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Bifurcation in a process of deformation of elastic-plastic body at finite homogeneous deformations

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FUNDAMENTAL equations governing the behaviour of an elastic-plastic body under homogeneous deformation are solved within the framework of the theory of dynamic stability, finite subcritical plastic deformations being taken into account. The problem of stability of homogeneous elastic-plastic bodies in a rectangular coordinate system is studied, on the example of the surface instability phenomenon.

W ramach teorii stateczności dynamicznej rozwiązano podstawowe równania rządzące zachowaniem się ciał sprężysto-plastycznych poddanych jednorodnej deformacji z uwzględnieniem skończonych plastycznych odkształceń podkrytycznych. Na przykładzie zjawiska niestateczności powierzchniowej przedyskutowano zagadnienie stateczności jednorodnych ciał sprężysto-plastycznych w prostokątnym układzie współrzędnych.

В рамках теории динамической устойчивости решены основные уравнения, описывающие поведедение упруго-пластических тел подвергнутых однородной деформации с учетом конечных пластических докритических деформаций. На примере явления поверхностной неустойчивости обсуждена задача устойчивости однородных упруго-пластических тел в прямоугольной системе координат.

THE first examinations concerning stability deformations of the perfectly-plastic material at large subcritical deformations were presented in the papers [1, 2]. In particular, the steady flow of the axisymmetric cylinder subjected to large plastic deformations was considered. Then, in the paper [3], deformation stability of the elastic-viscoplastic work-hardening material was investigated in a three-dimensional formulation. In a case of the homogeneous subcritical state, the general solution of the stability equations in a system of rectangular coordinates was obtained. The independent investigations of the deformation stability of the plates and shells at finite plastic deformations were presented, among others, in the papers [4–7]. The general relations were derived on the basis of formal transformations of the equilibrium equations and additivity of the elastic and plastic strains rates.

1.

Let us consider the three states of the deformed elastic-plastic body, moving with respect to a certain system of coordinates x^{α}: the initial state corresponding to the lack of stresses, the state of total deformation and the state which is obtained from a given deformed state by removing all the internal sresses. The three states mentioned above may be considered as the continuous manifolds the particular points of which are determined uniquely by the same Lagrangian coordinates X^i. Let us denote the components of the metric tensors in these three sates by $\mathring{g}_{ij}$, $\hat{g}_{ij}$ and $\dot{g}_{ij}$, respectively. It is well known [8] that for an arbitrary state of the finitely deformed elastic-plastic body the three pairs of the strain tensors may be introduced, namely the plastic, elastic and total strain tensor with the components

$$(1.1)\qquad \varepsilon^p_{ij} = \frac{1}{2}(\dot{g}_{ij}-\mathring{g}_{ij}), \quad \varepsilon^e_{ij} = \frac{1}{2}(\hat{g}_{ij}-\dot{g}_{ij}), \quad \varepsilon_{ij} = \frac{1}{2}(\hat{g}_{ij}-\mathring{g}_{ij}).$$

The formulae (1.1) imply

$$(1.2)\qquad \varepsilon_{ij} = \varepsilon^e_{ij}+\varepsilon^p_{ij}.$$

Notice that in the case of finite deformations the additivity property does not hold for the components having different structure of indexes in the accompanying system of coordinates since Eq. (1.2) relates the components of the tensors in different bases. Taking this fact into account, from Eqs. (1.2) one may obtain the components with a mixed structure of the indices

$$(1.3)\qquad \hat{\varepsilon}^i_j = \hat{\varepsilon}^{ei}_j+\dot{\varepsilon}^{pi}_j-2\dot{\varepsilon}^{pn}_j\hat{\varepsilon}^{ei}_n.$$

The plastic flow rule is assumed in a form similar to that of [9]

$$(1.4)\qquad \dot{e}^{pi}_j = \frac{3\dot{e}_u}{2\dot{\sigma}_u}\dot{S}^i_j, \quad \dot{e}^{pi}_j = \dot{\varepsilon}^{pi}_j-\frac{1}{3}\dot{\varepsilon}^{pk}_k g^i_j, \quad \dot{S}^i_j = \dot{\sigma}^i_j-\frac{1}{3}\dot{\sigma}^k_k g^i_j,$$

$$\sigma_k = \varphi(e_u), \quad \sigma_u = \sqrt{\frac{3}{2}S^j_iS^i_j}, \quad e_u = \sqrt{\frac{2}{3}e^{pj}_i e^i_j}.$$

The elastic strains are related to the stresses by Hooke's law

$$(1.5)\qquad \hat{\varepsilon}^{ei}_j = \frac{1}{E}[(1+\varkappa)\hat{\sigma}^i_j-\varkappa\hat{\sigma}^k_k g^i_j].$$

The condition of plastic incompressibility is

$$(1.6)\qquad \mathring{g}\dot{g}^{-1} = 1, \quad g = \det\|g_{ij}\|.$$

Following [10], the equations of motion and the boundary conditions may be written in the form

$$(1.7)\qquad \hat{\sigma}^i_{j,i}-\hat{\Gamma}^k_{ij}\hat{\sigma}^i_k+\hat{\Gamma}^k_{ik}\hat{\sigma}^i_j+\varrho\hat{F}_j = \varrho\hat{f}_j, \quad \hat{\sigma}^i_jn_i = \hat{P}_j,$$

where Γ^k_{ij} denote the Christoffel symbols of the second kind and $\varrho\hat{F}_j$, $\varrho\hat{f}_j$ and $\hat{p}_j$ are the mass forces, the inertia forces and the surface forces, respectively.

Since the relations (1.4) describe completely the plastic deformation under simple loading [9], our further considerations will be restricted to the case of homogeneous subcritical stress-strain state in a body and solely the simple tension (compression) will be studied, i.e.

$$(1.8)\qquad \varepsilon^j_i = 0, \quad i \neq j; \quad \varepsilon^j_i = \text{const}, \quad i = j.$$

It is obvious that if the metric tensor of the deformed state is assumed as the unity, then the components of the metric tensor in the undeformed state may be presented in the form

$$(1.9)\qquad \mathring{g}_{11} = \lambda_1^{-2}, \quad \mathring{g}_{22} = \lambda_2^{-2}, \quad \mathring{g}_{33} = \lambda_3^{-2}, \quad \mathring{g}_{ij} = 0 \quad (i \neq j).$$

Besides, the following relations result from the Eqs. (1.1) and (1.2)

$$2\hat{\varepsilon}_1^1 = 1-\lambda_1^{-2}, \quad 2\hat{\varepsilon}_2^2 = 1-\lambda_2^{-2}, \quad 2\hat{\varepsilon}_3^3 = 1-\lambda_3^{-2}. \tag{1.10}$$

Superposing the small additional motion determined by the displacement vector $\gamma w'(X^i, t)$ (γ is small parameter) on the basic motion described by the relations (1.1)–(1.7), the linearized equations may be derived. With the accuracy to the linear terms we have:

The elastic, plastic and total strains

$$\hat{\varepsilon}_j^{e'i} = \frac{1}{E}[(1+\varkappa)\hat{\sigma}_j'^i - \varkappa\hat{\sigma}_k'^k g_j^i], \quad \overset{*}{e}{}_j^{p'i} = \frac{3}{2}A\overset{*}{S}{}_j'^i + B\overset{*}{e}{}_j^{pi}\overset{*}{e}{}_n^{pm}\overset{*}{S}{}_m'^n,$$

$$A = \frac{\overset{*}{e}_u}{\overset{*}{\sigma}_u}, \quad B = \frac{1}{\overset{*}{e}_u}\left(\frac{1}{K\overset{*}{e}_u} - \frac{1}{\varphi(\overset{*}{\varepsilon}_u)}\right), \quad K = \left.\frac{d\varphi}{d\tilde{e}_u}\right|_{\tilde{e}_u=\overset{*}{e}_u}, \tag{1.11}$$

$$\hat{\varepsilon}_j'^i = \hat{\varepsilon}_j^{e'i} + \overset{*}{\varepsilon}{}_j^{p'i} - 2\overset{*}{\varepsilon}{}_j^{p'n}\hat{\varepsilon}_n^{ei} - 2\overset{*}{\varepsilon}{}_j^{pn}\hat{\varepsilon}_n^{e'i}.$$

The metric and strain tensors

$$\hat{g}_{ij}' = \hat{\nabla}_i w_j' + \hat{\nabla}_j w_i', \quad 2\hat{\varepsilon}_i'^k = \hat{g}^{kr}(\hat{\nabla}_r w_s' + \hat{\nabla}_s w_r')(g_i^s - 2\hat{\varepsilon}_i^s), \quad 2\hat{\varepsilon}_{ij}' = \hat{g}_{ij}'. \tag{1.12}$$

The velocity and acceleration vectors

$$v_i' = \dot{w}_i' + v_n \nabla_i w'^n, \quad \dot{v}_i' = \ddot{w}_i' + \dot{v}_n \nabla_i w'^n. \tag{1.13}$$

The condition of plastic incompressibility

$$\overset{*}{\varepsilon}{}^{p'}{}_k^k - 2\alpha_k \overset{*}{\varepsilon}{}^{p'}{}_k^k = 0, \; \alpha_k = \overset{*}{\varepsilon}{}^{pk-1}_{k-1} + \overset{*}{\varepsilon}{}^{pk+1}_{k+1} - 2\overset{*}{\varepsilon}{}^{pk-1}_{k-1}\overset{*}{\varepsilon}{}^{pk+1}_{k+1}. \tag{1.14}$$

The summation here is modulo three.

The equilibrium equations and the boundary conditions for the stress tensor increments are, respectively,

$$\nabla_n\hat{\sigma}_i'^n - \sigma_n^m \nabla_i \nabla_m \hat{w}'^n + \sigma_i^n \nabla_m \nabla_n \hat{w}'^m + \varrho' F_i + \varrho \hat{F}_i' = \varrho\ddot{w}_i' + \varrho\dot{v}_n \nabla_i \hat{w}'^n + \varrho'\dot{v}_i, \tag{1.15}$$

$$\hat{\sigma}_i'^m n_m + \sigma_i^m \hat{n}_m' = \hat{p}_i',$$

where $\hat{n}_i'$ and $\hat{p}_i'$ are the increments of the unit normal vector and of the surface load, respectively. Since the far advanced plastic flow is considered, the elastic strains may be neglected as small in comparison to large plastic deformations. This assumption allows us to identify the metric tensors $\hat{g}_{ij}$ and $\overset{*}{g}_{ij}$ and all quantities with the corresponding indices which will be omitted in further considerations. Thus, excluding $\varepsilon^{p'}{}_j^i$ and $\varepsilon^{e'}{}_j^i$ form Eqs. (1.11) and taking into account Eqs. (1.12) and (1.8), one obtains

$$\sigma_j'^i = g_j^i a_{j\alpha} g^{\alpha\alpha} \nabla_\alpha w_\alpha' + (1-g_j^i) G_{ij} g^{ii} (\nabla_i w_j' + \nabla_j w_i'), \; (Ki, j\Sigma\alpha), \tag{1.16}$$

where

$$a_{kj} = (\tau_{kj} - \frac{2}{3}\alpha_j \tau_{km})(1-2\varepsilon_j^j), \quad \Sigma_m, \; \tau_{in} = \frac{1}{\det\|d_{kl}\|}\frac{\partial}{\partial d_{ni}}\det\|d_{kl}\|,$$

$$G_{ji} = \frac{1-2\varepsilon_j^j}{3A + 2(1+\varkappa)(1-2\varepsilon_j^j)/E}, \tag{1.17}$$

$$d_{kj} = \left[\frac{3}{2}A + (1-2\varepsilon_k^k)\frac{1+\varkappa}{E}\right] g_k^j + Be_k^k e_j^j + \frac{2(1+\varkappa)}{3E}\varepsilon_j^j - \frac{\varkappa}{E}(1-2e_k^k) - \frac{1}{2}A.$$

Assuming linearized law relating the stresses and the displacements in the form of Eqs. (1.16), the unloading is disregarded. Such approximate approach has been assumed in a majority of papers concerning stability. Thus, Eqs. (1.15) written in terms of variations, together with Eqs. (1.16), lead to the closed-form system of equations. The undisturbed state of the body will be stable or unstable, depending on the behaviour of the disturbances in time.

2.

Consider a slow steady-state process of deformation of the three-dimensional bodies. The inertia forces in the basic state are neglected. Choosing the accompanying system of coordinates in such a way that in the moment of linearization it coincides with the fixed Cartesian coordinate system x^α as an arbitrary three-axial subcritical stress-strain state, one obtains from Eqs. (1.15) and (1.16) the following Eqs.:

$$L_{ij}w'_j = 0 \quad (j, i, j = 1, 2, 3),$$

$$L_{ij} = \delta_{ij}\left(M_{in}\frac{\partial^2}{\partial x_n^2} - \varrho\frac{\partial^2}{\partial t^2}\right) + (1-\delta_{ij})F_{ij}\frac{\partial^2}{\partial x_i \partial x_j}, \tag{2.1}$$

$$K_{i,j}, \Sigma_n, \quad M_{in} = \{a_{ii}, i = n; \quad G_{in}, i \neq n\},$$

$$F_{ij} = a_{ij} + G_{ij} + \frac{2}{3A}(\varepsilon_i^i - \varepsilon_j^j).$$

The Eqs. (2.1) represent the system of differential equations expressed in terms of displacements. General solution of this system may be presented in the form

$$w'_1 = (L_{22}L_{33} - L_{23}L_{32})\Phi, \quad w'_2 = (L_{31}L_{23} - L_{21}L_{33})\Phi, \tag{2.2}$$

$$w'_3 = (L_{21}L_{32} - L_{31}L_{22})\Phi.$$

The function Φ is determined from the equation

$$\sum_{n,m,p,l}^{n+m+p+l=6} A_{n,m,p,l}\frac{\partial^6}{\partial x_1^n \partial x_2^m \partial x_3^p \partial t^l}\Phi = 0 \quad (n, m, p, l\text{—even}), \tag{2.3}$$

where

$$A_{6,0,0,0} = a_{11}G_{21}G_{31}, \quad A_{4,2,0,0} = a_{11}G_{21}G_{32} + a_{11}a_{22}G_{31} + G_{12}G_{21}G_{31} - G_{31}F_{12}F_{21},$$

$$A_{2,2,2,0} = a_{11}a_{22}a_{33} + a_{11}G_{23}G_{32}$$

$$+ G_{13}G_{21}G_{32} + a_{22}G_{13}G_{13} + a_{33}G_{12}G_{21} + G_{12}G_{23}G_{31} - a_{11}F_{23}F_{32}$$

$$+ F_{21}F_{32}F_{13} - a_{33}F_{12}F_{21} + F_{31}F_{12}F_{23} - a_{22}F_{13}F_{31}, \quad A_{0,0,0,6} = -\varrho^3, \tag{2.4}$$

$$A_{2,0,0,4} = \varrho^2(a_{11} + G_{21} + G_{31}), \quad A_{4,0,0,2} = -\varrho(a_{11}G_{21} + G_{31}G_{21} + a_{11}G_{31}),$$

$$A_{2,2,0,2} = -\varrho(a_{11}a_{22} + G_{12}G_{21} + a_{22}G_{31} + a_{32}G_{21} + a_{11}G_{32} + G_{12}G_{31} - F_{12}F_{21}).$$

The values of the remaining constants $A_{n,p,m,l}$ appearing in Eq. (2.3) may be evaluated from the Eqs. (2.4) in the following manner. For example, to evaluate $A_{2,0,4,0}$ it is necessary to exchange in the right-hand side of the expression $A_{4,2,0,0}$ the indices 1 with 2, 2 with 3 and 3 with 1.

Notice that in the particular case of homogeneous subcritical state, the results analogous to [11] may be obtained, i.e., general solutions of the equations of static and dynamic stability for the elastic-plastic bodies subject to large deformations may be written down.

3.

Consider the surface instability of the half-space ($x_3 < 0$) at compression under "dead" load acting along the axes $0x_1$ and $0x_2$ ($0 \leqslant x_1 \leqslant a; 0 \leqslant x_2 \leqslant b$). Since the boundary of the half-space is free of load then, according to Eq. (1.15), the following boundary conditions are obtained at $x_3 = 0$

$$\sigma'^3_3 = 0, \quad \sigma'^3_2 = 0, \quad \sigma'^3_1 = 0. \tag{3.1}$$

The conditions (3.1), when expressed in terms of the function Φ, together with the relations (1.16) and (2.2) lead to

$$\begin{aligned}
&\sum_{n,m,p,l}^{n+m+p+l=4} A^3_{n,m,p,l} \frac{\partial^4}{\partial x_1^n \partial x_2^m \partial x_3^p \partial t^l} \frac{\partial}{\partial x_1} \Phi = 0,\\
&\sum_{n,m,p,l}^{n+m+p+l=2} A^2_{n,m,p,l} \frac{\partial^2}{\partial x_1^n \partial x_2^m \partial x_3^p \partial t^l} \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} \Phi = 0,\\
&\sum_{n,m,p,l}^{n+m+p+l=4} A^1_{n,m,p,l} \frac{\partial^4}{\partial x_1^n \partial x_2^m \partial x_3^p \partial t^l} \frac{\partial}{\partial x_3} \Phi = 0.
\end{aligned} \tag{3.2}$$

Here

$$\begin{aligned}
&A^3_{4,0,0,0} = a_{31} G_{21} G_{31}, \quad A^3_{2,2,0,0} = a_{31}(G_{21} G_{32} + G_{31} a_{22}) - a_{32} F_{21} G_{31},\\
&A^3_{2,0,2,0} = a_{31}(a_{33} G_{21} + G_{23} G_{31}) - a_{33} G_{21} F_{31},\\
&A^3_{0,4,0,0} = G_{32}(a_{31} a_{32} - a_{32} F_{21}), \quad A^3_{0,2,2,0} = a_{31}(a_{22} a_{33} + G_{23} G_{32})\\
&\qquad - a_{31} F_{23} F_{32} + a_{32}(F_{31} F_{23} - a_{33} F_{21}) + a_{33}(F_{21} F_{32} - a_{22} F_{31}),\\
&A^3_{0,0,4,0} = a_{33} G_{23}(a_{31} - F_{31}), \quad A^3_{2,0,0,2} = -\varrho a_{31}(G_{21} + G_{31}),\\
&A^3_{0,2,0,2} = \varrho[a_{32} F_{21} - a_{31}(a_{22} + G_{32})], \quad A^3_{0,0,2,2} = \varrho[a_{33} F_{31} - a_{31}(G_{23} + a_{33})],\\
&A^3_{0,0,0,4} = \varrho^2 a_{31}, \quad A^2_{2,0,0,0} = -(F_{21} G_{31} + F_{31} G_{21}),\\
&A^2_{0,2,0,0} = F_{21}(F_{32} - G_{32}) - a_{22} F_{31}, \quad A^2_{0,0,2,0} = F_{31} F_{23} - a_{33} F_{21} - F_{31} G_{23},\\
&A^2_{0,0,0,2} = \varrho(F_{21} + F_{31}), \quad A^1_{4,0,0,0} = G_{21}(G_{31} - F_{31}),\\
&A^1_{2,2,0,0} = G_{21} G_{32} + F_{21} F_{32} + a_{22}(G_{31} - F_{31}), \quad A^1_{2,0,2,0} = a_{33} G_{21} + G_{23}(G_{31} - F_{31}),\\
&A^1_{0,4,0,0} = a_{22} G_{32}, \quad A^1_{0,2,2,0} = a_{22} a_{33} + G_{23} G_{32} - F_{23} F_{32},\\
&A^1_{0,0,4,0} = a_{33} G_{23}, \quad A^1_{2,0,0,2} = \varrho(F_{31} - G_{21} - G_{31}),\\
&A^1_{0,2,0,2} = -\varrho(a_{22} + G_{32}), \quad A^1_{0,0,2,2} = -\varrho(a_{33} + G_{23}), \quad A^1_{0,0,0,4} = \varrho^2.
\end{aligned} \tag{3.3}$$

The solution of the Eq. (2.3) is sought for in the form

(3.4) $$\Phi = f(x_3)\cos k\frac{\pi}{a}x_1 \sin n\frac{\pi}{b}x_2 \exp st.$$

The choice of such form of solution ensures the fulfilment of the hinge-support boundary conditions at $x_1 = (0; a)$ and $x_2 = (0; b)$ in the integral sense.

The Eqs. (2.3)–(2.4) and (3.4) lead to the following differential equation determining $f(x_3)$

(3.5) $$\sum_{n=0}^{6} C_n \frac{d^{(n)}f}{dx_3^n} = 0 \quad (n\text{ — even}),$$

where

(3.6) $$\begin{gathered} C_6 = A_{0,0,6,0}, \quad C_4 = -(A_{0,2,4,0}N^2 + A_{2,0,4,0}K^2 - A_{0,0,4,2}S^2), \\ C_2 = A_{2,2,2,0}K^2N^2 + A_{0,4,2,0}N^4 + A_{4,0,2,0}K^4 + A_{0,0,2,4}S^4 \\ -A_{2,0,2,2}K^2S^2 - A_{0,2,2,2}N^2S^2, \\ C_0 = -(A_{6,0,0,0}K^6 + A_{4,2,0,0}K^4N^2 + A_{2,4,0,0}K^2N^4 + A_{0,6,0,0}N^6 - A_{0,0,0,6}S_6 \\ +A_{0,2,0,4}N^2S^4 + A_{2,0,0,4}K^2S^4 - A_{4,0,0,2}K^4S^2 - A_{0,4,0,2}N^4S^2 - A_{2,2,0,2}K^2N^2S^2), \\ K = k\frac{\pi}{a}, \quad N = n\frac{\pi}{b}. \end{gathered}$$

From the general solution of Eq. (3.5) we choose the solution satisfying at $x_3 \to -\infty$ the damping conditions.

Substituting the solution mentioned above into the boundary conditions (3.2) and taking into account the Eq. (3.4) we obtain by the usual procedure the equation which determines the critical value of the compression

(3.7) $$\det||\alpha_{ij}|| = 0 \quad (i, j = 1, 2, 3),$$

where

(3.8) $$\begin{gathered} \alpha_{3i} = A^3_{4,0,0,0}K^4 + A^3_{2,2,0,0}K^2N^2 - A^3_{2,0,2,0}K^2\eta_i^2 + A^3_{0,4,0,0}N^4 \\ -A^3_{0,2,2,0}N^2\eta_i^2 + A^3_{2,0,4,0}\eta_i^4 - A^3_{2,0,0,2}K^2S^2 - A^3_{0,2,0,2}N^2S^2 + A^3_{0,0,2,2}S^2\eta_i^2 + A^3_{0,0,0,4}S^4, \\ \alpha_{2i} = (-A^2_{2,0,0,0}K^2 - A^2_{0,2,0,0}N^2 + A^2_{0,0,2,0}\eta_i^2 + A^2_{0,0,0,2}S^2)\eta_i, \\ \alpha_{1i} = (A^1_{4,0,0,0}K^4 + A^1_{2,2,0,0}K^2N^2 \\ -A^1_{2,0,2,0}K^2\eta_i^2 + A^1_{0,4,0,0}N^4 - A^1_{0,2,2,0}N^2\eta_i^2 + A^1_{0,0,4,0}\eta_i^4 \\ -A^1_{2,0,0,2}K^2S^2 - A^1_{0,2,0,2}N^2S^2 + A^1_{0,0,2,2}S^2\eta_i^2 + A^1_{0,0,0,4}S^4)\eta_i, \end{gathered}$$

η_i^2 are the characteristic roots of the Eq. (3.5) (all different from each other).

4.

In the analogous way the plane strain problem may be considered. The compressive dead load is acting in the plane $x_1 0 x_2$ along the axis $0x_1$. Let the body occupy the lower half-space $x_2 < 0$. The general solution of the system of Eqs. (2.1) in this case has the form

(4.1) $$u_1 = L_{22}\Phi, \quad u_2 = -L_{21}\Phi.$$

The function Φ is determined by the equation

$$\sum_{n,m,l}^{n+m+l=4} B_{n,m,l} \frac{\partial^4}{\partial x_1^n \partial x_2^m \partial t^l} \Phi = 0 \quad (n, m, e \text{ — even}), \tag{4.2}$$

where

$$\begin{aligned} &B_{4,0,0} = a_{11} G_{21}, \quad B_{2,2,0} = G_{12} G_{21} + a_{11} a_{22} - F_{12} F_{21}, \quad B_{0,4,0} = a_{22} G_{12}, \\ &B_{2,0,2} = -\varrho(a_{11} + G_{21}), \quad B_{0,2,2} = -\varrho(a_{22} + G_{12}), \quad B_{0,0,4} = \varrho^2. \end{aligned} \tag{4.3}$$

The boundary conditions on this part of the surface $x_2 = 0$ which is free of loading are expressed in terms of the function Φ at $x_2 = 0$ as

$$\begin{aligned} &\left[(G_{21} - F_{21}) \frac{\partial^2}{\partial x_1^2} + a_{22} \frac{\partial^2}{\partial x_2^2} - \varrho \frac{\partial^2}{\partial t^2}\right] \frac{\partial}{\partial x_2} \Phi = 0, \\ &\left[a_{21} G_{21} \frac{\partial^2}{\partial x_1^2} + a_{22}(a_{21} - F_{21}) \frac{\partial^2}{\partial x_2^2} - a_{21} \varrho \frac{\partial^2}{\partial t^2}\right] \frac{\partial}{\partial x_1} \Phi = 0. \end{aligned} \tag{4.4}$$

The solution of Eq. (4.2) is chosen in the form

$$\Phi = f(x_2) \sin \frac{m\pi}{e} x_1 \exp st, \tag{4.5}$$

which ensures the fulfilment of the hinge-clamped conditions at $x = 0; l$. As it follows from the Eqs. (4.2) and (4.5), the function $f(x_2)$ satisfies the equation

$$\sum_{n=0}^{4} D_n \frac{d^{(n)} f}{d x_2^n} = 0 \qquad (n \text{ — even}), \tag{4.6}$$

where

$$\begin{aligned} &D_4 = B_{0,4,0}, \quad D_2 = B_{0,2,2} S^2 - B_{2,2,0} M^2, \\ &D_0 = B_{0,4,0} M^4 - B_{2,0,2} M^2 S^2 + B_{0,0,4} S^4, \quad M = \frac{m\pi}{l}. \end{aligned} \tag{4.7}$$

Then, as in Sect. 3, the equation for evaluation of the critical value of compression is found to be

$$\det \|\beta_{ij}\| = 0 \quad (i, j = 1, 2), \tag{4.8}$$

where

$$\begin{aligned} &\beta_{1i} = [-(G_{21} - F_{21}) M^2 + a_{22} \xi_i^2 - \varrho S^2] \xi_i, \\ &\beta_{2i} = -a_{21} G_{21} M^2 + a_{22}(a_{21} - F_{21}) \xi_i^2 - a_{21} \varrho S^2, \end{aligned} \tag{4.9}$$

$\xi_{i|}^2$ are the roots of the characteristic equation (4.6). The Eqs. (3.7) and (4.8) contain the quantity s. Following [12], the stability condition is assumed in the form

$$\max \{Re S_k\} < 0, \tag{4.10}$$

where s_k are the roots of the characteristic equation (3.7) and Eq. (4.8).

However, Eqs. (3.7) and (4.8) may be reduced, with an arbitrary degree of accuracy, to a polynomial, the general form of which for the arbitrary value of k may be written as

$$\sum_{n=0}^{k} \alpha_n s^{2n} = 0. \tag{4.11}$$

Applying the Rauss-Hurvitz criterion one finds out that $s = 0$. Thus, to evaluate the critical values of the strain, the condition $s = 0$ should be introduced into Eqs. (3.7) and (4.8). This result is connected with the fact that the boundary-value problem (1.15)–(1.16) (time factor is distinguished) subjected to the conservative loads is self-adjoint.

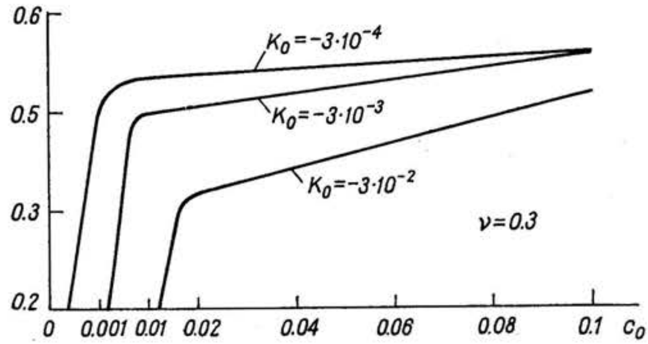

FIG. 1.

The solution of the characteristic equation (4.8) shows that, for weakly work-hardening materials ($c_0 \leqslant 0.1$), the surface instability may arise. The dependence of the critical degree of compression on the work-hardening parameter $0 \leqslant c_0 \leqslant 0.1$ for different values of the plastic yield limit k_0 ($c_0 = cE^{-1}$, E is Young modulus, $k_0 = \sigma_T E^{-1}$) is shown in Fig. 1. The relation between σ_u and e_u is chosen in the form $\sigma_u = \sigma_T + ce_u$. As it is seen from this figure, the value of critical strain decreases with the increase of the work-hardening. The surface instability phenomenon appears in a fully developed process of the plastic flow (c_0 is small) at very large deformations. However, the numerical values of critical loads, computed for this case, are unreal. Therefore the surface instability practically does not occur that confirms the results of the paper [3].

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