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BRIEF NOTES

A theorem for limiting lines in a perfectly plastic material

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IN SOLVING certain problems of the theory of plasticity it is often found that a postulated plastic region is bounded by a limiting line, that is by an envelope of stress characteristics which is not itself a stress characteristic. The author has shown in 1970 that for a homogeneous isotropic material, the stress field can not be extended beyond a limiting line without violating the yield condition, and thus a limiting line can only exist at the boundary of the material. The purpose of this paper is to extend this result and show that it remains true even when the material is both anisotropic and inhomogeneous.

1. Introduction

IT IS OFTEN found, when solving a problem in the theory of plasticity, that a postulated plastic region is bounded by a limiting line, that is by an envelope of stress characteristics which is not itself a stress characteristic. Such limiting lines appear to have first been introduced into the theory of plasticity, in order to satisfy the conditions at a perfectly rough interface, by PRANDTL (1923) in his study of the compression of a plastic layer between rough rigid plates. More recently OSTROWSKA (1967) and BYKOVTSEV (1962) have used limiting lines as the boundary between plastic and rigid material.

The properties of limiting lines have been studied by PRAGER and HODGE (1951) for an isotropic material with constant shear strength and by GEIRINGER (1958) for an isotropic weightless material. BOOKER (1970) has shown that for a homogeneous isotropic material, the stress field cannot be extended beyond a limiting line without violating the yield condition, and thus a limiting line can only exist at the boundary of the material. It is the purpose of this paper to extend this result and show that it remains true even when the material is both anisotropic and inhomogeneous.

2. Basic equations

Let us consider the general plane problem of plasticity [GEIRINGER (1958), p. 367]. The curve C shown in Fig. 1 is assumed to have continuous curvature K at 0; it proves convenient to introduce the curvilinear coordinates u, v shown in this figure (this is always permissible at least in some neighbourhood of 0). In terms of these coordinates the equations of equilibrium are:

$$\frac{\partial\sigma_v}{\partial v}+\frac{\partial\tau_{uv}}{\partial u}+K(\sigma_v-\sigma_u)+F_v=0, \tag{2.1}$$

$$\frac{\partial \tau_{uv}}{\partial v}+\frac{\partial \sigma_u}{\partial u}+2K\tau_{uv}+F_u = 0, \tag{2.2}$$

here F_u, F_v are the u, v components of the body force. It will be assumed that the yield criterion may be written in the form:

$$g(\sigma_u, \sigma_v, \tau_{uv}, u, v) = 0. \tag{2.3}$$

The yield criterion is completely arbitrary save that it is convex and all its partial derivatives of the second order exist and are continuous.

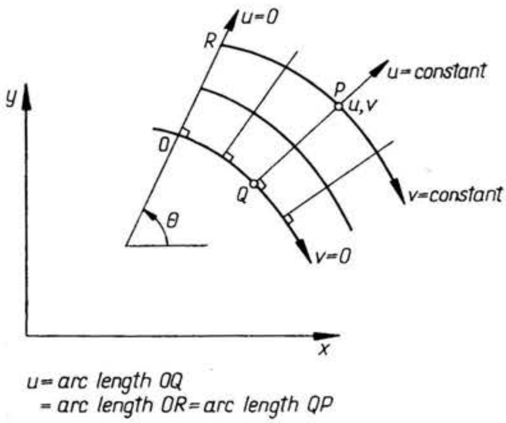

FIG. 1.

The characteristic equations (2.1)–(2.3) may be developed by the method due to HILL (1950), p. 294; and it may be shown that the curve C is a characteristic provided

$$\begin{gathered}\frac{\partial g}{\partial \sigma_u} = 0,\\ \frac{\partial g}{\partial \sigma_v}\frac{\partial \sigma_v}{\partial v}+\frac{\partial g}{\partial \tau_{uv}}\frac{\partial \tau_{uv}}{\partial v}+\frac{\partial g}{\partial v} = 0,\end{gathered} \tag{2.4}$$

and a limiting line (envelope of characteristics which is itself not a characteristic line) wherever

$$\begin{gathered}\frac{\partial g}{\partial \sigma_u} = 0,\\ \frac{\partial g}{\partial \sigma_v}\frac{\partial \sigma_v}{\partial v}+\frac{\partial g}{\partial \tau_{uv}}\frac{\partial \tau_{uv}}{\partial v}+\frac{\partial g}{\partial v} \neq 0.\end{gathered} \tag{2.5}$$

3. Proof of the theorem

Suppose that the curve C is a limiting line, the region below C ($v < 0$) is assumed to be in a plastic state and is denoted by D^-, and the region above C ($v > 0$) is denoted by D^+. As a first stage in the proof we will show that the stress state is completely continuous across C. Let us suppose that the values of $(\sigma_u, \sigma_v, \tau_{uv})$ in D^+, D^- are denoted by $(\sigma_u^+, \sigma_v^+, \tau_{uv}^+)$, $(\sigma_u^-, \sigma_v^-, \tau_{uv}^-)$, respectively. The tangential and normal tractions on C must be continuous and thus

$$\sigma_v^+ = \sigma_v^-, \; \tau_{uv}^+ = \tau_{uv}^-. \tag{3.1}$$

We shall suppose that the stress state in D^- is known and will try to determine the stress state in D^+. It follows from Eq. (3.1) that the stress state in D^+, when plotted in $(\sigma_u, \sigma_v, \tau_{uv})$ — space, may be considered to lie on the straight line L $\sigma_v = \sigma_v^-$, $\tau_{uv} = \tau_{uv}^-$. This line passes through the point $(\sigma_u^-, \sigma_v^-, \tau_{uv}^-)$ which lies on the yield surface and has the tangent plane,

$$\frac{\partial g^-}{\partial \sigma_u}(\sigma_u - \sigma_u^-) + \frac{\partial g^-}{\partial \sigma_v}(\sigma_v - \sigma_v^-) + \frac{\partial g^-}{\partial \tau_{uv}}(\tau_{uv} - \tau_{uv}^-) = 0,$$

where $\partial g^-/\partial \sigma_u$ denotes the value of $\partial g/\partial \sigma_u$ evaluated at $\sigma_u = \sigma_u^-$, $\sigma_v = \sigma_v^-$, $\tau_{uv} = \tau_{uv}^-$ etc. Since C is a limiting line, $\partial g^-/\partial \sigma_u = 0$, and thus this tangent plane has the equation

$$\frac{\partial g^-}{\partial \sigma_v}(\sigma_v - \sigma_v^-) + \frac{\partial g^-}{\partial \tau_{uv}}(\tau_{uv} - \tau_{uv}^-) = 0. \tag{3.2}$$

Equation (3.2) implies that the line L must lie in this tangent plane. Because it was assumed that the yield surface was convex, no point in this tangent plane can lie within the yield surface. The tangent plane will however touch the yield surface at the point $(\sigma_u^-, \sigma_v^-, \tau_{uv}^-)$ and may in fact touch the yield surface along the segment of some line, or element of some plane which contains the point $(\sigma_u^-, \sigma_v^-, \tau_{uv}^-)$. It is thus clear that since the line L lies in this plane and passes through $(\sigma_u^-, \sigma_v^-, \tau_{uv}^-)$ and since the yield criterion cannot be violated, the following statements must be true:

(a) The line L touches the yield surface at a single point, in which $\sigma_u^+ = \sigma_u^-$ and continuity is proven.

(b) The line L touches the yield surface at every point of a segment of this line, i.e., in some range $\sigma_u' \leqslant \sigma_u \leqslant \sigma_u''$ where of course $\sigma_u' \leqslant \sigma_u^- \leqslant \sigma_u''$. It may be shown that in this case Eq. $(2.5)_1$ is automatically satisfied and thus C is not a limiting line, contrary to hypothesis. This possibility need be considered no further.

Having established continuity of the stresses across the line C we may now assume without loss of generality that their values are known on the curve C. It now follows from Eqs. (2.1), (2.2) that

$$\begin{aligned} \frac{\partial \sigma_v}{\partial v} &= -\left(\frac{\partial \tau_{uv}}{\partial u} + K(\sigma_v - \sigma_u) + F_v\right), \\ \frac{\partial \tau_{uv}}{\partial v} &= -\left(2K\tau_{uv} + F_u + \frac{\partial \sigma_u}{\partial u}\right). \end{aligned} \tag{3.3}$$

These equations show that the normal derivatives $\partial \sigma_v/\partial v$, $\partial \tau_{uv}/\partial v$, exist and are continuous on both sides of C and thus, to sufficient accuracy,

$$\begin{aligned} \sigma_v(0, v) - \sigma_v(0, 0) &= \Delta \sigma_v = Av, \\ \tau_{uv}(0, v) - \tau_{uv}(0, 0) &= \Delta \tau_{uv} = Bv, \end{aligned} \tag{3.4}$$

where A, B are known constants which may be determined from Eq. (3.3). These equations are valid in both D^+ and D^-.

It is convenient to define

$$G = \frac{\partial g^-}{\partial \sigma_u}\Delta\sigma_u + \frac{\partial g^-}{\partial \sigma_v}\Delta\sigma_v + \frac{\partial g^-}{\partial \tau_{uv}}\Delta\tau_{uv} + \frac{\partial g^-}{\partial v}v + \frac{1}{2}\left(\frac{\partial^2 g^-}{\partial \sigma_u}\Delta\sigma_u^2 + 2\frac{\partial^2 g^-}{\partial \sigma_u \partial \sigma_v}\Delta\sigma_u\Delta\sigma_v + \ldots + \frac{\partial^2 g^-}{\partial v^2}v^2\right)$$

it being understood that all partial derivatives in this expression are evaluated at $(\sigma_u, \sigma_v, \tau_{uv}, u, v) = (\sigma_u(0,0), \sigma_v(0,0), \tau_{uv}(0,0), 0, 0)$ and where $\Delta\sigma_u = \sigma_u(0,v) - \sigma_u(0,0)$. Then since it was assumed that all the second order partial derivatives of the function g exist and are continuous, we may assert that when $u = 0$ and v is small,

$$g = G \text{ (to sufficient accuracy)}.$$

This equation may be simplified by using Eqs. (3.4) and the fact that C is a limiting line, i.e. $\partial g^-/\partial\sigma_u = 0$.

We then find

$$G = av + b\Delta\sigma_u^2 + 2h\Delta\sigma_u \cdot v + cv^2, \tag{3.5}$$

where

$$a = A\frac{\partial g^-}{\partial \sigma_v} + B\frac{\partial g^-}{\partial \tau_{uv}} + \frac{\partial g^-}{\partial v} \neq 0 \quad \text{(since } C \text{ is a limiting line);}$$

$$b = \frac{1}{2}\frac{\partial^2 g^-}{\partial \sigma_u^2} \geqslant 0 \quad \text{(since the yield surface is convex).}$$

We will now show that Eq. (3.5) implies that $av \leqslant 0$ when $v \geqslant 0$. We may assume that $\Delta\sigma_a$ is a continuous function of v which vanishes when $v = 0$; there are several cases to consider

Case I $b = 0$; then

$$G = av + 2h\Delta\sigma_u \cdot v = av + 0(v);$$

clearly the behaviour of G is dominated by the behaviour of av for small v and thus the condition that $G < 0$ implies that $av < 0$.

Case II $b > 0$, $bc - h^2 \geqslant 0$; then

$$G = av + \left[b\left(\Delta\sigma_u + \frac{h}{b}v\right)^2 + \frac{bc-h^2}{b^2}v^2\right].$$

The bracketed term is necessarily non-negative and thus if $G \leqslant 0$ then we must certainly have $av \leqslant 0$.

Case III $b > 0$, $bc - h^2 < 0$.

Let

$$\phi = b\left(\Delta\sigma_u + \frac{h}{b}v\right)^2 - \left(\frac{h^2 - bc}{b^2}v^2\right),$$

ϕ is a continuous function of v which vanishes at $v = 0$. This implies that in some interval $0 < v < \varepsilon, \phi$ does not change sign.

(a) Suppose $\phi > 0$; then we have a situation similar to Case II and thus $av \leqslant 0$.

(b) Suppose $\phi < 0$; then

$$b\left(\Delta\sigma_u + \frac{hv}{b}\right)^2 \leqslant \left(\frac{h^2-bc}{b^2}\right)v^2,$$

and thus

$$|\phi| \leqslant \left(\frac{h^2-bc}{b^2}\right)v^2$$

so that

$$G = av + 0(v);$$

again the behaviour of G is dominated by av and thus $av \leqslant 0$.

We have now established that $av \leqslant 0$ when $v > 0$. Likewise we can show that $av \leqslant 0$ when $v < 0$. These two results are only compatible when $a = 0$, that is unless C is a characteristic. This is contrary to hypothesis and thus the result is proven.

4. Conclusions

It has been shown that a limiting line cannot exist in the interior of a perfectly plastic material as it is impossible to satisfy equilibrium without violating the yield criterion beyond the limiting line. The proof of this result depends only on the equations of equilibrium and the yield condition and thus remains true no matter what flow rule is adopted, or what conditions are postulated in non plastic regions. The proof specifically excludes the case of jump inhomogeneities, i.e., the junction of two different materials and thus it may be conjectured that a limiting line may lie along such a function.

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