^{u\sqrt{(-1)}}\} = \alpha_0 + 2\Sigma_1^{\infty}\alpha_s\cos su \quad \ldots\ldots(6).$$

Again, putting

$$\frac{1}{e}\{1-\sqrt{1-e^2}\} = \lambda \quad \ldots\ldots(7),$$

we have

$$\frac{\sqrt{1-e^2}}{1-e\cos u} = 1 + 2\Sigma_1^{\infty}\lambda^p\cos pu \quad \ldots\ldots(8).$$

Multiplying these two series, and effecting the integration, we obtain

$$\frac{1}{\pi}\int_0^{\pi}\frac{\sqrt{1-e^2}f\{\epsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\epsilon^{u\sqrt{(-1)}}\}du}{1-e\cos u} = 2\{\tfrac{1}{2}\alpha_0 + \Sigma_1^{\infty}(\alpha_s\lambda^s)\} \quad \ldots\ldots(9),$$

and the second side of this equation being obviously derived from the expansion of $f\lambda\cos r\chi\lambda$ by rejecting negative powers of λ and dividing by 2, the term independent of λ may conveniently be represented by the notation

$$\underbrace{2f\lambda\cos r\chi\lambda} \quad \ldots\ldots(10);$$

where in general, if $\Gamma\lambda$ can be expanded in the form

$$\Gamma\lambda = \Sigma_{-\infty}^{\infty}(A_s\lambda^s), \quad [A_{-s} = A_s] \quad \ldots\ldots(11),$$

we have

$$\underbrace{\Gamma\lambda} = \tfrac{1}{2}A_0 + \Sigma_1^{\infty}A_s\lambda^s \quad \ldots\ldots(12).$$

(By what has preceded, the expansion of $\Gamma\lambda$ in the above form is always possible in a certain sense; however, in the remainder of the present paper, $\Gamma\lambda$ will always be of a form to satisfy the equation $\Gamma\left(\frac{1}{\lambda}\right) = \Gamma\lambda$, except in cases which will afterwards be considered, where the condition $A_{-s} = A_s$ is unnecessary.)

Hence, observing the equations (4), (9), (10),

$$\frac{\sqrt{1-e^2}f\{\epsilon^{u\sqrt{(-1)}}\}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)} = \Sigma_{-\infty}^{\infty}\cos rm\underbrace{2\cos r\chi\lambda f\lambda} \quad \ldots\ldots(13);$$

from which, assuming a system of equations analogous to (1), and representing by $\Pi(\Phi)$ the product $\Phi_1\Phi_2\ldots$, it is easy to deduce

$$\Pi\left\{\frac{\sqrt{1-e^2}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)}\right\}f\{\epsilon^{u_1\sqrt{(-1)}},\ \epsilon^{u_2\sqrt{(-1)}}\ldots\}$$

$$= \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\Pi\cos rm\underbrace{\Pi(2\cos r\chi\lambda)f(\lambda_1,\ \lambda_2\ldots)} \quad \ldots\ldots(14),$$

where $\Gamma(\lambda_1, \lambda_2 \ldots)$ being expansible in the form

$$\Gamma(\lambda_1, \lambda_2 \ldots) = \Sigma_{-\infty}^{\infty} \Sigma_{-\infty}^{\infty} \ldots A_{s_1, s_2 \ldots} \lambda_1^{s_1} \lambda_2^{s_2} \ldots [A_{s_1, s_2 \ldots} = A_{-s_1, -s_2 \ldots}] \ldots\ldots (15),$$

$$\Gamma(\lambda_1, \lambda_2 \ldots) = \Sigma_0^{\infty} \Sigma_0^{\infty} \ldots \frac{1}{2^N} A_{s_1, s_2 \ldots} \lambda_1^{s_1} \lambda_2^{s_2} \ldots, \ldots\ldots (16),$$

N being the number of exponents which vanish.

The equations (13) and (14) may also be written in the forms

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \Sigma_{-\infty}^{\infty} \cos rm \, \overbrace{2 \cos r\chi\lambda \frac{\sqrt{-1}\, \chi'\lambda \{1 - \tfrac{1}{2}e(\lambda + \lambda^{-1})\}}{\sqrt{1-e^2}} f\lambda} \ldots\ldots (17),$$

$f\{\epsilon^{u_1\sqrt{(-1)}}, \epsilon^{u_2\sqrt{(-1)}} \ldots\}$

$$= \Sigma_{-\infty}^{\infty} \Sigma_{-\infty}^{\infty} \ldots \Pi(\cos rm) \, \overbrace{\Pi\left\{2 \cos r\chi\lambda \frac{\sqrt{-1}\, \chi'\lambda \{1 - \tfrac{1}{2}e(\lambda + \lambda^{-1})\}}{\sqrt{1-e^2}}\right\} f(\lambda_1, \lambda_2 \ldots)} \ldots (18).$$

As examples of these formulæ, we may assume

$$\chi\{\epsilon^{u\sqrt{(-1)}}\} = m = u - e \sin u \ldots\ldots\ldots\ldots (19).$$

Hence, putting

$$\lambda^r \epsilon^{-\frac{re}{2}(\lambda - \lambda^{-1})} + \lambda^{-r} \epsilon^{\frac{re}{2}(\lambda - \lambda^{-1})} = \Lambda_r \ldots\ldots\ldots\ldots (20),$$

and observing the equation

$$\sqrt{-1}\, \epsilon^{u\sqrt{(-1)}} \chi'\{\epsilon^{u\sqrt{(-1)}}\} = 1 - e \cos u \ldots\ldots\ldots\ldots (21),$$

the equation (17) becomes

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \Sigma_{-\infty}^{\infty} \cos rm \, \overbrace{\Lambda_r \frac{\{1 - \tfrac{1}{2}e(\lambda + \lambda^{-1})\}^2}{\sqrt{(1-e^2)}} f\lambda} \ldots\ldots\ldots\ldots (22).$$

Thus, if

$$\theta - \varpi = \cos^{-1} \frac{\cos u - e}{1 - e \cos u} \ldots\ldots\ldots\ldots (23),$$

assuming

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \frac{\cos u - e}{1 - e \cos u} \ldots\ldots\ldots\ldots (24),$$

$$\cos(\theta - \varpi) = \Sigma_{-\infty}^{\infty} \frac{1}{\sqrt{1-e^2}} \cos rm \, \overbrace{\{1 - \tfrac{1}{2}e(\lambda + \lambda^{-1})\}\{\tfrac{1}{2}(\lambda + \lambda^{-1}) - e\} \Lambda_r} \ldots (25),$$

the term corresponding to $r = 0$ being

$$\frac{1}{2\sqrt{1-e^2}} \{2\lambda - 2e - e(\lambda^2 + 1) + 2e^2\lambda\}, \; = -e \ldots\ldots\ldots\ldots (26).$$

Again, assuming

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \frac{d\theta}{dm} = \frac{\sqrt{1-e^2}}{(1 - e \cos u)^2} \ldots\ldots\ldots\ldots (27),$$

and integrating the resulting equation with respect to m,

$$\theta - \varpi = \Sigma_{-\infty}^{\infty} \frac{\sin rm}{r} \overbrace{\underbrace{\Lambda_r}} = m + 2\Sigma_1^{\infty} \frac{\sin rm}{r} \overbrace{\underbrace{\Lambda_r}} \qquad (28),$$

a formula given in the fifth No. of the *Mathematical Journal*, and which suggested the present paper.

As another example, let

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \cos(\theta - \varpi) \frac{d\theta}{dm} = \frac{\sqrt{1-e^2}\,(\cos u - e)}{(1 - e\cos u)^3} \qquad (29).$$

Then integrating with respect to m, there is a term

$$2m \overbrace{\underbrace{\frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{1}{2}e(\lambda + \lambda^{-1})}}} \qquad (30),$$

which it is evident, *à priori*, must vanish. Equating it to zero, and reducing, we obtain

$$\frac{e}{1-e^2} = \overbrace{\underbrace{\frac{\lambda + \lambda^{-1}}{1 - \frac{1}{2}e(\lambda + \lambda^{-1})}}} \qquad (31),$$

that is

$$\frac{e}{1-e^2} = \lambda + \frac{e}{2}(\lambda^2 + 1) + \frac{e^2}{4}(\lambda^3 + 3\lambda) + \frac{e^3}{8}(\lambda^4 + 4\lambda^2 + 3) + \ldots \qquad (32),$$

a singular formula, which may be verified by substituting for λ its value: we then obtain

$$\sin(\theta - \varpi) = 2\Sigma_1^{\infty} \frac{\sin rm}{r} \overbrace{\underbrace{\Lambda_r \frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{1}{2}(\lambda + \lambda^{-1})}}} \qquad (33).$$

The expansions of $\sin k(\theta - \varpi)$, $\cos k(\theta - \varpi)$, are in like manner given by the formulæ

$$\cos k(\theta - \varpi) = \Sigma_{-\infty}^{\infty} \overbrace{\underbrace{\Lambda_r L' \cos kL}} \cos rm \qquad (34),$$

$$\sin k(\theta - \varpi) = \Sigma_{-\infty}^{\infty} \overbrace{\underbrace{\Lambda_r \frac{1}{kr} \cos kL}} \frac{\sin rm}{r} \qquad (35),$$

where, to abbreviate, we have written

$$\cos^{-1}\left\{\frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{1}{2}e(\lambda + \lambda^{-1})}\right\} = L \qquad (36),$$

$$\frac{\{1 - \frac{1}{2}e(\lambda + \lambda^{-1})\}^2}{\sqrt{1-e^2}} = L' \qquad (37).$$

Forming the analogous expressions for

$$\cos k(\theta' - \varpi'), \qquad \sin k(\theta' - \varpi'),$$

substituting in

$$\cos k(\theta-\theta')=\cos k(\varpi-\varpi')\{\cos k(\theta-\varpi)\cos k(\theta'-\varpi')+\sin k(\theta-\varpi)\sin k(\theta'-\varpi')\}$$
$$-\sin k(\varpi-\varpi')\{\sin k(\theta-\varpi)\cos k(\theta'-\varpi')-\sin k(\theta'-\varpi')\cos k(\theta-\varpi)\},$$

and reducing the whole to multiple cosines, the final result takes the very simple form

$$\cos k(\theta-\theta')=\Sigma_{-\infty}^{\infty}\cos\{r'm'-rm+k(\varpi-\varpi')\}\,\underbrace{\overbrace{\Lambda_r\Lambda'_{r'}\cos kL\cos kL'\left(L-\frac{1}{kr}\right)\left(L'-\frac{1}{kr'}\right)}}\;\ldots(38).$$

Again, formulæ analogous to (14), (18), may be deduced from the equation

$$\Gamma(m_1,\ m_2\ldots)$$
$$=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\left\{\begin{aligned}&\cos(r_1m_1+r_2m_2\ldots)\int_0^{2\pi}\frac{dm_1}{2\pi}\int_0^{2\pi}\frac{dm_2}{2\pi}\ldots\cos(r_1m_1+r_2m_2\ldots)\,\Gamma(m_1,\ m_2\ldots)\\&+\sin(r_1m_1+r_2m_2\ldots)\int_0^{2\pi}\frac{dm_1}{2\pi}\int_0^{2\pi}\frac{dm_2}{2\pi}\ldots\sin(r_1m_1+r_2m_2\ldots)\,\Gamma(m_1,\ m_2\ldots)\end{aligned}\right.\quad(39),$$

which holds from $m_1=0$ to $m_1=2\pi$, &c., but in many cases universally. In this case, writing for $\Gamma(m_1,\ m_2\ldots)$ the function

$$\Pi\left\{\frac{1}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\,\chi'\{\epsilon^{u\sqrt{(-1)}}\}}\,\frac{\sqrt{1-e^2}-e\sin u\sqrt{-1}}{1-e\cos u}\right\}f\{\epsilon^{u_1\sqrt{(-1)}},\ \epsilon^{u_2\sqrt{(-1)}}\ldots\}\ldots(40);$$

and observing

$$\frac{\sqrt{1-e^2}-e\sin u\sqrt{-1}}{1-e\cos u}=\frac{1+\lambda\epsilon^{-u\sqrt{(-1)}}}{1-\lambda\epsilon^{-u\sqrt{(-1)}}}=1+2\Sigma_1^{\infty}\{\cos su-\sqrt{-1}\sin su\}\lambda^s\ldots\ldots(41),$$

an exactly similar analysis, (except that in the expansion

$$\Gamma(\lambda_1,\ \lambda_2\ldots)=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots A_{s_1,\,s_2}\ldots\lambda_1^{s_1}\lambda_2^{s_2}\ldots,$$

the supposition is not made that $A_{s_1,\,s_2}\ldots=A_{-s_1,\,-s_2}\ldots$), leads to the result

$$f\{\epsilon^{u_1\sqrt{(-1)}},\ \epsilon^{u_2\sqrt{(-1)}}\ldots\}\,\Pi\left\{\frac{\sqrt{1-e^2}-e\sin u\sqrt{-1}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\,\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)}\right\}$$
$$=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\left\{\begin{aligned}&\cos(r_1m_1+r_2m_2\ldots)\,\underbrace{\overbrace{2^{\mathfrak{n}}\cos(r_1\chi_1\lambda_1+r_2\chi_2\lambda_2\ldots)f(\lambda_1,\ \lambda_2\ldots)}}\\&+\sin(r_1m_1+r_2m_2\ldots)\,\underbrace{\overbrace{2^{\mathfrak{n}}\sin(r_1\chi_1\lambda_1+r_2\chi_2\lambda_2\ldots)f(\lambda_1,\ \lambda_2\ldots)}}\end{aligned}\right.\ldots(42),$$

($\mathfrak{n}$) being the number of variables $u_1,\ u_2\ldots.$ Hence also $f\{\epsilon^{u_1\sqrt{(-1)}},\ \epsilon^{u_2\sqrt{(-1)}}\ldots\}$

$$=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\left\{\begin{aligned}&\cos(r_1m_1+\ldots)\underbrace{\overbrace{\cos(r_1\chi_1\lambda_1+\ldots)\,\Pi\left\{\frac{2\sqrt{-1}\,\chi'\lambda\{1-\tfrac12e(\lambda+\lambda^{-1})\}}{\sqrt{1-e^2}-\tfrac12e(\lambda-\lambda^{-1})}\right\}f(\lambda_1,\ \lambda_2\ldots)}}\\&+\sin(r_1m_1+\ldots)\underbrace{\overbrace{\sin(r_1\chi_1\lambda_1+\ldots)\,\Pi\left\{\frac{2\sqrt{-1}\,\chi'\lambda\{1-\tfrac12e(\lambda+\lambda^{-1})\}}{\sqrt{1-e^2}-\tfrac12e(\lambda-\lambda^{-1})}\right\}f(\lambda_1,\ \lambda_2\ldots)}}\end{aligned}\right.$$

$$\ldots\ldots\ldots(43).$$

By choosing for $f\{\epsilon^{u_1\sqrt{(-1)}}, \epsilon^{u_2\sqrt{(-1)}} \ldots\}$, functions expansible without sines, or without cosines, a variety of formulæ may be obtained: we may instance

$$\overbrace{\frac{(\lambda-\lambda^{-1})\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\Lambda_r}{\sqrt{1-e^2}-\frac{1}{2}e(\lambda-\lambda^{-1})}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(44),$$

Λ_r having the same meaning as before.

Also,

$$\overbrace{\frac{\{\frac{1}{2}(\lambda+\lambda^{-1})-e\}\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\Lambda'_r}{\sqrt{1-e^2}-\frac{1}{2}e(\lambda-\lambda^{-1})}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(45),$$

where

$$\Lambda'_r = \lambda^r \epsilon^{-\frac{re}{2}(\lambda-\lambda^{-1})} - \lambda^{-r}\epsilon^{\frac{re}{2}(\lambda-\lambda^{-1})} \ldots\ldots\ldots\ldots\ldots\ldots(46).$$

Again,

$$\overbrace{\frac{\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}(\lambda-\lambda^{-1})\Lambda'_r}{1-\frac{1}{2}\frac{e}{\sqrt{1-e^2}}(\lambda-\lambda^{-1})}} + \frac{2}{r}\overbrace{\Lambda_r} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(47),$$

and

$$\overbrace{\frac{\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\{(\lambda+\lambda^{-1})-\frac{1}{2}e\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{1-e^2}}(\lambda-\lambda^{-1})}} = \overbrace{\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\{(\lambda+\lambda^{-1})-\frac{1}{2}e\}\Lambda_r}, \ldots(48);$$

or, what is the same thing,

$$\overbrace{\frac{(\lambda-\lambda^{-1})\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\{(\lambda+\lambda^{-1})-\frac{1}{2}e\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{1-e^2}}(\lambda-\lambda^{-1})}} = 0 \ldots\ldots\ldots\ldots(49);$$

or, comparing with (44),

$$\overbrace{\frac{(\lambda^2-\lambda^{-2})\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{1-e^2}}(\lambda-\lambda^{-1})}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(50),$$

which are all obtained by applying the formula (43) to the expansion of $\frac{\sin}{\cos}(\theta-\varpi)$, and comparing with the equations (25), (33).