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On the Galilean invariance of balance equation for a singular surface in continuum

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THE PAPER contains the derivation of the Galilean invariant form of the balance equations for a singular surface. It is proved that such an invariance requires a certain structure of the surface sources. We illustrate the formal results, applying them in the theory of capillarity, the theory of very strong shock waves and the theory of Müller's material. In the last case, we derive an inequality, limiting the strength of non-adiabatic shock waves.

Praca zawiera wyprowadzenie równań bilansu dla powierzchni osobliwej w postaci niezmienniczej względem transformacji Galileusza. Dowodzi się, że powyższe żądanie niezmienniczości jest spełnione, jeśli źródła powierzchniowe mają pewną szczególną budowę. Formalne wyniki są zilustrowane przykładami zastosowań do teorii włoskowatości, teorii bardzo silnych fal uderzeniowych i teorii materiałów Müllera. W tym ostatnim przypadku wyprowadzono nierówność, ograniczającą natężenie nieadiabatycznej fali uderzeniowej.

Работа содержит вывод уравнений баланса для сингулярной поверхности в инвариантном виде по отношению к преобразованию Галилея. Доказывается, что вышеупомянутое требование инвариантности удовлетворено, если поверхностные источники обладают некоторым частным строением. Формальные результаты иллюстрированы примерами применения в теории капиллярности, в теории очень сильных ударных волн и в теории материалов Мюллера. В этом последнем случае выведено неравенство, ограничивающее интенсивность неадиабатической ударной волны.

1. Introduction

THE CLASSICAL theory of the balance equations for a singular surface in a continuous medium is based on the well-known Kotchine's condition, for example see [3]. However, the common approach excludes many cases of great practical importance. This limitation follows from the assumption on the smoothness of the field in question. Namely, it is assumed that the true value of the field on the surface of discontinuity is of no importance and the Kotchine's condition describes only the jump of limits of this field. In many cases, such as strong shock waves, cracks, phase changes, etc., this assumption is strongly violated. Some examples are presented below. In the present paper we extend the Kotchine's condition to cover such cases. To do so we include two phenomena: strong discontinuities of the field and surface sources.

The former is connected, for instance, with the existence of surface energy of defects, while the latter can describe the mass production due to the phase change, the entropy production due to the presence of the singular surface, local exchanges in mixtures etc. Some aspects of this problem have been discussed in [7] and in my earlier papers [4, 5, 6].

The second section of the paper contains the derivation of the generalized Kotchine's condition and a discussion of its Galilean invariance. It is problematic whether such an invariance should be required. However, the only hint we have in this matter is the invariance of the classical Rankine-Hugoniot's conditions. It is quite obvious that the classical jump conditions of mass, momentum, moment of momentum and energy are not Galilean invariant, when taken separately. For instance, the kinetic energy violates the invariance of energy balance. However, the whole set of conditions is invariant in this sense that the balance of mass, used in the balance of momentum leads to the invariant condition, and similarly—for the balance of energy. The third section is devoted to particular balance equations, corresponding to these conditions. We deliver the Galilean invariant form of these equations (in the sense to be explained further) under comparatively strong assumptions on the properties of quantities present in the equations. On the other hand, we were able to establish the invariance conditions for the surface, when the latter is considered to be embedded in the three-dimensional Euclidean space. As pointed out by D. G. B. EDELEN [2], we can expect some further intrinsic limitations.

In the fourth section we deal with the second law of thermodynamics for a singular surface.

2. Jump conditions

2.1. General balance equations for a singular surface

Let $\chi(\mathscr{P}, t)$ be a region of $\mathscr{E}^3$, occupied by a material at the instant of time t, and $\partial\chi(\mathscr{P}, t)$ be the outward oriented surface of $\chi(\mathscr{P}, t)$. Interactions of this material with the universe are described by the system of balance equations. Each of them has the following form (long-range actions neglected):

$$\frac{d\Phi}{dt}(\mathscr{P}, t) = \oint_{\partial\chi(\mathscr{P},t)} \mu d\mathscr{A} + \int_{\chi(\mathscr{P},t)} \lambda^e d\mathscr{V} + \overset{\bullet}{\Lambda}(\mathscr{P}, t). \tag{2.1}$$

The function Φ stands here for any function of a thermodynamic site of $\mathscr{P}$. Usually it is assumed that Φ is volume-continuous. However, in the presence of surface concentrations this assumption must be weakened to include the singularity on this surface. If σ is the present configuration of such a surface, the assumptions on σ being specified further in this section, then the continuity of Φ is as follows:

$$\bigwedge_{\mathscr{P}} |\Phi(\mathscr{P}, t)| \leqslant \alpha\mathscr{V}\big(\chi(\mathscr{P}, t)\big) + \beta\mathscr{A}\big(\sigma \cap \chi(\mathscr{P}, t)\big), \tag{2.2$_1$}$$

where α and β are constants, while $\mathscr{V}$ and $\mathscr{A}$ are volume and surface measures respectively. Similar conditions for fluxes have already been used in axiomatic thermodynamics (see, for instance, [5]) and lead to the following representation of Φ:

$$\Phi(\mathscr{P}, t) = \int_{\chi(\mathscr{P},t)} \varphi_v d\mathscr{V} + \int_{\sigma\cap\chi(\mathscr{P},t)} \varphi_s d\mathscr{A}. \tag{2.2$_2$}$$

Similar assumptions for the sources $\overset{\bullet}{\Lambda}$ lead to the representation

$$\overset{\bullet}{\Lambda}(\mathscr{P}, t) = \int_{\chi(\mathscr{P},t)} \overset{\bullet}{\lambda} d\mathscr{V} + \int_{\sigma\cap\chi(\mathscr{P},t)} \overset{\bullet}{\nu} d\mathscr{A}. \tag{2.2$_3$}$$

We assume that the volume density φ_v, the surface density φ_s, as well as the surface flux μ, the surface source $\overset{*}{\nu}$, the volume supply λ^e and the volume source $\overset{*}{\lambda}$ satisfy the balance equation (2.1) and certain constitutive relations. The only property of these relations used in the following sections is their invariance with respect to the constant shifting and constant rotation of an inertial reference frame in the configuration space.

Surface fluxes satisfy Noll's theorem

$$\mu_{\partial\chi(\mathscr{P},t)} = -\mu_{-\partial\chi(\mathscr{P},t)}, \tag{2.3}$$

where $-\partial\chi(\mathscr{P}, t)$ is the surface of $\chi(\mathscr{P}, t)$ with the orientation opposite to $\partial\chi(\mathscr{P}, t)$.

We assume that σ is an oriented surface of class C^2, given by the relations

$$\mathbf{x} = \boldsymbol{\xi}(a^\Delta, t), \quad \mathbf{x} \in \sigma \subset \operatorname{Int}\chi(\mathscr{P}, t), \quad \Delta = \mathrm{I}, \mathrm{II}, \tag{2.4}$$

where a^Δ are any admissible surface coordinates on σ. It is convenient to introduce a particular parametrization of $\mathscr{E}^3$, connected with the reference frame given on σ. Namely, we assume that σ is a subregion of a parametric surface $z = z_0$, while for the arbitrary point $\mathbf{x}$ of $\mathscr{E}^3$ we have

$$\mathbf{x} = \mathbf{x}_0(a^\Delta, z_0) + \mathbf{n}(z - z_0) \tag{2.5}$$

and $\mathbf{n}$ is the unit normal vector to the surface $z = z_0$. At the same time we assume that the coordinates a^Δ move with the surface σ, i.e. any point of σ has the same coordinates a^Δ during the motion. In such a case the motion of σ is given by the relation

$$z_0 = z_0(a^\Delta, t), \quad (a^\Delta) \in \sigma \tag{2.6}$$

and the speed of σ perpendicular to the surface is

$$c(a^\Delta, t) := \frac{\partial z_0}{\partial t}(a^\Delta, t); \tag{2.7}$$

c is called the speed of displacement of σ.

Let us return to the discussion of the balance equation (2.1). Bearing in mind Eqs. (2.2), we obtain the following representation for Eq. (2.1):

$$\frac{d}{dt}\int\limits_{\chi(\mathscr{P},t)} \{\varphi_v(a^\Delta, z, t) + \varphi_s(a^\Delta, z, t)\,\delta(z - z_0)\}\, d\mathscr{V} = \oint\limits_{\partial\chi(\mathscr{P},t)} \mu(a^\Delta, z, t)\, d\mathscr{A}$$
$$+ \int\limits_{\chi(\mathscr{P},t)} \overset{*}{\nu}(a^\Delta, z, t)\,\delta(z - z_0)\, d\mathscr{V} + \int\limits_{\chi(\mathscr{P},t)} \{\lambda^e(a^\Delta, z, t) + \lambda^*(a^\Delta, z, t)\}\, d\mathscr{V}, \tag{2.8}$$

where $\delta(z - z_0)$ is the Dirac function for the surface σ. We assume that the balance equation (2.8) holds for any material subregion $\chi(\mathscr{P}, t)$, either containing σ in $\operatorname{Int}\chi(\mathscr{P}, t)$ or having no common points with σ.

The field φ_v is said to be smooth in $\chi(\mathscr{P}, t)$ if:

i) it is continuously differentiable in $\chi^+(\mathscr{P}, t)$ and $\chi^-(\mathscr{P}, t)$;

ii) it approaches finite limits $\varphi^+(a^\Delta, z_0, t)$ and $\varphi_v^-(a^\Delta, z_0, t)$ for $\mathscr{A}$—almost every point of σ;

iii) the fields λ^e and λ^* are continuous in $\chi(\mathscr{P}, t)$;

iv) surface fluxes μ approach finite limits $\mu^{\pm}(a^1, z_0, t)$ for $\mathscr{A}$—almost every point of σ;

v) the velocity field $\dot{\mathbf{x}}$ approaches finite limits $\dot{\mathbf{x}}^{\pm}(a^1, z_0, t)$ for $\mathscr{A}$—almost every point of σ;

vi) the motion ψ approaches the finite limits $\psi^{\pm}(a^1, z_0, t)$ for $\mathscr{A}$—almost every point of σ.

The regions $\chi^+(\mathscr{P}, t)$ and $\chi^-(\mathscr{P}, t)$ are supplementary parts of $\chi(\mathscr{P}, t)$, divided by any smooth extension of σ (Fig. 1).

The surface σ is said to be singular with respect to the balance equation (2.1) if either $\varphi_s \neq 0$, or $\dot{\nu} \neq 0$, or at least one of the above fields had different limits on the positive and negative sides of σ.

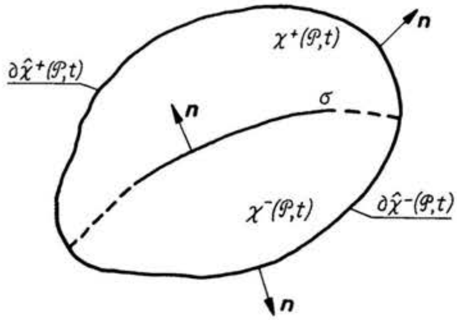

FIG. 1.

The balance equation for such a surface can be derived in a standard manner using Eq. (2.8). Let us write the field φ_v in the following form:

$$\varphi_v(a^1, z, t) = \eta(z-z_0)\varphi_v^+(a^1, z, t) + [1-\eta(z-z_0)]\varphi_v^-(a^1, z, t), \tag{2.9}$$

where η is the Heaviside function, φ_v^+ is an extension of $\varphi_v|_{\chi^+(\mathscr{P},t)}$ to $\chi(\mathscr{P}, t)$ and similarly for φ_v^-. The time differentiation in Eq. (2.8) yields

$$\frac{d}{dt}\int\limits_{\chi(\mathscr{P},t)} \{\varphi_v(a^1, z, t) + \varphi_s(a^1, z, t)\,\delta(z-z_0)\}\,d\mathscr{V} = \frac{d}{dt}\int\limits_{\chi(\mathscr{P},t)} \{\eta(z-z_0)\varphi_v^+(a^1, z, t)$$
$$+ [1-\eta(z-z_0)]\varphi_v^-(a^1, z, t) + \varphi_s(a^1, z, t)\,\delta(z-z_0)\}\,d\mathscr{V}$$
$$= \int\limits_{\chi(\mathscr{P},t)} \Big\{-\delta(z-z_0)\varphi_v^+(a^1, z, t)\dot{z}_0(a^1, t) + \eta(z-z_0)\frac{\partial\varphi_v^+}{\partial t}(a^1, z, t)$$
$$+ \delta(z-z_0)\varphi_v^-(a^1, z, t)\dot{z}_0(a^1, t) + [1-\eta(z-z_0)]\frac{\partial\varphi_v^-}{\partial t}(a^1, z, t) + \frac{\partial\varphi_s}{\partial t}(a^1, z, t)\,\delta(z-z_0)$$
$$-\varphi_s(a^1, z, t)\,\delta'(z-z_0)\dot{z}_0(a^1, t)\Big\}\,d\mathscr{V} + \oint\limits_{\partial\chi(\mathscr{P},t)} \{\eta(z-z_0)\varphi_v^+(a^1, z, t)$$
$$+ [1-\eta(z-z_0)]\varphi_v^-(a^1, z, t) + \varphi_s(a^1, z, t)\,\delta(z-z_0)\}\,\dot{x}_{\mathbf{n}}\,d\mathscr{A}.$$

Hence

$$(2.10)\qquad \frac{d\Phi}{dt}(\mathscr{P},t) = \int_{\sigma}\left\{-[\![\varphi_v(a^\Gamma,t)]\!]\dot{z}_0(a^\Gamma,t)+\frac{\partial\varphi_s}{\partial t}(a^\Gamma,z,t)+\frac{\partial\varphi_s}{\partial z}(a^\Gamma,z_0,t)\dot{z}_0(a^\Delta,t)\right\}d\mathscr{A}$$

$$+\oint_{\partial\chi(\mathscr{P},t)}\varphi_v(a^\Gamma,z,t)\dot{x}_{\mathbf{n}}d\mathscr{A}+\int_{\chi(\mathscr{P},t)}\frac{\partial\varphi_v}{\partial t}(a^\Gamma,z,t)d\mathscr{V},$$

where

$$(2.11)\qquad [\![\varphi_v(a^\Gamma,t)]\!]: = \varphi_v^+(a^\Gamma,z_0,t)-\varphi_v^-(a^\Gamma,z_0,t).$$

Simultaneously with Christoffel's brackets above we use further the following notation

$$(2.12)\qquad \langle\varphi_v(a^\Gamma,t)\rangle: = \frac{1}{2}\{\varphi_v^+(a^\Gamma,z_0,t)+\varphi_v^-(a^\Gamma,z_0,t)\},$$

and similarly for other quantities.

Now, shrinking down $\chi^+(\mathscr{P},t)$ and $\chi^-(\mathscr{P},t)$ to σ and bearing in mind the above continuity assumptions, we obtain from Eqs. (2.10) and (2.8)

$$(2.13)\qquad \int_{\sigma}\left\{-[\![\varphi_v]\!]c+\frac{\partial\varphi_s}{\partial t}-\Delta c+[\![\varphi_v\dot{x}_{\mathbf{n}}]\!]\right\}d\mathscr{A} = \int_{\sigma}\{[\![\mu]\!]+\overset{*}{\nu}\}d\mathscr{A},$$

where we have made use of (2.7) and introduced the notation

$$(2.14)\qquad \Delta(a^\Gamma,t): = -\frac{\partial\varphi_s}{\partial z}(a^\Gamma,z_0,t).$$

Here the arguments a^Γ, t are neglected to simplify the notation. If we assume that the integrands in the above relation are surface continuous, then we obtain the local relation of the form

$$(2.15)\qquad [\![\varphi_v U]\!]+\Delta c+[\![\mu]\!]+\overset{*}{\nu}-\frac{\partial\varphi_s}{\partial t} = 0,\quad \mathscr{A}\text{—a.e.}$$

where

$$(2.16)\qquad U^{\pm}: = c-\dot{x}_{\mathbf{n}}^{\pm}$$

are the speeds of propagation of the surface σ relative to the material particles instantaneously located on the positive and negative sides of σ.

It is convenient to deal rather with densities per unit mass than per unit volume. Therefore we define the following fields:

$$(2.17)\qquad \varrho^{\pm}\varphi^{\pm}: = \varphi_v^{\pm},$$

$\varrho^{\pm}$ being the limits of the mass density on σ. Hence

$$(2.18)\qquad [\![\varrho U\varphi]\!]+\Delta c+[\![\mu]\!]+\overset{*}{\nu}-\frac{\partial\varphi_s}{\partial t} = 0,\quad \mathscr{A}\text{—a.e.}$$

The above formula is the balance equation for a singular surface.

In the simplest case

$$\varphi_s \equiv 0 \quad (\Rightarrow \varDelta \equiv 0), \qquad \overset{*}{\nu} \equiv 0, \tag{2.19}$$

we obtain from Eq. (2.18) the classical Kotchine's jump condition

$$[\![\varrho U \varphi]\!] + [\![\mu]\!] = 0. \tag{2.20}$$

2.2. Galilean invariance

As we have already mentioned, we do not specify any particular constitutive relations to be satisfied by the quantities of the relation (2.18). However, some of the variables, opening these relations must be listed to find the conditions for the Galilean invariance of Eq. (2.18). Therefore we assume that among other variables we take into account the following ones:

$$[\![\dot{\mathbf{x}}]\!],\ \langle \dot{\mathbf{x}} \rangle,\ U^{\pm}. \tag{2.21}$$

With respect to the relation (2.16), we have

$$c = \langle U \rangle + \langle \dot{x}_n \rangle, \tag{2.22}$$

and

$$\dot{\mathbf{x}}^+ = \langle \dot{\mathbf{x}} \rangle + 0.5[\![\dot{\mathbf{x}}]\!], \qquad \dot{\mathbf{x}}^- = \langle \dot{\mathbf{x}} \rangle - 0.5[\![\dot{\mathbf{x}}]\!]. \tag{2.23}$$

It means that the fields (2.21) define all remaining velocity fields on σ. Let us notice that in the points $\mathbf{x} \in \chi(\mathscr{P}, t) \setminus \sigma$ we have $[\dot{\mathbf{x}}] \equiv 0$, $\langle \dot{\mathbf{x}} \rangle = \dot{\mathbf{x}}$ and hence we may consider the fields (2.21) as describing the velocity fields on $\chi(\mathscr{P}, t)$. According to these remarks we have the following constitutive relations:

$$\begin{aligned}
\varrho(\mathbf{x}, t) &= \mathscr{R}([\![\dot{\mathbf{x}}]\!], \langle \dot{\mathbf{x}} \rangle, U^+, U^- \ldots),\\
\varphi(\mathbf{x}, t) &= \mathscr{F}([\![\dot{\mathbf{x}}]\!], \langle \dot{\mathbf{x}} \rangle, U^+, U^-, \ldots),\\
\mu(\mathbf{x}, t) &= \mathscr{M}([\![\dot{\mathbf{x}}]\!], \langle \dot{\mathbf{x}} \rangle, U^+, U^-, \ldots),\\
\varDelta(\mathbf{x}, t) &= \mathscr{D}([\![\dot{\mathbf{x}}]\!], \langle \dot{\mathbf{x}} \rangle, U^+, U^-, \ldots),\\
\overset{*}{\nu}(\mathbf{x}, t) &= \mathscr{N}([\![\dot{\mathbf{x}}]\!], \langle \dot{\mathbf{x}} \rangle, U^+, U^-, \ldots),\\
\varphi_s(\mathbf{x}, t) &= \mathscr{F}_s([\![\dot{\mathbf{x}}]\!], \langle \dot{\mathbf{x}} \rangle, U^+, U^-, \ldots),
\end{aligned} \tag{2.24}$$

where dots stay for all remaining independent constitutive variables, while $\mathscr{R}, \mathscr{F}, \mathscr{M}, \mathscr{D}, \mathscr{N}, \mathscr{F}_s$ are constitutive functionals which satisfy certain additional restrictions. This has been discussed in many papers on the thermodynamic theory of materials.

It is a matter of simple calculations to find the general conditions for Eq. (2.8) to be invariant with respect to the change of the inertial reference frame in the three-dimensional configuration space. Generally speaking we have to make the proper choice of the form of surface sources.

Let us perform these calculations for the infinitesimal transformation

$$\mathbf{x} \to \mathbf{x} + \boldsymbol{\epsilon} t, \quad \boldsymbol{\epsilon} = \text{const}, \quad \frac{1}{|\dot{\mathbf{x}}|}|\boldsymbol{\epsilon}| \ll 1. \tag{2.25}$$

Then, we have

$$\begin{aligned} \dot{\mathbf{x}} &\to \dot{\mathbf{x}} + \boldsymbol{\epsilon}, \\ \mathbf{x}^{\pm} &\to \dot{\mathbf{x}}^{\pm} + \boldsymbol{\epsilon}, \\ U^{\pm} &\to U^{\pm}, \\ c &\to c + \boldsymbol{\epsilon} \cdot \mathbf{n}, \\ [\![\dot{\mathbf{x}}]\!] &\to [\![\dot{\mathbf{x}}]\!], \\ \langle \dot{\mathbf{x}} \rangle &\to \langle \dot{\mathbf{x}} \rangle + \boldsymbol{\epsilon}. \end{aligned} \tag{2.26}$$

Eliminating c from Eq. (2.18) using Eq. (2.22) and taking into account the constitutive relations (2.24), we see that the change of the relation (2.18) due to the transformation in Eq. (2.25) is connected with the presence of $\langle \dot{\mathbf{x}} \rangle$. If $\delta_{\langle \dot{\mathbf{x}} \rangle}$ is the Frechet's derivative, then the Galilean invariance condition of Eq. (2.18) takes the following form:

$$[\![\delta_{\langle \dot{\mathbf{x}} \rangle}(\varrho\varphi) U]\!] + \delta_{\langle \dot{\mathbf{x}} \rangle} \Delta c + \Delta \mathbf{n} + \delta_{\langle \dot{\mathbf{x}} \rangle} \overset{\bullet}{\nu} + [\![\delta_{\langle \dot{\mathbf{x}} \rangle} \mu]\!] - \delta_{\langle \dot{\mathbf{x}} \rangle} \frac{\partial \varphi_s}{\partial t} = 0. \tag{2.27}$$

I have not been able to find the general solution of this differential condition and I doubt if it could be done without any further assumptions on the form of ϱ, φ, Δ, $\overset{\bullet}{\nu}$, μ and φ_s. The next Section contains some particular solutions of Eq. (2.27), which cover the most common cases.

3. Rankine-Hugoniot's conditions

3.1. Balance of mass

In this case the functions of Eq. (2.18) are as follows:

$$\varphi =: 1, \quad \Delta =: \beta, \quad \mu \equiv 0, \quad \overset{\bullet}{\nu} = \overset{\bullet}{\varrho}, \quad \varphi_s =: \varrho_s \tag{3.1}$$

and the balance equation takes the form

$$[\![\varrho U]\!] + \beta \langle U + \dot{x}_{\mathbf{n}} \rangle + \overset{\bullet}{\varrho} - \frac{\partial \varrho_s}{\partial t} = 0. \tag{3.2}$$

In the above relations β is the mass concentration on σ, $\varrho^{\pm}$ are limits of the mass density on σ, $\overset{\bullet}{\varrho}$ is a surface mass source. Such a term occurs, for instance, in the case of a phase change. ϱ_s is the singularity of mass density on σ due to changes in time of the mass concentration on σ.

The sufficient condition for the Galilean invariance of Eq. (3.2) can be written in the following form:

$$[\![U \delta_{\langle \dot{\mathbf{x}} \rangle} \varrho]\!] + \langle U + \dot{x}_{\mathbf{n}} \rangle \delta_{\langle \dot{\mathbf{x}} \rangle} \beta + \beta \mathbf{n} + \delta_{\langle \dot{\mathbf{x}} \rangle} \overset{\bullet}{\varrho} - \delta_{\langle \dot{\mathbf{x}} \rangle} \frac{\partial \varrho_s}{\partial t} = 0 \tag{3.3}$$

for any $\dot{\mathbf{x}}^+$ and $\dot{\mathbf{x}}^-$. We find the solution of this relation assuming that

$$\delta_{\langle\dot{\mathbf{x}}\rangle}\varrho \equiv 0, \quad \delta_{\langle\dot{\mathbf{x}}\rangle}\beta \equiv 0, \quad \delta_{\langle\dot{\mathbf{x}}\rangle}\frac{\partial\varrho_s}{\partial t} \equiv 0. \tag{3.4}$$

It seems to cover all known cases which appear in physical applications. On the other hand, for $\delta_{\langle\dot{\mathbf{x}}\rangle}\beta \neq 0$ we can expect, due to the relation (2.14), that $\delta_{\langle\dot{\mathbf{x}}\rangle}\frac{\partial\varrho_s}{\partial t} \neq 0$. The latter poses many difficulties connected with the existence of the time derivatives of $\langle\dot{\mathbf{x}}\rangle$ on the singular surface σ. I have not been able to solve this problem.

The dependence of ϱ on $\langle\dot{\mathbf{x}}\rangle$ does not change the procedure of the considerations below and leads to the same results. We neglect this term in Eq. (3.3) to simplify the calculations.

Bearing in mind Eqs. (3.4), we have

$$\beta\mathbf{n} + \delta_{\langle\dot{\mathbf{x}}\rangle}\overset{\bullet}{\varrho} = 0. \tag{3.5}$$

To find the solution of Eq. (3.5) we assume in addition that $\overset{\bullet}{\varrho}$ is an analytic function with respect to $\langle\dot{\mathbf{x}}\rangle$, i.e. it can be expanded into the uniformly convergent power series with respect to $\langle\dot{x}\rangle$

$$\overset{\bullet}{\varrho} = \overset{\bullet}{\varrho}_0 + \overset{\bullet}{\boldsymbol{\rho}}_1\cdot\langle\dot{\mathbf{x}}\rangle + \langle\dot{\mathbf{x}}\rangle\cdot\overset{\bullet}{\boldsymbol{\rho}}_2\langle\dot{\mathbf{x}}\rangle + \ldots \tag{3.6}$$

where $\overset{\bullet}{\boldsymbol{\rho}}_0, \overset{\bullet}{\boldsymbol{\rho}}_1, \overset{\bullet}{\boldsymbol{\rho}}_2, \ldots$ depend on all variables of the relations (2.24) but $\langle\dot{\mathbf{x}}\rangle$. The substitution of Eq. (3.6) in Eq. (3.5) yields

$$\beta\mathbf{n} + \overset{\bullet}{\boldsymbol{\rho}}_1 + (\overset{\bullet}{\boldsymbol{\rho}}_2 + \ldots)\langle\dot{\mathbf{x}}\rangle = 0 \tag{3.7}$$

for all $\langle\dot{\mathbf{x}}\rangle$. It means

$$\beta\mathbf{n} + \overset{\bullet}{\boldsymbol{\rho}}_1 = 0, \quad \overset{\bullet}{\boldsymbol{\rho}}_2 = \ldots = 0. \tag{3.8}$$

Finally, the balance law (3.2) takes the form

$$[\![\varrho U]\!] + \beta\langle U\rangle + \overset{\bullet}{\varrho}_0 - \frac{\partial\varrho_s}{\partial t} = 0. \tag{3.9}$$

The relation (3.9) is the general invariant form of the mass surface balance under the assumption (3.4) in the presence of both surface sources and concentrations. We do not pretend to introduce physical interpretations of β, $\overset{\bullet}{\varrho}_0$ and ϱ_s. However, two particular cases are obvious. First, for β and ϱ_s being identically zero, we have

$$[\![\varrho U]\!] + \overset{\bullet}{\varrho}_0 = 0, \quad \overset{\bullet}{\varrho}([\![\dot{\mathbf{x}}]\!]) = \text{const}. \tag{3.10}$$

This relation describes the speed of propagation of the surface dividing two phases of the material, one of the density ϱ^-, the other of ϱ^+, when the speed of the phase change is $\overset{\bullet}{\varrho}_0$.

On the other hand, the relation (3.9) can be used in the case of very high gradients of the density occurring, for instance, in nuclear explosions after a short time-lapse. We demonstrate this example of a shock wave at the end of this section. In such a case, we assume

$$\overset{\bullet}{\varrho}_0 = -\alpha[\![\dot{x}_n]\!] = \alpha[\![U]\!], \quad \alpha([\![\dot{\mathbf{x}}]\!]) = \text{const}. \tag{3.11}$$

If we define

$$\bar{\varrho}^{+} := \varrho^{+}+0.5\alpha, \quad \bar{\varrho}^{-} := \varrho^{-}+0.5\alpha, \tag{3.12}$$

then

$$[\![\varrho U]\!]+\overset{\bullet}{\varrho}_0 = [\![\varrho U]\!]+\alpha[\![U]\!] = \langle\varrho\rangle[\![U]\!]+\alpha[\![U]\!]+[\![\varrho]\!]\langle U\rangle = \langle\bar{\varrho}\rangle[\![U]\!]+[\![\bar{\varrho}]\!]\langle U\rangle = [\![\bar{\varrho}U]\!].$$

Hence

$$[\![\bar{\varrho}U]\!] = 0. \tag{3.13}$$

Let us introduce the following notation:

$$\bar{m} := -\langle\bar{\varrho}U\rangle, \quad \bar{\mathscr{V}}^{\pm} := (\bar{\varrho}^{\pm})^{-1}. \tag{3.14}$$

Making use of Eqs. (3.14) in Eq. (3.13), we find

$$[\![U]\!] = -\bar{m}[\![\bar{\mathscr{V}}]\!] \Rightarrow [\![\dot{x}_n]\!] = \bar{m}[\![\bar{\mathscr{V}}]\!]. \tag{3.15}$$

Hence the linear dependence of the mass sources on $[\![\dot{\mathbf{x}}]\!]$, given by Eq. (3.11), leads to the linear relation between the discontinuity of the normal component of the particle velocity and the jump of the value of the specific volume, modified by 0.5α, which describes the influence of the strong discontinuity of the mass density on the surface.

3.2. Balance of momentum

In this case we have

$$\varphi =: \dot{\mathbf{x}}, \quad \Delta =: \mathbf{p}, \quad \mu = \mathbf{Tn}, \quad \overset{\bullet}{\nu} =: \overset{\bullet}{\mathbf{t}}, \quad \varphi_s =: \boldsymbol{\tau}_s, \tag{3.16}$$

where $\mathbf{p}$ describes the concentration of momentum on the surface σ, $\mathbf{Tn}$ is the limit of traction from both sides of σ, while $\mathbf{T}$ is Cauchy's stress tensor; $\overset{\bullet}{\mathbf{t}}$ is the surface source of momentum and $\partial\boldsymbol{\tau}_s/\partial t$ describes the time changes of the momentum concentration carried by σ. In general the jump condition (2.18) for momentum takes the following form:

$$[\![\varrho U\dot{\mathbf{x}}]\!]+\mathbf{p}\langle U+\dot{x}_n\rangle+[\![\mathbf{Tn}]\!]+\overset{\bullet}{\mathbf{t}}-\frac{\partial\boldsymbol{\tau}_s}{\partial t} = 0. \tag{3.17}$$

The Galilean invariance condition for Eq. (3.17) is as follows:

$$[\![\varrho U]\!]\mathbf{1}+\delta_{\langle\dot{\mathbf{x}}\rangle}\mathbf{p}\langle U+\dot{x}_n\rangle+\mathbf{p}\otimes\mathbf{n}+[\![\delta_{\langle\dot{\mathbf{x}}\rangle}\mathbf{Tn}]\!]+\delta_{\langle\dot{\mathbf{x}}\rangle}\overset{\bullet}{t}-\delta_{\langle\dot{\mathbf{x}}\rangle}\frac{\partial\boldsymbol{\tau}_s}{\partial t} = 0. \tag{3.18}$$

Now we make two assumptions. First of all, we assume that the jump condition for mass in its invariant form is the sufficient condition for Eq. (3.17) to be invariant. It is a matter of elementary calculations to prove that such a condition holds in the case of classical relations:

$$[\![\varrho U]\!] = 0 \Rightarrow \langle\varrho U\rangle[\![\dot{\mathbf{x}}]\!]+[\![\mathbf{Tn}]\!] = 0.$$

At the same time, we assume that $\mathbf{p}, \mathbf{Tn}, \dfrac{\partial\boldsymbol{\tau}_s}{\partial t}$ do not depend on $\langle\dot{\mathbf{x}}\rangle$. Hence

$$[\![\delta U]\!]\mathbf{1}+\mathbf{p}\otimes\mathbf{n}+\delta_{\langle\dot{\mathbf{x}}\rangle}\overset{\bullet}{\mathbf{t}} = 0$$

for any $\langle\dot{\mathbf{x}}\rangle$.

After considerations similar to those of Sect. 3.1, we get

$$\overset{\bullet}{\mathbf{t}} = \overset{\bullet}{\mathbf{t}}_0 - [\![\varrho U]\!] \langle \dot{\mathbf{x}} \rangle - \mathbf{p} \langle \dot{x}_{\mathbf{n}} \rangle. \tag{3.19}$$

Making use of Eq. (3.19) in Eq. (3.17), we obtain

$$\langle \varrho U \rangle [\![\dot{\mathbf{x}}]\!] + \mathbf{p} \langle U \rangle + [\![\mathbf{Tn}]\!] + \overset{\bullet}{\mathbf{t}}_0 - \frac{\partial \boldsymbol{\tau}_s}{\partial t} = 0. \tag{3.20}$$

The relation (3.20) is the invariant jump condition for momentum; it reduces to the classical condition for $\boldsymbol{\tau}_s \equiv 0$, $\overset{\bullet}{\mathbf{t}}_0 \equiv 0$ and $\mathbf{p} \equiv 0$.

3.3. Balance of moment of momentum

In the case of a classical continuum (without couple stresses) we have

$$\varphi =: \mathbf{r} \wedge \dot{\mathbf{x}}, \quad \Delta = \mathbf{r} \wedge \mathbf{p}, \quad \mu =: \mathbf{r} \wedge \mathbf{Tn}, \quad \overset{\bullet}{\nu} =: \mathbf{r} \wedge \overset{\bullet}{\mathbf{t}}, \quad \varphi_s =: \mathbf{r} \wedge \boldsymbol{\tau}_s \tag{3.21}$$

where $\mathbf{r}$ is the position vector with respect to a chosen origin in the configuration space $\mathscr{E}^3$. From Eq. (2.18) it follows that

$$[\![\varrho U \mathbf{r} \wedge \dot{\mathbf{x}}]\!] + \mathbf{r} \wedge \mathbf{p} c + [\![\mathbf{r} \wedge \mathbf{Tn}]\!] + \mathbf{r} \wedge \overset{\bullet}{\mathbf{t}} - \frac{\partial}{\partial t}(\mathbf{r} \wedge \boldsymbol{\tau}_s) = 0$$

or, taking into account the balance of momentum (3.17),

$$\mathbf{n} \wedge \boldsymbol{\tau}_s = 0. \tag{3.22}$$

We have made use of the formulae

$$[\![\mathbf{r}]\!] \equiv 0, \quad \frac{\partial \mathbf{r}}{\partial t} = c\mathbf{n}, \tag{3.23}$$

the second one following from Eqs. (2.5) and (2.7).

The relation (3.22) is obviously Galilean invariant under the assumptions made for the balance of momentum, that is

$$\delta_{\langle \dot{\mathbf{x}} \rangle} \boldsymbol{\tau}_s \equiv 0 \tag{3.24}$$

and means that the momentum singularity $\boldsymbol{\tau}_s$ must be tangent to σ.

3.4. Balance of energy

Again, considering the classical continuous medium we have the following definitions

$$\varphi =: e + \frac{1}{2}\dot{x}^2, \quad \Delta =: \varkappa, \quad \mu =: \dot{\mathbf{x}} \cdot \mathbf{Tn} + q, \quad \overset{\bullet}{\nu} =: \overset{\bullet}{q}, \quad \varphi_s =: e_s, \tag{3.25}$$

where e is the specific energy, $\varkappa$ surface energy, q heat flux, $\overset{\bullet}{q}$ energy surface source, e_s surface singularity of energy. The substitution of Eqs. (3.25) in Eq. (2.18) yields the following result:

$$\left[\!\!\left[\varrho U \left(e + \frac{1}{2}\dot{x}^2 \right) \right]\!\!\right] + \varkappa \langle U + \dot{x}_{\mathbf{n}} \rangle + [\![\dot{\mathbf{x}} \cdot \mathbf{Tn} + q]\!] + \overset{\bullet}{q} - \frac{\partial e_s}{\partial t} = 0. \tag{3.26}$$

The Galilean invariance of this relation is ensured by the following condition:

$$(3.27)\qquad \delta_{\langle\dot{\mathbf{x}}\rangle}\left[\!\!\left[\varrho U\left(e+\frac{1}{2}\dot{x}^2\right)\right]\!\!\right]+\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa\langle U+\dot{x}_{\mathbf{n}}\rangle+\varkappa\mathbf{n}+(\delta_{\langle\dot{\mathbf{x}}\rangle}\langle\mathbf{Tn}\rangle)^T[\![\dot{\mathbf{x}}]\!]$$
$$+[\![\mathbf{Tn}]\!]+(\delta_{\langle\dot{\mathbf{x}}\rangle}[\![\mathbf{Tn}]\!])^T\langle\dot{\mathbf{x}}\rangle+\delta_{\langle\dot{\mathbf{x}}\rangle}\overset{*}{q}-\delta_{\langle\dot{\mathbf{x}}\rangle}\frac{\partial e_s}{\partial t}=0.$$

As previously, we make two assumptions: i) the jump conditions for mass and momentum are sufficient for the invariance required by Eq. (3.27); ii) $e, \varkappa, q$ and e_s are independent of $\langle\dot{\mathbf{x}}\rangle$.

In such a case, the condition (3.27) takes the following form:

$$(3.28)\qquad [\![\varrho U\dot{\mathbf{x}}]\!]+[\![\mathbf{Tn}]\!]+\varkappa\mathbf{n}+\delta_{\langle\dot{\mathbf{x}}\rangle}\overset{*}{q}=0.$$

If $\overset{*}{q}$ can be expanded into the uniformly convergent power series with respect to $\langle\dot{\mathbf{x}}\rangle$, the solution of Eq. (3.28) is as follows:

$$(3.29)\qquad \overset{*}{q}=\overset{*}{q}_0-\varkappa\langle\dot{x}_{\mathbf{n}}\rangle-\langle\varrho U\rangle[\![\dot{x}^2]\!]-\frac{1}{2}[\![\varrho U]\!]\langle\dot{\mathbf{x}}\rangle^2-\langle\dot{x}\rangle\cdot[\![\mathbf{Tn}]\!].$$

The substitution of (3.29) in (3.26) yields the invariant form of the energy balance

$$(3.30)\qquad [\![\varrho Ue]\!]+\frac{1}{8}[\![\varrho U]\!][\![\dot{x}]\!]^2+\varkappa\langle U\rangle+[\![\dot{\mathbf{x}}]\!]\cdot\langle\mathbf{Tn}\rangle+[\![q]\!]+\overset{*}{q}_0-\frac{\partial e_s}{\partial t}=0.$$

Assuming that all sources are equal to zero, we obtain the classical condition

$$(3.31)\qquad \langle\varrho U\rangle[\![e]\!]+[\dot{\mathbf{x}}]\cdot\langle\mathbf{Tn}\rangle+[q]=0.$$

Let us notice that even the presence of mass sources alone either leads to non-invariant conditions for momentum and energy or requires the existence of certain momentum and energy sources to balance the non-invariant terms. For instance, if we put $\overset{*}{t}=\boldsymbol{\tau}_s\equiv 0$, $\overset{*}{q}=e_s\equiv 0$ and keep $[\![\varrho U]\!]\neq 0$, the relations (3.17) and (3.26) take the form

$$[\![\varrho U]\!]\langle\dot{\mathbf{x}}\rangle+\langle\varrho U\rangle[\![\dot{\mathbf{x}}]\!]+[\![\mathbf{Tn}]\!]=0,$$

$$[\![\varrho Ue]\!]+\frac{1}{8}[\![\varrho U]\!][\![\dot{x}]\!]^2+\langle\varrho U\rangle\langle\dot{x}\rangle\cdot[\![\dot{\mathbf{x}}]\!]+\frac{1}{2}[\![\varrho U]\!]\langle\dot{x}\rangle^2$$
$$+\varkappa\langle U+x_{\mathbf{n}}\rangle+[\![\dot{\mathbf{x}}]\!]\cdot\langle\mathbf{Tn}\rangle+\langle\dot{\mathbf{x}}\rangle\cdot[\mathbf{Tn}]+[\![q]\!]=0,$$

which evidently cannot be transformed into invariant conditions.

3.5. Surface energy

It is worth mentioning a particular case in which the only strong discontinuity, present in the jump conditions is $\varkappa$. In such a case the relations (3.2), (3.17) and (3.26) have the following form:

$$(3.32)\qquad [\![\varrho U]\!]+\overset{*}{\varrho}=0,$$

$$(3.33)\qquad [\![\varrho U\mathbf{x}]\!]+[\![\mathbf{Tn}]\!]+\overset{*}{\mathbf{t}}=0,$$

$$(3.34)\qquad \left[\!\!\left[\varrho U\left(e+\frac{1}{2}\dot{x}^2\right)\right]\!\!\right]+\varkappa c+[\![\dot{\mathbf{x}}\cdot\mathbf{Tn}+q]\!]+\overset{*}{q}=0.$$

As we can see further, this case yields non-trivial surface sources. The mass source $\overset{*}{\varrho}$ does not vanish in spite of the absence of any phase transition. This case seems to be of special interest in the theory of defects.

On the other hand, we do not have to demand that

$$\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa = 0, \tag{3.35}$$

as opposed to the cases described in the preceding subsections. However, we replace this condition by the following one

$$\delta_{\langle\dot{\mathbf{x}}\rangle}\overset{*}{q} = 0, \tag{3.36}$$

which may be read as the pure thermal character of the energy surface sources. The influence of kinematics is taken care of by the dependence of $\varkappa$ on $[\![\dot{\mathbf{x}}]\!]$ and, certainly, by a possible dependence of both $\overset{*}{q}$ and $\varkappa$ on $U^{\pm}$.

Let us make use of the Galilean invariance condition in Eq. (3.34). Then

$$[\![\varrho U\dot{\mathbf{x}}+\mathbf{Tn}]\!]+\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa c+\varkappa\mathbf{n} = 0. \tag{3.37}$$

Using Eq. (3.33) we obtain

$$\overset{*}{\mathbf{t}} = c\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa+\varkappa\mathbf{n}. \tag{3.38}$$

Hence the surface energy $\varkappa$ is a "potential" for surface sources of momentum.

Now, the Galilean invariance of Eq. (3.33) is ensured by the condition

$$[\![\varrho U]\!]\mathbf{1}+\delta_{\langle\dot{\mathbf{x}}\rangle}\overset{*}{\mathbf{t}} = 0. \tag{3.39}$$

Hence, making use of Eq. (3.32) we get

$$\overset{*}{\varrho} = \frac{1}{3}\operatorname{tr}\delta_{\langle\dot{\mathbf{x}}\rangle}\overset{*}{t}. \tag{3.40}$$

If we substitute Eq. (3.38) in Eq. (3.40), we obtain

$$\overset{*}{\varrho} = \frac{1}{3}\operatorname{tr}(c\delta_{\langle\dot{\mathbf{x}}\rangle}\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa+\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa\otimes\mathbf{n}), \tag{3.41}$$

which means that the surface energy $\varkappa$ creates certain mass surface sources. Taking into account Eqs. (3.38) and (3.41), we finally obtain the following invariant jump conditions for this case:

$$\begin{gathered}
[\![\varrho U]\!]+\frac{1}{3}\operatorname{tr}(c\delta_{\langle\dot{\mathbf{x}}\rangle}\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa+\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa\otimes\mathbf{n}) = 0,\\
[\![\varrho U\dot{\mathbf{x}}]\!]+[\![\mathbf{Tn}]\!]+c\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa+\varkappa\mathbf{n} = 0,\\
\left[\!\!\left[\varrho U\left(e+\frac{1}{2}\dot{x}^2\right)\right]\!\!\right]+\varkappa c+[\![\dot{\mathbf{x}}\cdot\mathbf{Tn}+q]\!]+\overset{*}{q} = 0.
\end{gathered} \tag{3.42}$$

Let us consider an even more particular case of the singular surface σ, which is not a shock wave (i.e. $[\![\dot{\mathbf{x}}]\!]\equiv 0$), and the surface energy gradients do not suffer a jump (i.e. $\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa = \delta_{\langle\dot{\mathbf{x}}\rangle}\delta_{\langle\dot{\mathbf{x}}\rangle}\varkappa \equiv 0$). Then

$$\begin{gathered}
[\![\varrho]\!] = 0,\\
[\![\mathbf{Tn}]\!]+\varkappa\mathbf{n} = 0,\\
m([\![\varrho e]\!]+\varkappa)-[\![q]\!]-\overset{*}{q} = 0, \quad m := -\varrho U.
\end{gathered} \tag{3.42}$$

It is easily seen that the above conditions lead directly to basic results of the theory of capillarity. Making the assumption on the purely mechanical character of this phenomenon

(3.43) $$[q]+\overset{*}{q} = 0,$$

we have

(3.44) $$\varkappa = -\mathbf{n}\cdot[\![\mathbf{Tn}]\!], \quad \mathbf{n}\wedge[\![\mathbf{Tn}]\!] = 0,$$
$$[\![\varrho e]\!]+\varkappa = 0, \quad m \neq 0.$$

The above relations mean that the surface σ, under the assumptions described, has the following properties:

i) a tangent component of the stress vector **Tn** is continuous through σ;

ii) the surface energy $\varkappa$ is numerically equal to the surface tension (Maxwell's "surface pressure")—$\mathbf{n}\cdot[\![\mathbf{Tn}]\!]$; for fluids, the relation $(3.44)_1$ takes the form

(3.45) $$\varkappa = [p],$$

where p is a pressure;

iii) the surface energy $\varkappa$ describes the change of internal energy $[\![e]\!]$ due to the capillarity which is considered to be a formation of a new surface. Otherwise, the surface σ is material and $m \equiv 0$, which means that the energy balance equation $(3.42)_3$ holds trivially.

3.6. Example

The last result of the previous subsection seems to be sufficiently impressive to support the idea of the surface sources. However, I was not able to deliver a numerical example in which $\varkappa$ would be specified explicitly. Therefore, the calculations presented below

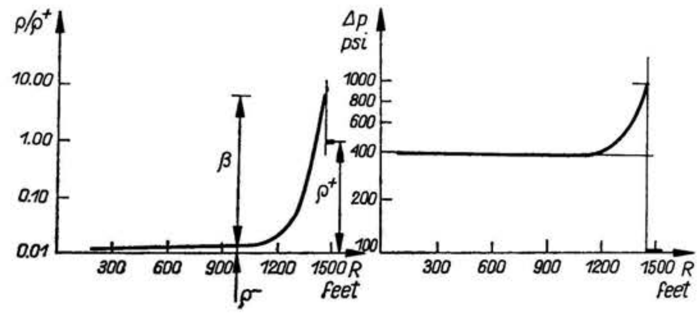

FIG. 2.

concern a nuclear explosion of 1MT power on sea level. The numerical data taken from the paper of H. L. BRODE [1], prove that the classical Kotchine's condition does not work in this case, while the relations of this paper give reasonable results. As an example we present the computation of the speed of the shock wave front with respect to the origin placed in the centre of explosion after the time-lapse $t = .074$ sec following the explosion. The data required in the jump conditions of mass and momentum are as follows (Fig. 2):

(3.46) $$\varrho^- = 0.01\times10^{-3} \quad g\times cm^{-3},$$

(3.46) [cont.]
$$\varrho^- + \beta = 6.00\times 10^{-3} \quad \mathrm{g}\times\mathrm{cm}^{-3},$$
$$\varrho^+ = 1.20\times 10^{-3} \quad \mathrm{g}\times\mathrm{cm}^{-3},$$
$$\mathbf{n}\cdot([\![\mathbf{Tn}]\!]+\overset{\bullet}{\mathbf{t}}_0) = 0.703\times 10^{8} \quad \mathrm{g}\times\mathrm{cm}^{-1}\times\mathrm{sec}^{-2}.$$

Making use of the jump conditions for mass and momentum, we obtain

$$c = 2.71 \ \mathrm{km/sec} \tag{3.47}$$

while the experimental value is ~ 3 km/sec. It means that in spite of all simplifications the model leads to quantitatively reasonable results. The classical equations are useless in this case; some models, based on the notion of the front of finite thickness, have been used.

4. Balance of entropy

Let us return again to the general balance equation (2.18). In the case of the entropy function we have

$$\varphi =: \eta, \quad \Delta =: \lambda, \quad \mu =: h, \quad \overset{\bullet}{\nu} =: \overset{\bullet}{h}, \quad \varphi_s =: \eta_s. \tag{4.1}$$

Hence

$$[\![\varrho U\eta]\!] + \lambda c + [\![h]\!] + \overset{\bullet}{h} - \frac{\partial \eta_s}{\partial t} = 0. \tag{4.2}$$

The interpretation of terms in this relation is similar to the one presented above. The requirement of the Galilean invariance of Eq. (4.2) joined with the assumption

$$\delta_{\langle\dot{\mathbf{x}}\rangle}\eta \equiv 0, \quad \delta_{\langle\dot{\mathbf{x}}\rangle}\lambda \equiv 0, \quad \delta_{\langle\dot{\mathbf{x}}\rangle}h \equiv 0, \quad \delta_{\langle\dot{\mathbf{x}}\rangle}\frac{\partial \eta_s}{\partial t} \equiv 0 \tag{4.3}$$

gives the condition

$$\lambda\mathbf{n} + \delta_{\langle\dot{\mathbf{x}}\rangle}\overset{\bullet}{h} = 0, \tag{4.4}$$

for any $\langle\dot{\mathbf{x}}\rangle$. Assuming again that $\overset{\bullet}{h}$ is an analytic function with respect to $\langle\dot{\mathbf{x}}\rangle$, we easily find the solution of Eq. (4.4) in the form

$$\overset{\bullet}{h} = \overset{\bullet}{h}_0 - \lambda\langle\dot{x}_{\mathbf{n}}\rangle, \tag{4.5}$$

where $\overset{\bullet}{h}_0$ does not depend on $\langle\dot{\mathbf{x}}\rangle$. Hence, the jump condition for the entropy in the invariant form is as follows:

$$[\![\varrho U\eta]\!] + \lambda\langle U\rangle + [\![h]\!] + \overset{\bullet}{h}_0 - \frac{\partial \eta_s}{\partial t} = 0. \tag{4.6}$$

Let us now focus our attention on the second law of thermodynamics. It is a matter of simple calculations [4, 6] to prove that this law yields the following inequality for a singular surface:

$$\overset{\bullet}{h} \geqslant 0. \tag{4.7}$$

Making use of Eqs. (4.5) and (4.6) we obtain

$$[\![\varrho U\eta]\!]+\lambda\langle U\rangle+[\![h]\!]-\frac{\partial\eta_s}{\partial t}\leqslant -\lambda q\langle\dot{x}_{\mathbf{n}}\rangle. \tag{4.8}$$

It is evident that the above inequality is not invariant with respect to the Galilean transformation under the condition (4.3) unless

$$\lambda\equiv 0. \tag{4.9}$$

Although the surface entropy sources $\overset{\bullet}{h}$ are still allowed in such a case, the distribution of entropy in the vicinity of σ must be sufficiently smooth, i.e. the concentration of entropy on σ cannot be affected by the motion of σ through the material. Moreover its total change is described by the intrinsic term $\partial\eta_s/\partial t$. For $\partial\eta_s/\partial t\equiv 0$, we obtain from Eqs. (4.8) and (4.9) a so-called second law of thermodynamics for a singular surface

$$[\![\varrho U\eta]\!]+[\![h]\!]\leqslant 0. \tag{4.10}$$

Let us derive more specific results for Müller's material, which satisfies the following relation between heat and entropy fluxes

$$h=\Lambda q,\quad h=\mathbf{h}\cdot\mathbf{n},\quad q=\mathbf{q}\cdot\mathbf{n}, \tag{4.11}$$

where $\Lambda>0$ is the coldness of the material. In the thermodynamic equilibrium the coldness Λ becomes the inverse of absolute temperature.

The substitution of Eqs. (4.11) in the inequality (4.10) gives

$$[\![\varrho U\eta]\!]+[\![\Lambda]\!]\langle q\rangle+[\![q]\!]\langle\Lambda\rangle\leqslant 0. \tag{4.12}$$

Let us eliminate the jump of the heat flux by using the jump condition of energy. Then

$$[\![\varrho UF]\!]+\varkappa\langle U\rangle+[\![\dot{\mathbf{x}}]\!]\cdot\langle\mathbf{Tn}\rangle-\frac{[\![\Lambda]\!]}{\langle\Lambda\rangle}\langle q\rangle+\frac{1}{8}[\![\varrho U]\!][\![\dot{x}]\!]^2+\overset{\bullet}{q}_0-\frac{\partial e_s}{\partial t}\geqslant 0, \tag{4.13}$$

where

$$F:=e-\frac{1}{\langle\Lambda\rangle}\eta. \tag{4.14}$$

The inequality (4.13) is called a reduced entropy inequality for a singular surface, while the function F corresponds to the Helmholtz free energy.

To illustrate of the physical meaning of this inequality, let us consider again the above mentioned nuclear explosion. This time, to simplify our considerations, we discuss the wave after a longer time-lapse. We assume it to be $t=1.40$ sec after the explosion. The corresponding distribution of the temperature is shown in Fig. 3.

With respect to the symmetry of phenomenon we can assume

$$\frac{d}{d\xi}[\![\theta]\!]=0,$$

where θ is the temperature and ξ is a parametrization of an arbitrary curve on σ. Therefore we can make use of Maxwell's theorem

$$[\![\mathrm{grad}\theta]\!]=a_\theta\mathbf{n}, \tag{4.15}$$

where a_θ is a so-called thermal amplitude. Regarding the relatively long time-lapse we can neglect the surface sources and concentrations. Hence, the second law of thermodynamics is as follows:

$$m[\![\eta]\!] - [\![\Lambda K \operatorname{grad}\theta]\!] \cdot \mathbf{n} \geqslant 0,$$

where K is a thermal conductivity.

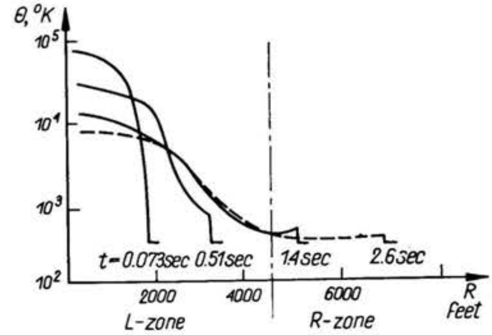

FIG. 3.

However, it is easily seen that

$$(\operatorname{grad}\theta)^+ \cdot \mathbf{n} = 0$$

and therefore

(4.16) $$a_\theta = -(\operatorname{grad}\theta)^- \cdot \mathbf{n}.$$

Making use of Eq. (4.16) in Eq. (4.15), we obtain

$$m[\![\eta]\!] - K\langle\Lambda\rangle a_\theta + 0.5K[\![\Lambda]\!]a_\theta \geqslant 0,$$

and finally

(4.17) $$m[\![\eta]\!] - K a_\theta \Lambda^- \geqslant 0.$$

Let us consider two cases (see: Fig. 3):

i) L-zone: $(\operatorname{grad}\theta)^- \cdot \mathbf{n} < 0 \Rightarrow a_\theta > 0$; hence

$$m[\![\eta]\!] > 0,$$

which is a classical condition of the thermodynamic stability of the shock wave.

ii) R-zone: $(\operatorname{grad}\theta) \cdot \mathbf{n} > 0 \Rightarrow a_\theta < 0$; hence

$$m[\![\eta]\!] + K|a_\theta|\Lambda^- \geqslant 0.$$

If we assume in addition $m[\eta] > 0$, then $(m < 0)$ the strength Δ of the wave satisfies the following inequality

(4.18) $$\Delta := \frac{[\![\varrho]\!]}{\varrho^+} \leqslant 1 + \frac{K(\operatorname{grad}\theta)^- \cdot n\Lambda^-}{\varrho^+ U^- |[\![\eta]\!]|}, \quad U^- > 0,$$

which means that the distribution of the temperature as shown in Fig. 3 is possible for sufficiently weak shocks; it is in agreement with other results presented in the same figure.

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