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Propagation of nonlinear dispersive and dissipative waves*)

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TANIUTI's reductive perturbation theory, valid for quasi-linear systems of partial differential equations of the form $U_t+A(U)U_x+B(U)=0$, is applied for the analysis of small but finite nonlinear harmonic wave propagation in dispersive and dissipative solids. In this theory the stretching transformation allows for separating in the asymptotic expansion solution the rapidly oscillating harmonic dependence from the slowly varying amplitude and determining at the lowest approximation the modulation of the amplitude. Two examples of wave propagation in solids, namely, the elastic transverse waves in a taut string laying on a uniformly distributed nonlinear elastic support and longitudinal elastic viscoplastic waves in a thin long rod, are discussed in detail. The former problem is purely dispersive and the latter purely dissipative. It was demonstrated that in a case of the string the amplitude modulation is governed by the nonlinear Schrödinger equation and in a case of viscoplastic rod by the generalized Schrödinger equation. Analysis of the coefficients of these equations may say if the amplitude is modulationally stable, bounded or the solitary wave solution exists.

Zastosowano metodę Taniuti i współpracowników, słuszną dla quasi-liniowych układów równań różniczkowych cząstkowych pierwszego rzędu typu $U_t+A(U)U_x+B(U)=0$, do analizy zagadnień rozprzestrzeniania się fal harmonicznych o małej lecz skończonej amplitudzie w ośrodkach dyspersyjnych i dysypatywnych. Zastosowanie w tej teorii transformacji rozciągnięcia pozwala na rozdzielenie szybko zmieniającej się części harmonicznej od powolnej zmiany amplitudy i określenie z pierwszej aproksymacji modulacji amplitudy. W pracy rozważono szczegółowo dwa przykłady propagacji fal — fale poprzeczne w napiętej strunie podpartej nieliniowo-sprężyście i lepkoplastyczne fale podłużne w cienkim długim pręcie. Pierwszy problem jest czysto dyspersyjny, drugi — czysto dysypatywny. Wykazano, że równaniem różniczkowym amplitudy dla struny i pręta są odpowiednio nieliniowe i uogólnione równania Schrödingera. Analiza współczynników tych równań pozwala stwierdzić, czy rozwiązanie jest stateczne, ograniczone, lub czy istnieją rozwiązania typu "solitary wave".

Применен метод Таньюти и сотрудников, справедливый для квазилинейных систем дифференциальных уравнений в частных производных первого порядка типа $U_t+A(U)U_x+B(U)=0$, для анализа задач распространения гармонических волн малой, но конечной амплитуды в дисперсионных и диссипативных средах. Применение в этой теории трансформации растяжения позволяет разделить быстро меняющуюся гармоническую часть от медленного изменения амплитуды и определить из первой аппроксимации модуляцию амплитуды. В работе рассмотрены подробно два примера распространения волн — поперечные волны в натянутой струне подпертой нелинейно-упругим образом и вязкопластические продольные волны в тонком, длинном стержне. Первая задача чисто дисперсионная, вторая — чисто диссипативная. Показано, что дифференциальным уравнением амплитуды для струны и стержня являются соответственно нелинейное и обобщенное уравнение Шредингера. Анализ коэффициентов этих уравнений позволяет констатировать является ли решение устойчивым, ограниченным или существуют ли решения типа „solitary wave".

1. Introduction

THE STUDY of waves in nonlinear continua has received tremendous attention recently. The concept of a wave as a repetitive or periodic phenomena ceases to be of much value in nonlinear problems as compared to its use in linear problems. This is so since the Fourier analysis plays here a little role. In nonlinear wave problems the approach based

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on the moving singular surfaces across which some variables or their derivatives suffer discontinuity has proved to be extremely fruitful. Applying compatibility conditions to the field equations one can obtain an equation determining the possible speeds of wave propagation, study the growth and decay of the amplitude and analyze the formation of the shock waves. One serious limitation of this theory is the difficulty in satisfying the boundary conditions and therefore these studies are restricted mainly to the unbounded media and in particular, to one-dimensional problems.

The waves propagating in the waveguides and in dispersive and dissipative nonlinear media are of special theoretical and practical interest in many fields of physics and mechanics. Nonlinear dispersive waves have been known for a century in the water-wave theory. The Korteweg-De Vries equation describing the waves in shallow water has been up till now of particular interest because of its peculiar solution in the form of periodic, cnoidal and solitary waves. It was recently shown that apart from the Korteweg-De Vries equation there are other scalar partial differential equations describing dispersion or dissipation effects of nonlinear waves such as Burgers equation, Schrödinger equation, Klein-Gordon equation, Boussinesq equation which exhibit solitary waves solutions and in some particular cases solitons—qualitatively new waves which preserve their shape and velocity upon colliding with other solitary waves. Studies in seeking solitary waves are widely carried out in plasma physics, quantum mechanics and in the gas and fluid flow theory [1–5]. An extensive review of these works may be found in [6]. A particular lack of analogical studies in solid mechanics has been observed.

The first work on this subject in solid mechanics seems to belong to NARIBOLI who, in [7], studied the nonlinear longitudinal dispersive waves in an isotropic homogeneous elastic cylindrical rod. The governing equation he obtained is the analog of the Korteweg-De Vries equation in fluid. However he asserted the existence of different types of waves: the Airy, Boussinesq, cnoidal and solitary waves.

In [8] NARIBOLI and SEDOV extended these ideas to dispersive dissipative waves for viscoelastic materials of the rate type. They derived the generalized Burgers-Korteweg-De Vries equation describing the effect of dissipation and dispersion on waves of small but finite amplitude in rods and plates. SPENCE [9] also studied the propagation of one-dimensional waves of small finite amplitude in nonlinear materials with memory and showed that the evolution of the amplitude is described by the Burgers equation.

Just recently TEYMUR and SUHUBI [10] have applied TANIUTI and WEI'S [11] method, i.e., a method similar to the one we shall develop in this paper, to the analysis of dispersive and dissipative waves in solids. In particular they considered one-dimensional waves in a nonlinear Kelvin-Voigt solid and in a finite linear viscoelastic half-space. They found that longitudinal waves in a half-space are characterized by the Burgers equation while the shear waves by the generalized Burgers equation. Further, Braun using a similar two-timing perturbation technique investigated nonlinear effects on harmonic waves propagating in a finitely-deformed elastic material, [12].

In the papers mentioned above interesting analytical solutions were obtained by reducing the corresponding problems to the one of scalar equations described earlier. However, up to now there is still no general rule of reducing the dynamic continuum mechanics problems to these scalar equations. In this respect the use of the Schrödinger equation seems to be promising as, under some conditions, it admits the solitary wave solution. The scalar differential equations in the past were relied on a number of physical arguments. Recently, an attempt was undertaken by TANIUTI and his co-workers [13, 14, 15] to elaborate the general method of reducing the broad class of boundary-value problems governed by the system of nonlinear partial differential equations to a single second-order nonlinear differential equation. Using the coordinate stretching approach they constructed a theory for the propagation of a plane wave of small wave number and low frequency as well as for finite amplitude and high frequency waves. When the amplitude changes gradually and insignificantly in the vibration period, the stretching transformation allows the system to be separated on the part which changes rapidly and connected with oscillations and the part of the amplitude which changes slowly. In asymptotic expansion the lowest approximation determines the modulation of the amplitude. It was suggested [16] that this method may be successfully adopted for investigating the dispersion and dissipation of nonlinear elastic and non-elastic waves and that the results allready obtained be used in other fields.

The notion of dispersion and dissipation of harmonic waves is determined uniquely for linear systems but not precisely for nonlinear ones. In a linear system the field quantities associated with linear waves may be resolved into Fourier components. For example, in the case of one-dimensional plane waves the governing equations have an elementary solution of the form $a\exp i(kx-i\omega t)$, where x denotes a one-dimensional space coordinate, t the time, a the amplitude, k the wave number and ω the angular frequency. The wave-length Λ is defined to be $\Lambda = 2\pi/k$. A consistency condition which requires that the solution should be not trivial leads to the relationship between ω and k of the form $\omega = \omega(k)$ which is known as a dispersion relation. The general solution is then given in terms of Fourier integrals of the form $\int_0^\infty A(k)\exp[i(kx-\omega t)]dk$, where the spectrum function $A(k)$ is determined by appropriate initial or boundary conditions. We define the phase velocity c_p and the group velocity λ of the wave by the following known relations: $c_p = \omega/k$ and $\lambda = \partial\omega/\partial k$. From the dispersion relation it results that the phase velocity and the group velocity are generally not equal to each other. In other words the phase velocity c_p depends on the wave number and we say that the wave is dispersive or that the wave has dispersion in a wider sense. In dispersive systems the dispersion relation generally gives a complex ω for a real k. Therefore, not only does the phase velocity depend on the wave length, but also the effective amplitude of the wave will be attenuated with time if $\mathrm{Im}(\omega) < 0$. We call such attenuation dissipation. On the other hand if $\mathrm{Im}(\omega) > 0$, the effective amplitude of the wave will grow without bound in the course of time. We call this instability. If, in particular, $\omega = \omega(k)$ is a real function of a real k, where $\partial c_p/\partial k \neq 0$, then neither dissipation not instability occurs. We call this pure dispersion or dispersion in the narrower sense.

On the other hand, in a nonlinear wave system, the frequency of the wave is not only the function of the wave number but also of other parameters the amplitude, for example, which are all assumed to be small in the linear approximation. As opposed to linear systems, it seems that a precise definition of dispersion has not yet been established for nonlinear systems. In many cases, however, we can obtain corresponding equations of a linear dispersive system by linearizing the governing equations which are originally nonlinear. In these cases we can consider the dispersion of nonlinear waves. Following JEFFREY and KAKUTANI [4] we shall say that the system is dispersive if its linearized form exhibits dispersion in the sense of linear waves.

The aim of this paper is to reduce the system of equations governing the dynamics of nonlinear dispersive and dissipative systems to a single nonlinear equation for the wave amplitude and obtain the special solutions in an explicit form for the lowest order of perturbation. In Sect. 2 a brief description of Taniuti's reductive method is given. For the purpose of illustrating the method, problems of propagation of transverse waves in elastically supported stretched string and viscoplastic waves in a thin rod are discussed in detail in Sects. 3 and 4 respectively. The object of our study is to investigate how the plane wave is modulated by nonlinear effects.

2. Taniuti's reductive method

In the one-dimensional Taniuti's reductive perturbation approach the basic system of equations has the form

$$\frac{\partial \mathbf{U}}{\partial t}+\mathbf{A}(x, t, \mathbf{U})\frac{\partial \mathbf{U}}{\partial x}+\mathbf{B}(x, t, \mathbf{U}) = 0, \tag{2.1}$$

where $\mathbf{U}$ is a column vector with n components $u_1, u_2, \ldots, u_n$; the $n\times n$ matrix $\mathbf{A}$ and the column vector $\mathbf{B}$ are the functions of u_i's being sufficiently smooth. The system of Eqs. (2.1) contains a very wide class of boundary-value problems of nonlinear wave propagation in continuous media.

Let $U^{(0)}$ be a constant solution satisfying

$$\mathbf{B}(\mathbf{U}^{(0)}) = 0 \tag{2.2}$$

and define the matrices $\mathbf{A}_0$ and $\Delta\mathbf{B}_0$ by

$$\mathbf{A}_0 = \mathbf{A}(\mathbf{U}^{(0)}), \quad (\nabla\mathbf{B}_0)_{ij} = \left(\frac{\partial \mathbf{B}_i}{\partial u_j}\right)_{\mathbf{U}=\mathbf{U}^{(0)}}.$$

Then Eq. (2.1) linearized about $\mathbf{U}^{(0)}$ takes the form

$$\frac{\partial \mathbf{U}}{\partial t}+\mathbf{A}_0\frac{\partial \mathbf{U}}{\partial x}+\nabla\mathbf{B}_0\cdot\mathbf{U} = 0, \tag{2.3}$$

which admits a high frequency plane wave solution $\sim \exp\pm\{i(kx-i\omega t)\}$ which is characterized by the dispersion relation

$$\det W_{\pm 1} = \det[\mp i\omega\mathbf{I}\pm ik\mathbf{A}_0+\nabla\mathbf{B}_0] = 0, \tag{2.4}$$

$\mathbf{I}$ is the unit matrix, assumed to have for any real k at least one real root ω which changes smoothly as k changes, but

$$\det W_l = \det[-il\omega\mathbf{I}+ilk\mathbf{A}_0+\nabla\mathbf{B}_0] \neq 0 \quad \text{for} \quad l\neq 1. \tag{2.4'}$$

For a wave with small but finite amplitude of the order ε, U is then expanded in the neighbourhood of $\mathbf{U}^{(0)}$ in terms of small parameter ε and harmonics $\sim \exp[il(kx-i\omega t)]$ as

$$\mathbf{U} = \sum_{\alpha=0} \varepsilon^\alpha \mathbf{U}^{(\alpha)}, \quad \mathbf{U}^{(\alpha)} = \sum_{l=-\infty}^{\infty} \mathbf{U}_l^{(\alpha)}(\tau, \xi)\exp[il(kx-\omega t)], \quad \text{for} \quad \alpha\geqslant 1,$$

(2.5) or

$$\mathbf{U} = \mathbf{U}^{(0)} + \sum_{\alpha=1}^{\infty} \varepsilon^{\alpha} \sum_{l=-\infty}^{+\infty} U_l^{(\alpha)}(\tau, \xi)\exp[il(kx - i\omega t)],$$

$\mathbf{U}_l^{(\alpha)} = \mathbf{U}_{-l}^{(\alpha)*}$, (the asterisk denotes the complex conjugation) where τ and ξ are the slow variables introduced through stretching

$$\tau = \varepsilon^2 t,$$
$$\xi = \varepsilon(x - \lambda t), \quad \lambda = \frac{\partial\omega}{\partial k}. \tag{2.6}$$

In this way the rapidly oscillating harmonic dependence is separated from the amplitude modulation, slowly varying as indicated by the scaled coordinates, Eqs. (2.6).

Substituting this expansion, together with the expanded terms

$$\mathbf{A} = \mathbf{A}_0 + \varepsilon\nabla\mathbf{A}_0\cdot\mathbf{U}^{(1)} + \varepsilon^2\left\{\nabla\mathbf{A}_0\cdot\mathbf{U}^{(2)} + \frac{1}{2}\nabla\nabla\mathbf{A}_0 : \mathbf{U}^{(1)}\mathbf{U}^{(1)}\right\} + \ldots,$$
$$\mathbf{B} = \mathbf{B}_0 + \varepsilon\mathbf{B}_0\cdot\mathbf{U}^{(1)} + \varepsilon^2\left\{\nabla\mathbf{B}_0\cdot\mathbf{U}^{(2)} + \frac{1}{2}\nabla\nabla\mathbf{B}_0 : \mathbf{U}^{(1)}\mathbf{U}^{(1)}\right\}$$
$$\varepsilon^3\left\{\nabla\mathbf{B}_0\cdot\mathbf{U}^{(3)} + \nabla\nabla\mathbf{B}_0 : \mathbf{U}^{(1)}\mathbf{U}^{(2)} + \frac{1}{6}\nabla\nabla\nabla\mathbf{B}_0 : \mathbf{U}^{(1)}\mathbf{U}^{(1)}\mathbf{U}^{(1)}\right\} + \ldots, \tag{2.7}$$

where the following notation was assumed

$$\nabla\mathbf{A}_0\cdot\mathbf{U}^{(1)} = \sum_i \left(\frac{\partial\mathbf{A}}{\partial u_i}\right)_{\mathbf{U}=\mathbf{U}_0} u_i^{(1)},$$

$$\nabla\nabla\mathbf{A}_0 : \mathbf{U}^{(1)}\mathbf{U}^{(1)} = \sum_{i,j} \left(\frac{\partial^2\mathbf{A}}{\partial u_i\,\partial u_j}\right)_{\mathbf{U}=\mathbf{U}_0} u_i^{(1)}u_j^{(1)},$$

into the original system of equations and equating to zero the coefficients of the various powers of ε for the same harmonics gives in first order the linear dispersion relation. In second order λ is determined to be the group velocity ($\lambda = \partial\omega/\partial k$). And in the third order it is found that the nonlinear modulation is represented by a nonlinear Schrödinger equation for the first-order amplitude $\varphi^{(1)}$ (where $\mathbf{U}_1^{(1)} = \varphi^1\mathbf{R}$ and $W_1\mathbf{R} = 0$)

$$\alpha\frac{\partial\varphi^{(1)}}{\partial\tau} + \beta\frac{\partial^2\varphi^{(1)}}{\partial\xi^2} + \gamma|\varphi^{(1)}|^2\varphi^{(1)} = 0, \tag{2.8}$$

where

$$\alpha = \mathbf{L}\cdot\mathbf{R},$$
$$\beta = \mathbf{L}(-\lambda\mathbf{I} + \mathbf{A}_0)Z(-\lambda\mathbf{I} + \mathbf{A}_0)\mathbf{R},$$
$$\gamma = \mathbf{L}\left[ik\left\{2(\nabla\mathbf{A}_0\cdot\mathbf{R}^*)\mathbf{R}_2^{(2)} + (\nabla\mathbf{A}_0\cdot\mathbf{R}_0^{(2)})\mathbf{R} - (\nabla\mathbf{A}_0\mathbf{R}_2^{(2)})\mathbf{R}^* + (\nabla\nabla\mathbf{A}_0 : \mathbf{R}\mathbf{R}^*)\mathbf{R}\right.\right.$$
$$\left.\left. - \frac{1}{2}(\nabla\nabla\mathbf{A}_0 : \mathbf{R}\mathbf{R})\mathbf{R}^* + \nabla\nabla\mathbf{B}_0 : (\mathbf{R}\mathbf{R}_0^{(2)} + \mathbf{R}^*\mathbf{R}_2^{(2)}) + \frac{1}{2}\nabla\nabla\nabla\mathbf{B}_0 \vdots \mathbf{R}\mathbf{R}^*\mathbf{R}\right\}\right]. \tag{2.9}$$

Here the row vector $\mathbf{L}$ is determined from the relation $\mathbf{L}W_1 = 0$ and $\mathbf{Z}$ is the following matrix

$$(2.10) \qquad Z_{ik} = \left(\frac{\partial D_{ik}}{\partial p}\right)_{p=\omega} \Bigg/ \left(\frac{\partial D}{\partial p}\right)_{p=\omega},$$

in which $D_{ik}(p)$'s are the co-factors of $D(p) = \det[-ip\mathbf{I}+ik\mathbf{A}_0+\nabla\mathbf{B}_0]$ and the vectors $\mathbf{R}_0^{(2)}$ and $\mathbf{R}_2^{(2)}$ are

$$(2.11) \qquad \begin{aligned} \mathbf{R}_0^{(2)} &= -W_0^{-1}\left\{ik(\nabla\mathbf{A}_0\cdot\mathbf{R}^*)\mathbf{R} - \text{c.c.} + \frac{1}{2}(\nabla\nabla\mathbf{B}_0:\mathbf{R}^*)\mathbf{R} + \text{c.c.}\right\},\\ \mathbf{R}_2^{(2)} &= -W_2^{-1}\left\{ik(\nabla\mathbf{A}_0\cdot\mathbf{R})\mathbf{R} + \frac{1}{2}\nabla\nabla\mathbf{B}_0:\mathbf{R}\mathbf{R}\right\},\end{aligned}$$

where "c.c." denotes the complex conjugation.

In some cases Eq. (2.8) admits further reduction to the Korteweg-De Vries equation $u_t+\varepsilon uu_x+\gamma u_{xx}+\delta u_{xxx} = 0$. In the solution (2.5)–(2.7) the approximations to be computed are: $\mathbf{U}_1^{(\alpha)}$, $\alpha = 1, 2$; $l = \pm 1, \pm 2, 0$. A crucial assumption of this theory is that for ω_0 is a real root of $\det W_{\pm 1} = 0$, $l\omega_0$ are not also roots of this equation. That is, a monofrequency oscillation state is assumed to exist and self-resonance is excluded. However, in most physical problems this conditions is not valid for $l = 0$. In application this particular difficulty may be avoided by means of subsidiary or boundary conditions.

As an application of the reductive perturbation method presented we shall examine in the next two sections the problem of elastic waves in a taut string lying on an elastic nonlinear foundation and elastic viscoplastic waves in a thin rod. These are the simplest examples of dispersive and dissipative wave motion, respectively. Moreover, in a linear case they have explicit form solutions.

3. Taut string on nonlinear elastic foundation

Consider an infinite elastic string supported laterally by a distributed spring of spring constant K_1 jointly with nonlinear support K_2y^3. The string is assumed to be under constant tension F. The lateral displacement of the string is denoted by y and the distance along the string by x, Fig. 1. The inclination angle of the string is assumed to be small enough so that its cosine may be approximated by one and its sine by its tangent y_x.

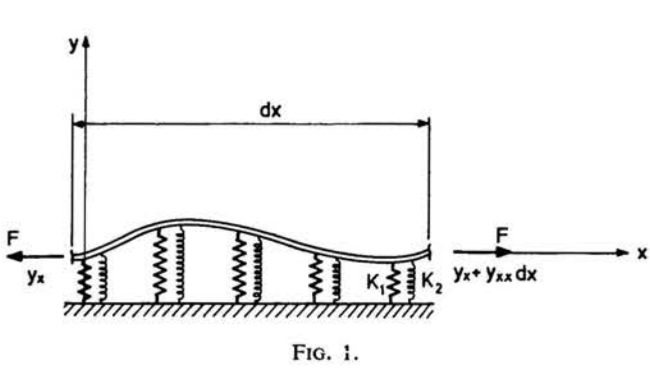

FIG. 1.

The string is always in equilibrium in the x direction and only the dynamics in the y direction is considered. The string is characterized by the cross section area A and the material density ϱ. The lateral acceleration y_{tt} times the mass $A\varrho dx$ of a section of a string balances the lateral force exerted on the section. This lateral force consists of the forces $-K_1 y$ and $-K_2 y^3$ exerted by the supports and lateral component of the stretching force Fy_{xx} acting on a differential segment of the string having a length dx. Combining all the forces we obtain the differential equation

$$\hat{A}\hat{\varrho}\hat{y}_{\hat{t}\hat{t}} - \hat{F}\hat{y}_{\hat{x}\hat{x}} + \hat{K}_1\hat{y} + \hat{K}_2\hat{y}^3 = 0, \tag{3.1}$$

where the "$\wedge$" denotes dimensional quantities.

Introducing dimensionless quantities

$$y = \frac{\hat{y}}{b}, \quad x = \frac{\hat{x}}{b}, \quad t = \frac{\hat{t}a}{b}, \quad K_1 = \frac{\hat{K}_1 b^2}{\hat{F}}, \quad K_2 = \frac{\hat{K}_2 b^4}{\hat{F}},$$

where $a^2 = \dfrac{\hat{F}}{\hat{A}\hat{\varrho}}$ and b is a characteristic length, Eq. (3.1) assumes now the form

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + K_1 y + K_2 y^3 = 0. \tag{3.1'}$$

This equation may be replaced by a system of first-order partial differential equations by introducing new variables u and v through the relations

$$\begin{aligned} u &= \frac{\partial y}{\partial x} \quad \text{or} \quad u - \frac{\partial y}{\partial x} = 0,\\ v &= \frac{\partial y}{\partial t} \quad \text{or} \quad v - \frac{\partial y}{\partial t} = 0. \end{aligned} \tag{3.2}$$

Differentiating Eq. (3.2_1) with respect to time we obtain

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \tag{3.3}$$

which, together with Eqs. (3.1′) and (3.2_2), form the system of the governing equations

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} + K_1 y + K_2 y^3 &= 0,\\ \frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} &= 0,\\ \frac{\partial y}{\partial t} - v &= 0. \end{aligned} \tag{3.4}$$

This system of equations may be easily transformed to the compact matrix form, Eq. (2.1), and $\mathbf{U}, \mathbf{A}$ and $\mathbf{B}$ now become

$$\mathbf{U} = \begin{bmatrix} v \\ u \\ y \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -K_1 y - K_2 y^3 \\ 0 \\ v \end{bmatrix}. \tag{3.5}$$

Equation $(3.2)_1$ is the subsidiary condition which will be used for evaluating the zero approximations of the solution $\mathbf{U}_0^{(\alpha)}$.

Let us expand the solution $\mathbf{U}$ in the series (2.5) around the constant state $\mathbf{U}^{(0)} = 0$. Then the matrix (2.4') $W_l = [-i\omega l\mathbf{I} + ikl\mathbf{A}_0 + \nabla\mathbf{B}_0]$ in the explicit form becomes

$$W_l = \begin{bmatrix} -il\omega & -ilk & 1 \\ -ilk & -il\omega & 0 \\ -1 & 0 & -il\omega \end{bmatrix}. \tag{3.6}$$

The dispersion equation, i.e., $\det W_1 = 0$, is

$$\omega^2 - k^2 - 1 = 0. \tag{3.7}$$

To derive the differential equation for the amplitude $\varphi^{(1)}$, the coefficients α, β, γ and the vectors $\mathbf{R}_0^{(2)}$ and $\mathbf{R}_2^{(2)}$ should be evaluated from Eqs. (2.9)–(2.11).

In our case we have

$$\mathbf{R} = \begin{bmatrix} \omega \\ -k \\ i \end{bmatrix}, \quad \mathbf{L} = [\omega, -k, i]. \tag{3.8}$$

However, since the determinant W_0 equals zero, the vector $\mathbf{R}_0^{(2)}$ can not be determined directly by this method. Instead, thc approximations $\mathbf{U}_0^{(\alpha)}$ should be found independently from the subsidiary equation (3.2_1) and the governing equations (3.4). After some calculations we obtain

$$\begin{aligned} v_0^{(1)} &= y_0^{(1)} = u_0^{(1)} = 0, \quad \text{i.e.} \quad \mathbf{U}_0^{(1)} = 0, \\ v_0^{(2)} &= y_0^{(2)} = u_0^{(2)} = 0, \quad \text{i.e.} \quad \mathbf{U}_0^{(2)} = 0 \end{aligned} \tag{3.9}$$

what finally leads to $\mathbf{R}_0^{(2)} = 0$.

Since $\nabla\mathbf{A}_0 = \nabla\nabla\mathbf{A}_0 = \nabla\nabla\mathbf{B}_0 = 0$, we can compute straightforwaidly that

$$\mathbf{R}_0^{(2)} = 0. \tag{3.10}$$

Substituting Eqs. (3.8)–(3.10) to the formulae (2.9), we obtain

$$\alpha = 2\omega^2, \quad \beta = -iK_1/\omega, \quad \gamma = 3i\omega K_2. \tag{3.11}$$

Hence the differential equation for the amplitude modulation is the nonlinear Schrödinger equation

$$i\frac{\partial\varphi^{(1)}}{\partial\tau} + p\frac{\partial^2\varphi^{(1)}}{\partial\xi^2} + q|\varphi^{(1)}|^2\varphi^{(1)} = 0 \tag{3.12}$$

where

$$p = \frac{K_1}{2\omega^3}, \quad q = -\frac{3K_2}{2\omega}.$$

Although it is not our aim to discus, the solution of Eq. (3.12) for the corresponding initial and boundary conditions, some interesting features of the solution of Eq. (2.8) should be remarked here. Consider the following type of the solution

$$\varphi^{(1)}(\xi, \tau) = A(\xi)\exp(i\nu\tau), \tag{3.13}$$

where $A(\xi)$ is a real function of ξ and ν is a real constant, satisfying the equation

$$i\frac{\partial\varphi^{(1)}}{\partial\tau} + p\frac{\partial^2\varphi^{(1)}}{\partial\xi^2} = q|\varphi^{(1)}|^2\varphi^{(1)},$$

provided that $A(\xi)$ fulfils the following condition

$$\frac{1}{2}\left(\frac{dA}{d\xi}\right)^2+V(A)=E_0, \quad V(A)=-\frac{q}{4p}-\frac{\nu}{2p}A^2,$$

where E_0 is an arbitrary constant. This is equivalent to the classical equation of motion for a unit mass with total energy E_0 under the potential $V(A)$. By virtue of this analogy the solution $A(\xi)$ depends on mutual relations between p, q, ν and E_0.

	p, q	ν, p, q	E_0, ν, p, q	$A(\xi)$
	$pq > 0$	$\nu p < 0$ $\nu q < 0$	$0 < E_0 < \frac{\nu^2}{4pq}$	$A(\xi)$ represents wave trains expressible by the Jacobian functions (comp. [6])
(3.14)			$E_0 = \frac{\nu^2}{4pq}$	$A(\xi)=\left(-\frac{\nu}{q}\right)^{\frac{1}{2}}\tanh\left[\left(-\frac{\nu}{2p}\right)^{\frac{1}{2}}\xi\right]$
	$pq > 0$	$\nu p > 0$ $\nu q > 0$		there is no bounded real solution
	$pq < 0$	$\nu p > 0$ $\nu q < 0$		two types of wave trains expressed by Jacobian eliptic functions
			$E_0 > 0$	1) large amplitude waves
			$\frac{\nu^2}{4pq} < E_0 < 0$	2) small amplitude waves
			$E_0 = 0$	solitary waves
(3.15)				$A(\xi)=\left(-\frac{2\nu}{q}\right)^{\frac{1}{2}}\operatorname{sech}\left[\left(-\frac{\nu}{p}\right)^{\frac{1}{2}}\xi\right].$
	$pq < 0$	$\nu p < 0$	$E_0 > 0$	$A(\xi)$ represents wave trains expressible by the Jacobian eliptic functions.

When $q = 0$ the nonlinear effect disappears and the amplitude equation reduces to the form

$$i\frac{\partial\varphi^{(1)}}{\partial\tau}+p\frac{\partial^2\varphi^{(1)}}{\partial\xi^{(2)}}=0, \tag{3.16}$$

which may be integrated through the quadratures, and its general solution is

$$\varphi^{(1)}(\xi,\tau)=e^{-i\nu\tau}(C_1e^{ik\xi}+C_2e^{-ik\xi}). \tag{3.17}$$

The solution is then sinusoidal in the phase ξ and the phase is linear in time. $q = 0$ corresponds to $K_2 = 0$ in Eq. (3.12) and we have the classical problem of transverse wave propagation in a taut string elastically supported.

4. Elastic viscoplastic waves in thin rod

The problem of the propagation of infinitesimal elastic viscoplastic waves in a long thin rod is described by the equation of motion

$$\frac{\partial\hat{\sigma}}{\partial\hat{x}}=-\varrho\frac{\partial\hat{v}}{\partial\hat{t}}, \tag{4.1}$$

the continuity equation

$$\frac{\partial \varepsilon}{\partial \hat{t}} = \frac{\partial \hat{v}}{\partial \hat{x}} \tag{4.2}$$

and the Malvern constitutive equation [17]

$$\frac{\partial \varepsilon}{\partial \hat{t}} = \frac{1}{E}\frac{\partial \hat{\sigma}}{\partial \hat{t}} + \hat{\Gamma}\Phi\left(\frac{\hat{\sigma}-\hat{\sigma}_0}{\hat{\sigma}_0}\right), \tag{4.3}$$

where $\hat{\sigma}$, $\hat{v}$ and ε are the dimensional stress, velocity and strain respectively, E is the Young modulus, Γ the viscosity coefficient and Φ is the nonlinear relaxation function depending on the excess of the stress beyond a static plastic limit. Introducing dimensionless quantities similarly as in Sect. 3,

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}c_0}{b}, \quad v = \frac{\hat{v}}{c_0}, \quad \sigma = \frac{\hat{\sigma}}{c_0^2}, \quad \Gamma = \frac{\hat{\Gamma}b}{c_0},$$

where b is a characteristic length and $c_0^2 = \dfrac{E}{\varrho}$ is the squared bar velocity, the system of Eqs. (4.1)–(4.3) may be written in the following matrix form:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}(\mathbf{U}) = 0 \tag{4.4}$$

where

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v \\ \sigma \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \Gamma\Phi\left(\frac{\sigma-\sigma_0}{\sigma_0}\right) \end{bmatrix} \tag{4.5}$$

To specify the relaxation function Φ we assume its power form $\Phi = \left(\dfrac{\sigma-\sigma_0}{\sigma_0}\right)^{\delta}$ as suggested by COWPER and SYMONDS for mild steel and aluminium [18].

Consider the constant state $U^{(0)}$ about which the expansion (2.5) is considered as

$$\mathbf{U}^{(0)} = \begin{bmatrix} 0 \\ \sigma_0 \end{bmatrix}, \quad \text{such that} \quad \mathbf{B}[\mathbf{U}^{(0)}] = 0. \tag{4.6}$$

However, to preserve the existence of the small amplitude viscoplastic loading and unloading wave we introduce a new quantity σ_0' which is the value of the stress just beyond σ_0. The assumption of such a prestressed state σ_0' under the quasi-static strain rate $\partial\varepsilon_0'/\partial t = \text{const}$ was discussed in detail in [19]. Introducing for the simplicity the notations

$$\sigma - \sigma_0' = \bar{\sigma}, \quad \frac{\partial \varepsilon}{\partial t} - \frac{\partial \varepsilon_0'}{\partial t} = \frac{\partial \bar{\varepsilon}}{\partial t}$$

we obtain the constitutive equation

$$\frac{\partial \bar{\varepsilon}}{\partial t} = \frac{1}{E}\frac{\partial \bar{\sigma}}{\partial t} + \Gamma\Phi\left(\frac{\bar{\sigma}}{\sigma_0}\right), \quad \Phi\left(\frac{\bar{\sigma}}{\sigma_0}\right) = \frac{\Gamma}{\sigma_0}\bar{\sigma}^{\delta} \tag{4.3'}$$

which is rather a viscoelastic than elastic viscoplastic constitutive relation.

Linearization of Eq. (4.4), in which Eq. (4.3′) is included (called further Eq. (4.4′)) around the state $\bar{\mathbf{U}}^{(0)} = 0$ gives

$$\mathbf{A}_0 = \mathbf{A} \quad \text{and} \quad \nabla \mathbf{B}_0 = \left(\frac{\partial B_i}{\partial u_j}\right)_{\mathbf{U}=\mathbf{U}_0} = 0 \quad \text{for} \quad i, j = 1, 2 \text{ and } \delta > 1.$$

Hence the linearized form of Eq. (4.4′) leads to a linear dispersionless system with the frequency equation

$$W_1 = \omega^2 - k^2 = 0. \tag{4.7}$$

For $\delta = 1$, Eq. (4.4′) is linear and the only non-zero element of $\nabla \mathbf{B}_0$ is $\dfrac{\partial B_2}{\partial u_2} = \dfrac{\Gamma}{\sigma_0}$. Its dispersion equation

$$\begin{bmatrix} -i\omega & ik \\ ik & -i\omega + 2\psi \end{bmatrix} = -\omega^2 + i\omega\psi + k^2 = 0, \quad 2\psi = \frac{\Gamma}{\sigma_0} \tag{4.8}$$

is not real. The transformation

$$\mathbf{U} = e^{-\psi t}\tilde{\mathbf{U}}(x, t) \tag{4.9}$$

leads to the system of Eqs. (4.5) but in which the term B is

$$\mathbf{B}^T = [-\psi v, -\psi\sigma]^T, \quad T = \text{transpose}. \tag{4.10}$$

The equation obtained does have the real dispersion relation

$$\begin{bmatrix} -i\omega - \psi & ik \\ ik & -i\omega + \psi \end{bmatrix} = -\omega^2 + k^2 - \psi^2 = 0, \tag{4.11}$$

or, in dimensional form,

$$\omega^2 = c_0^2 k^2 - \psi^2. \tag{4.12}$$

This relation has several anomalies. One of these is a low wave number cut off at $k = k_c = c_0^{-1}\psi$ below which the solution is not oscillatory in time and no wave propagates. Another is that the group velocity is equal to $c_0^2 k/\omega$, a quantity larger than c_0. Hence, unless $\psi \ll c_0 k$ the propagation is strongly dissipative and the utility of the concept of group velocity is lost. If $\psi \ll c_0 k$, the dissipation is weak and the term $\mathbf{B}$ in Eq. (4.5) with Eq. (4.9) may be neglected. It is in contrast with the method of characteristics according to which the wave always propagates with the velocity c_0 regardless how large the term $\mathbf{B}$ may be.

Let us notice that the system of Eqs. (4.5) with the vector $\mathbf{B}$ determined by Eq. (4.10) may easily be put in one second-order partial differential equation

$$\frac{\partial^2 \tilde{\sigma}}{\partial t^2} - \frac{\partial^2 \tilde{\sigma}}{\partial x^2} + 2\psi\tilde{\sigma} = 0. \tag{4.13}$$

which is the same as Eq. (3.1′) with $K_1 = 2\psi$ and $K_2 = 0$. Therefore, the amplitude equation and its general solution are given by Eqs. (3.16) and (3.17), respectively. We can thus conclude that the nonlinearity expressed in terms of the Cowper-Symonds constitutive relation was lost in our reductive perturbation method.

PERZYNA [20] in 1963 suggested two interesting expressions for $\Phi(F)$, the power series and the exponential series. The former, i.e.,

$$\Phi(F) = \sum_l a_l F^l, \quad F = \frac{\sigma - \sigma_0}{\sigma_0}, \tag{4.14}$$

which includes the model $\Phi(F) = F^\delta$ considered previously is more suitable for our analysis. We take into account only the first three terms of the series (4.14) since the remaining terms are excluded in the 3-rd order approximation procedure. We shall consider the governing system of Eqs. (4.4)–(4.5) with the constitutive relation

$$\Phi(\bar{\sigma}) = 2\psi\bar{\sigma} + a_2\bar{\sigma}^2 + a_3\bar{\sigma}^3, \tag{4.15}$$

where $a_1 = \dfrac{\Gamma}{\sigma_0}$ was assumed to be the same as in a previous case and the substitution $\bar{\sigma} = \sigma - \sigma_0'$ was performed. The dispersion relation determined from the linearized governing system around $\bar{U}_0$ is given by Eq. (4.11). The matrix W_1 now assumes the form

$$W_1 = \begin{bmatrix} -\psi - li\omega & lik \\ lik & \psi - li\omega \end{bmatrix}. \tag{4.16}$$

The coefficients α, β, γ in the amplitude equation (2.8) may be evaluated in a straightforward way from the formulae (2.9) since now $\det W_0 = -\psi^2$, is different from zero. Because of the symmetry of the matrix $\mathbf{W}_1$ the components of the column vector $\mathbf{R}$ and row vector $\mathbf{L}$ are the same and take the form

$$\mathbf{L} = [\psi - i\omega,\ ik]; \quad \mathbf{R} = \begin{bmatrix} \psi - i\omega \\ ik \end{bmatrix}. \tag{4.17}$$

The matrix $\mathbf{Z}$ is

$$\mathbf{Z} = \frac{i}{2\omega} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.18}$$

The vectors $\mathbf{R}_0^{(2)}$, $\mathbf{R}^*$ and $\mathbf{R}_2^{(2)}$ appearing in the expression for γ are

$$\mathbf{R}_0^{(2)} = \frac{a_2 k^2}{\psi} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{R}^* = \begin{bmatrix} \psi + i\omega \\ -ik \end{bmatrix}, \quad \mathbf{R}_2^{(2)} = \frac{a_2 k^2}{6\psi} \begin{bmatrix} 2ik \\ \psi + 2i\omega \end{bmatrix}. \tag{4.19}$$

Substituting Eqs. (4.17)–(4.19) into Eq. (2.9), we finally get

$$\alpha = -2\omega^2 - 2i\psi\omega, \tag{4.20}$$

$$\beta = \frac{\psi}{\omega^2}(3k^2 + \omega^2) + i\frac{k^2(\psi^2 + \omega^2)}{2\omega^3}, \tag{4.21}$$

$$\gamma = -\frac{k^4}{6}[(6 - \psi)a_2^2 + 3a_3] + i\frac{a_2^2 k^4 \omega}{3\psi}. \tag{4.22}$$

Hence for the first-order amplitude modulation of an elastic viscoplastic wave in a rod we obtained the generalized Schrödinger equation.

5. Concluding remarks

We have shown that Taniuti's reductive perturbation theory, valid for the quasi-linear system of partial differential equations of the form $\mathbf{U}_t+\mathbf{A}(\mathbf{U})\mathbf{U}_x+\mathbf{B}(\mathbf{U}) = 0$ may be successfully used for analyzing nonlinear harmonic wave propagation in dispersive and dissipative solids. Two examples of wave propagation, namely, the elastic transverse waves in a taut string lying on a uniformly distributed nonlinear elastic support and longitudinal elastic viscoplastic waves in a thin semi-infinite rod are discussed in detail.

The former problem is a purely dispersive one and the latter is purely dissipative. Real dispersion relations were obtained in both cases and the range of possible wave frequencies were estimated. It was demonstrated that in the case of a string the amplitude modulation is governed by the nonlinear Schrödinger equation and for viscoplastic waves in a rod by the generalized Schrödinger equation. An analysis of the coefficients of these equations (without solving them) makes it possible to say if the amplitude is modulationally stable, bounded or if the solitary wave solution exists.

The advantage of this method is its generality and mathematical elegance (in spite of some algebraic complexity in combined problems) and consequently, its applicability to a large class of interesting boundary-value problems of solid mechanics, including thermal effects [21]. The consideration of higher order approximations is straightforward. This theory may be extended to two dimensions.

In the two-dimensional generalization one considers a wave propagating along the unbounded x-direction and bounded in the transverse z-direction with the boundary condition given. The basic system of equations becomes $\mathbf{U}_t+\mathbf{A}^z(\mathbf{U})\mathbf{U}_z+\mathbf{A}^x(\mathbf{U})\mathbf{U}_x+\mathbf{B}(\mathbf{U}) = 0$. As it was the case before, the frequency is obtained from the linearized system and the amplitude modulation is governed by the generalized nonlinear Schrödinger equation.

In concluding this section, it is worth mentioning that the Krylov-Bogolubov-Mitropolsky method is also useful in obtaining the nonlinear Schrödinger equation for the amplitude modulation of monochromatic plane waves [22]. The only assumption used in this method is the annihilation of secular terms, and the method can suggest quite naturally a heuristic coordinate transformation on which Taniuti's perturbation method is based.

The results of this work may be useful in the analysis of stresses in a structure or machine part performed by new ultrasonic techniques referred to now as the acousto-elastic method.

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