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Optimal control in the unilateral thin plate theory(*)

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IN THE PRESENT paper the optimal control problem of unilateral thin plates is investigated. At the beginning, by means of a maximal monotone graph, the problems arising in the unilateral thin plate theory are incorporated in the same mathematical model — a variational inequality whose solution in sought over a closed convex set. Then, the optimal control problem is formulated and an abstract existence result is given. Two numerical approaches are given: the "regularization and penalization" approach and the "decomposition" approach. Finally, the developed theory is applied to the numerical computation of the optimal control of a plate with friction boundary conditions.

W pracy bada się problem sterowania optymalnego dla cienkich płyt z jednostronnymi więzami. Na wstępie sformułowano matematyczny model zagadnienia w postaci nierówności wariancyjnej, której rozwiązania poszukuje się w domkniętym obszarze wypukłym. Następnie sformułowano problem sterowania optymalnego i podano dowód istnienia funkcji sterowania optymalnego dla tego problemu. Zaproponowano dwie metody numeryczne: metodę "regularyzacji" i metodę "rozkładu". Na podstawie sformułowanej teorii podano przykład numeryczny optymalnego sterowania dla płyty z tarciowymi warunkami brzegowymi.

В работе исследуется проблема оптимального управления для тонких плит с односторонними связями. Вначале сформулирована математическая модель задачи в виде вариационного неравенства, решение которого ищется в замкнутой выпуклой области. Затем сформулирована проблема оптимального управления и дается доказательство существования функции оптимального управления для этой проблемы. Предложены два численных метода: метод „регуляризации'' и метод „разложения''. Опираясь на сформулированную теорию дается численный пример оптимального управления для плиты с граничными условиями с течением.

1. Introduction

IN RECENET years many inequality constrained problems in mechanics have been examined. These problems are called "unilateral problems", because the variations of the inequality-constrained quantities are "unilateral" [9, 13], i.e., they take place inward to the admissible set defined by the inequality conditions. Usually this set satisfies the closedness and convexity property in an appropriately chosen function space and can be defined by means of the notion of "subdifferential" [12]. In these problems, because of the unilateral character of variations, the "principles" of virtual (resp. complementary virtual) work are valid in an inequality form called "variational inequality" — or "abstract unilateral problem" in the terminology of [7] — the solution of which must be sought in a convex closed set. The mathematical theory of variational inequalities which are actually the weak formulation of the unilateral boundary value problems is developed by FICHERA

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in [6, 7] and by LIONS, STAMPACCHIA [10], BREZIS [3] etc. Except for the problem of SIGNORINI and some relative dynamical problems [14] many other unilateral problems of mechanics can be found in [4]. In static problems the variational inequalities are equivalent to the minimum problem of a functional over a convex set and, accordingly, after the discretization, the algorithms of non-linear optimization can be used. In dynamic problems the variational inequalities are approximated using a regularization technique by a sequence of appropriately defined variational equalities corresponding to non-linear differential equations.

In the present paper we consider the optimal control problem corresponding to the unilateral thin plate theory [4]. The system, whose optimal control has to be discussed, is governed by an elliptic variational inequality. The mathematical theory of optimal control problems governed by variational inequalities is still a largely unexplored field, except for some results already presented in [11, 17].

First, the unilateral boundary value problems of thin plates are incorporated in the same mathematical model by means of a general convex functional. Then, the optimal control problem is formulated and the existence of its solution is discussed. The theory is illustrated by means of a numerical example concerning the control of the displacement of a unilateral plate.

2. Unilateral thin plate theory presented in a unified manner

In this paper R^3 is a three-dimensional Euclidean space and $Ox_1x_2x_3$ is an orthonormal system. Let $\Omega \subset R^2$ be an open, bounded connected domain of the Ox_1x_2 — plane with boundary Γ belonging to C^2. $\bar{\Omega}$ denotes the region occupied by the middle surface of the plate which is assumed to have a small thickness h. In the sequel the summation convention for repeated indices will be employed and C are generic positive constants, which are not necessarily the same in any two places. Further, $\eta = (\eta_1, \eta_2)$ (resp. $\tau = (\tau_1, \tau_2)$) denotes the unit normal vector in an outward direction to the boundary Γ (resp. the unit tangential vector to the boundary, resulting from η through a rotation of $\pi/2$). Following the assumptions of the theory of thin plates [16], we obtain such relations for the considered problem [4]:

$$D\Delta^2 u = f_3 \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\bar{Q}_3 = Q_3 - \frac{\partial M_\eta}{\partial \tau} = -D\left\{\frac{\partial \Delta u}{\partial \eta} + (1-\nu)\frac{\partial}{\partial \tau}\left[\eta_1\eta_2\left(\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1^2}\right) + (\eta_1^2 - \eta_2^2)\frac{\partial^2 u}{\partial x_1 \partial x_2}\right\} \quad \text{on} \quad \Gamma, \tag{2.2}$$

$$M_\tau = -D\left\{\Delta u + (1-\nu\left(2\eta_1\eta_2\frac{\partial^2 u}{\partial x_1 \partial x_2} - \eta_2^2\frac{\partial^2 u}{\partial x_1^2} - \eta_1^2\frac{\partial^2 u}{\partial x_2^2}\right\} \quad \text{on} \quad \Gamma. \tag{2.3}$$

Here, $u(x_1, x_2)$ is the vertical displacement of the middle surface of the plate, $f_3(x_1, x_2)$ is the prescribed distribution of external loads acting vertically to the plane of the plate, D is the bending rigidity — assumed to be uniform — of the plate, ν is Poisson's ratio.

The biharmonic differential equation (2.1) is valid in every point of Ω. At every point of the boundary a shear force vector Q_3 and a moment vector, analysed in a bending moment M_τ parallel to τ, and a torsional moment M_η parallel to η appear. According to the simplifying hypothesis of Kirchhoff the influence of M_η is incorporated in the shear force vector Q_3 and thus the shear force $\bar{Q}_3 = Q_3 - \frac{\partial M_\eta}{\partial \tau}$ results. Expression (2.2) resp. (2.3) gives the relationship between $\bar{Q}_3$ (resp. (M_τ)) and the displacements of the plate. Assuming that $v \in C^2(\Omega)$ and using Green's identity, the relations (2.2), (2.3) can be written compactly in the variational form

$$a(u, v-u) = (f_3, v-u) + (\bar{Q}_3, v-u)_\Gamma - \left(M_\tau, \frac{\partial(v-u)}{\partial n}\right)_\Gamma, \quad \forall v \in C^2(\Omega), \tag{2.4}$$

where

$$v-u \in C^2(\Omega),$$

$$(\varphi, \psi) = \int_\Omega \varphi\psi d\Omega, \quad (\Phi, \psi)_\Gamma = \int_\Gamma \Phi\psi d\Gamma, \tag{2.5}$$

and the bilinear form $a(u, v-u)$ is

$$a(u, v-u) = D\int_\Omega \left\{\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2(v-u)}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2(v-u)}{\partial x_2^2} + \nu\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2(v-u)}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2(v-u)}{\partial x_1^2}\right) + 2(1-\nu)\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2(v-u)}{\partial x_2 \partial x_1}\right\} d\Omega. \tag{2.6}$$

Expression (2.4) is valid, by means of a functional extension, when functions u, v belong to the Sobolev space $W_2^2(\Omega)$ supplied with the norm

$$\|u\| = \Big(\sum_{|a|\leqslant 2} |D^a u|_{L_2}^2\Big)^{\frac{1}{2}}, \quad |a| = a_1 + a_2, \tag{2.7}$$

where $a = (a_1, a_2)$, a_1, a_2 are non-negative integers $D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \partial x_2^{a_2}}$ and $|\cdot|_{L_2}$ denotes the L_2 —norm.

According to he boundary conditions imposed, the solution will be sought in an appropriately chosen Hilbert space V, which has the basic structure of space $W_2^2(\Omega)$. Let V' be the dual space of V and let $(l, v) = \int_\Omega lv d\Omega$ be the duality pairing defined on $V \times V'$ ($v \in V$, $l \in V'$). The dual space is supposed to have the strong topology defined by the norm $\{\|l\| = \sup|(l, v)|, \|v\| \leqslant 1, v \in V, l \in V'\}$. Then the form (2.4) will have a meaning if $f_3 \in V'$. The forms $(\bar{Q}_3, v-u)$ and $\left(M_\tau, \frac{\partial(v-u)}{\partial \eta}\right)_\Gamma$ have a meaning if $\bar{Q}_3 \in W_2^{-3/2}(\Gamma)$ and $M_\tau \in W_2^{-1/2}(\Gamma)$ since, by the trace theorems, $v-u|_\Gamma \in W_2^{+3/2}(\Gamma)$ and $\left.\frac{\partial(v-u)}{\partial \eta}\right|_\Gamma \in W_2^{+1/2}(\Gamma)$.

The unilateral problems considered in this paper result from the following unilateral conditions:

1. Unilateral conditions with respect to the displacements u of the points of a subset $\Gamma_S \subset \Gamma$.

2. Unilateral conditions with respect to the rotation $\frac{\partial u}{\partial \eta}$ of the points of a subset $\Gamma_S \subset \Gamma$.

3. Unilateral conditions with respect to the displacements u of the points of a properly regular subset $\Omega_0 \subset \Omega$.

All these unilateral conditions can be interpreted in a unified manner by means of a "maximal monotone graph". Let us introduce a non-decreasing, possibly multivalued, mapping $B: R \to (-\infty + \infty]$, defining a relation on $R \times (-\infty + \infty]$. It is assumed that the effective domain of B, i.e., the set $D_B = \{\xi | B(\xi) < +\infty \xi \in R\}$, is not empty and that $B(0) < + +\infty$. The graph $(\xi, B(\xi)) \subset R^2$ is maximal monotone, i.e., every parallel to the second bisector of the Ox_1x_2 coordinate system has only one common point with the non-decreasing graph. The unilateral conditions 1), 2), 3) stated before can be written respectively in the form

$$-\bar{Q}_3 \in B(u), \quad \forall u \in \Gamma_S, \tag{2.8}$$

$$M_\tau \in B\left(\frac{\partial u}{\partial \eta}\right), \quad \forall u \in \Gamma_S, \tag{2.9}$$

$$-\bar{\bar{f}}_3 \in B(u), \quad \forall u \in \Omega_0. \tag{2.10}$$

In Eq. (2.10) $\bar{\bar{f}}_3 = f_3 - \bar{f}_3$, where $\bar{f}_3$ is the given external load and $\bar{\bar{f}}_3$ is the reaction introduced by the unilateral condition. The symbol $\in$ has been used because the mapping B is generally multivalued.

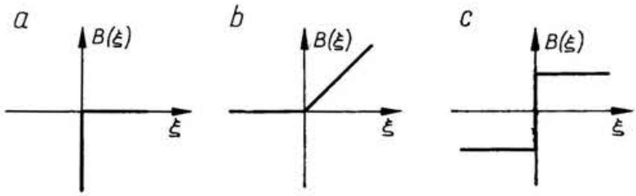

FIG. 1.

For example the Signorini's boundary conditions [6] are obtained by the graph (Fig. 1a)

$$B(\xi) = \begin{cases} 0 & \text{if} \quad \xi > 0, \\ (-\infty, 0] & \text{if} \quad \xi = 0, \\ \emptyset & \text{if} \quad \xi < 0, \end{cases} \tag{2.11}$$

and have the form

$$\bar{Q}_3 \geqslant 0, \quad u \geqslant 0, \quad \bar{Q}_3 u = 0 \quad \text{on} \quad \Gamma_S. \tag{2.12}$$

In Fig. 1b, c the graphs defining the elastic unilateral and the friction boundary conditions are given [3]. It results from the generality of the form of the graph that many other forms of unilateral conditions may be defined in such a manner.

In this paper the optimal control of the following boundary value problems will be examined:

1st problem:

$$u = 0, \quad M_\tau = 0 \left(\text{resp. } u = 0 \ \frac{\partial u}{\partial \eta} = 0\right) \quad \text{on} \quad \Gamma_V,$$

$$-\bar{Q}_3 \in B(u), \quad M_\tau = 0 \quad \text{on} \quad \Gamma_S = \Gamma - \Gamma_V;$$

$$\text{then } V = \{v | v \in W_2^2(\Omega), \quad v = 0 \quad \text{on} \quad \Gamma_V\},$$

$$\text{resp. } V = \left\{v | v \in W_2^2(\Omega), \quad v = 0, \quad \frac{\partial v}{\partial \eta} = 0 \quad \text{on} \quad \Gamma_V\right\}.$$

2nd problem:

$$u = 0, \quad M_\tau = 0 \quad \left(\text{resp. } u = 0, \quad \frac{\partial u}{\partial \eta} = 0\right) \quad \text{on} \quad \Gamma_V,$$

$$u = 0, \quad M_\tau \in B\left(\frac{\partial u}{\partial \eta}\right) \quad \text{on} \quad \Gamma_S = \Gamma - \Gamma_V,$$

$$\text{then} \quad V = \{v | v \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega)\},$$

$$\text{resp.} \quad V = \left\{v | v \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega), \quad \frac{\partial v}{\partial \eta} = 0 \quad \text{on} \quad \Gamma_V\right\}.$$

3rd problem:

$$u = 0, \quad M_\tau = 0 \quad \left(\text{resp. } u = 0, \frac{\partial u}{\partial \eta} = 0\right) \quad \text{on} \quad \Gamma = \Gamma_V,$$

$$\bar{f}_3 = \bar{f}_3 + \bar{\bar{f}}_3, \quad -\bar{\bar{f}}_3 \in B(u) \quad \text{on} \quad \Omega_0 \subset \Omega,$$

$$\text{then} \quad V = \{v | v \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega)\},$$

$$\text{resp.} \quad V = \{v | v \in \mathring{W}_2^2(\Omega)\}.$$

In these problems the Γ_V part of the boundary is assumed to be non-rectilinear only in the case in which the condition $M_\tau = 0$ is valid on Γ_V. The existence and uniqueness theory of the optimal control problem is based upon the coercivity of the bilinear form $a(u, v)$ on V. If boundary conditions other than the preceding ones are valid, or if Γ_V is non-rectilinear, there is the possibility that the form $a(u, v)$ will not be coercive, but semicoercive and thus the solution of the unilateral problem will not be unique(1) [7].

Because of the properties of the $(\xi, B(\xi))$ graph, there exists a convex function $j: R \to (-\infty, +\infty]$, $j \neq \infty$, such that the values of the posibly multivalued mapping B at the point $\xi_1 \in R$ constitute the subdifferential set of the function at the point ξ_1 [3, 15]. The subdifferential set $B(\xi)$ of a convex function $j(\xi)$ at point ξ_1 is given by

$$B(\xi_1) = \{\chi | j(\xi) - j(\xi_1) \geqslant \chi(\xi - \xi_1), \quad \forall \xi \in R\}, \tag{2.13}$$

i.e., for $\xi \to \xi_1, j(\xi_1) < \infty$, χ belongs to the "contingent" set

$$\left\{\lim_{\xi \to \xi_1} \frac{j(\xi) - j(\xi_1)}{\xi - \xi_1}\right\}. \tag{2.14}$$

(1) In this case many problems related to the general form of the solutions are, as far as we know, unsolved.

Elementarily it could be concluded that the following inequality is valid (for the 1st problem)

$$j(v)-j(u) \geqslant (-\bar{Q}_3)(v-u), \tag{2.15}$$

because of the convexity of function j which is defined as an integral of the monotone mapping B, i.e.,

$$j(\xi) = \int_0^{\xi} B(\xi)d\xi. \tag{2.16}$$

Now, a function $\Phi: V \to (-\infty, +\infty]$ is defined by means of

$$\Phi(v) = \begin{cases} \int_{\Gamma_S} j(v)d\Gamma & \text{if} \quad j(v) \in L^1(\Gamma) \\ \infty & \text{if} \quad j(v) \notin L^1(\Gamma) \end{cases} \tag{2.17}$$

Function Φ is convex, weakly lower semicontinuous (w.l.s.c) on V, and not identically $+\infty$. Moreover, the mapping $J: V \to V'$, which is defined by means of

$$\big(J(u), v\big)_{\Gamma} = \int_{\Gamma_S} \bar{B}(u)v d\Gamma, \quad \bar{B} \in B, \quad \forall v \in V \cap D_{\phi} \tag{2.18}$$

is an element of the subdifferential $\partial\Phi$ of Φ [3], i.e.,

$$J(u) \in \partial\Phi(u), \quad \forall u \in V \cap D_{\phi}, \tag{2.19}$$

and accordingly

$$\Phi(v)-\Phi(u) \geqslant \big(J(u), (v-u)\big) = \int_{\Gamma_S} -\bar{Q}_3(v-u)d\Gamma, \quad \forall v \in V \cap D_{\phi}. \tag{2.20}$$

From Eq. (2.20) it results that the admissible perturbations v of the solution u take place in the convex, closed set $V \cap D_{\Phi}$ defined by

$$V \cap D_{\Phi} = \{v | v \in V, \quad \Phi(v) < \infty\}. \tag{2.21}$$

For example, it follows in the case of the graph of Fig. 1a that $V \cap D_{\Phi} = \{v | v \in V, v \geqslant 0\}$.

Similarly we can obtain the functions j and Φ for the other considered problems as well. (Only for the 3rd problem the integrals in the expressions analogous to Eqs. (2.18) and (2.20) are extended over Ω_0, and instead $f_3, \bar{f}_3$ must be written). From Eq. (2.20) and the identity (2.4), after its functional extension in the space $W_2^2(\Omega)$, it follows that the variational inequality as presented below is valid:

$$a(u, v-u) \geqslant (f_3, v-u)-\Phi(v)+\Phi(u), \quad \forall v \in V \cap D_{\Phi}. \tag{2.22}$$

The converse is always true, i.e., conditions (2.1), (2.2), (2.3) and the boundary conditions can be deduced from inequality (2.22) [3].

Accordingly, the function $u \in V$ is a solution of the unilateral boundary value problem if and only if it is a solution of the variational inequality (2.22) over the convex set $V \cap D_{\Phi}$.

As it is known [4] the bilinear form $a(u, v)$ is continuous and coercive for $u, v \in V$, i.e., we may write

$$a(v, u) \geqslant C\|u\|_V^2. \tag{2.23}$$

According to [11] the problem (2.22) is equivalent to the optimization problem

(2.24)
$$\Pi(u) = \inf\left\{\Pi(v) = \frac{1}{2}a(v,v)+\Phi(v)-(f_3,v)\ (\text{resp. } \bar{f}_3 \ \text{ in problem 3}), \quad v\in V\cap D_\Phi\right\},$$

which admits a unique solution u for every $f_3 \in V'$ [11].

3. Definition and existence of the optimal control function

Until now we have considered the spaces V and V' and the duality pairing between them. It may be observed that the property

$$V \subset L^2(\Omega) \subset V' \tag{3.1}$$

is valid, where the injections of V into $L^2(\Omega)$ (resp. $L^2(\Omega)$ into V') are dense, i.e., $\bar{V} = L^2(\Omega)$ (resp. $L^2(\Omega) = V'$). The unique extension of the scalar product of $L_2(\Omega)$ coincides with the duality pairing on $V\times V'$. To formulate the control problem let us introduce the Hilbert space U of control functions ζ over Ω, and suppose that the controls ζ are constrained to belong to a convex, closed subset of U called U_{ad}. Define further two linear bounded operators $B: U \to V'$ and $M: V \to H$, where H is a Hilbert space of functions h called "observation functions". With every control ζ we associate a "state function" $u(\zeta)$ by means of the variational inequality

$$a\big(u(\zeta), v-u(\zeta)\big)+\Phi(v)-\Phi\big(u(\zeta)\big) \geqslant (f_3+B\zeta, v-u(\zeta)), \quad \forall v \in V\cap D_\Phi, \quad \forall\zeta\in U_{ad}, \tag{3.2}$$

which admits a unique solution $u(\zeta)$ for every ζ, and the "cost" to be optimized I

$$I(\zeta) = ||Mu-h||_H^2+(N\zeta,\zeta)_U. \tag{3.3}$$

Here, h is an element of H and $N: U\to U$ is linear bounded, symmetric and coercive operator, i.e.,

$$(N\zeta,\zeta)_U \geqslant C||\zeta||_U^2, \quad \forall\zeta\in U, \quad C>0. \tag{3.4}$$

Further, let us denote by v and z the perturbed values of the functions u and ζ respectively, which are constrained to remain in the convex closed sets $V\cap D_\Phi$ and U_{ad}, respectively.

The optimal control problem is: to find

$$\operatorname*{Inf}_{z\in U_{ad}} I(z). \tag{3.5}$$

The following proposition holds:

Proposition 1: If the state $u(\zeta)$ and the control ζ are defined by means of Eqs. (3.2) and (3.3), then there exists at least one optimal control ζ.

P r o o f: Let z_η be a minimizing sequence in U_{ad}, i.e.,

$$I(z_\eta) \to \inf_{z\in U_{ad}} I(z) \tag{3.6}$$

and let $v_\eta = v_\eta(z_\eta)$ the solution of Eq. (3.2), which belongs to $V\cap D_\Phi$. Because of Eqs. (2.23), (3.3), (3.4) the sequences z_η and v_η are bounded, i.e., $||z_\eta|| \leqslant C$ and $||v_\eta|| \leqslant C$.

Accordingly, we may extract subsequences $\{z_\mu\}$ and $\{v_\mu\}$ such that $z_\mu \to z_0$ weakly in U_{ad} and $v_\mu \to v_0$ weakly in $V \cap D_\Phi$. Since $V \cap D_\Phi$ and U_{ad} are convex sets, they are weakly closed, and thus $z_0 \in U_{ad}$ and $v_0 \in V \cap D_\Phi$. We must prove that v_0 and z_0 satisfy the variational inequality, i.e., that $v_0 = v_0(z_0)$. With Sobolev's imbedding theorems the injection from V into $L^2(\Omega)$ is compact and thus, if $v_\mu \to v_0$ in the weak topology of V, then $v_\mu \to v_0$ strongly in $L^2(\Omega)$.

Moreover, we may define according to Eq. (3.1) the linear bounded operator $\bar{B}: U \to \to L_2(\Omega)$, such that

(3.7) $$(Bz, v)_{V \times V'} = (\bar{B}z, v)_{L_2 \times L_2}$$

and accordingly

(3.8) $$\lim(\bar{B}z_\mu, v_\mu) = (Bz_0, v_0).$$

Thus if Eq. (3.2) is satisfied by v_μ and z_μ, we may take the limit and we obtain — using that $\Phi(v)$ is w.l.s.c. — that v_0 and z_0 are indeed solutions of Eq. (3.2). Now it will be proved that z_0 is the optimal control which is denoted by ζ. Functional $I(z)$ is w.l.s.c. in U and hence

(3.9) $$\liminf I(z_\mu) \geqslant I(z_0).$$

But

(3.10) $$\liminf I(z_\mu) = \inf I(z) \quad \text{for} \quad z \in U_{ad}$$

and thus $I(z_0) = \inf\, I(z) \forall z \in U_{ad}$, and we may take $z_0 = \zeta$ in the proposition q.e.d.

4. Regularization and penelization of the control problem

The control problem is not of a usual type and it cannot be attacked with the usual computational techniques of optimal control theory. The main difficulty is the variational inequality which gives the state function u as a function of the control ζ. Here, we replace— using the techniques of regularization and penalization — the variational inequality by a sequence of variational equalities which are weak formulations of respective operator equations.

Let us introduce a family of everywhere finite, convex, differentiable, functionals $\Phi_\varepsilon(v)$ depending upon a small parameter ε which is destined to tend to zero, and suppose that

(4.1) $$\lim_{\varepsilon \to 0} \Phi_\varepsilon(v) = \Phi(v), \quad \forall v \in V.$$

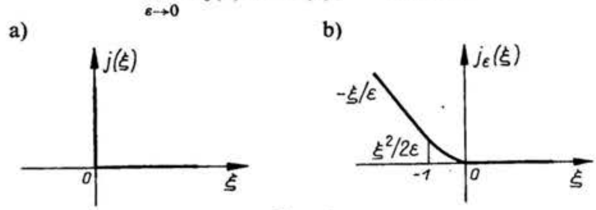

FIG. 2.

Functionals $\Phi_\varepsilon(v)$ can easily be derived by "penalizing" and/or "regularizing" function (ξ). To regularize (resp. penalize) a function $j(\xi)$ means to replace it in the neighbourhoods of the non-differentiable points (resp. points of infinity) by a differentiable curve (resp. a curve taking everywhere finite values), depending continuously upon ε. In Fig. 2a the

$j(\xi)$ function of Signorini's problem is given and in Fig. 2b its regularized and penalized form is given.

The regularized and penalized graph $(\xi, B_\varepsilon(\xi))$ permits us to construct by a limit procedure [8] the convex functional j which has B as subdifferential set.

Let us define further by $u_\varepsilon(\zeta)$ the solution of the variational inequality

$$a(u_\varepsilon(\zeta), v-u_\varepsilon(\zeta)+\Phi_\varepsilon(v)-\Phi_\varepsilon(u_\varepsilon(\zeta)) \geqslant (f_3+B\zeta, v-u_\varepsilon(\zeta)), \quad \forall v \in V, \quad \zeta \in U_{ad}. \tag{4.2}$$

Following the methods of [4] it is easy to prove that Eq. (4.2) is fully equivalent to the variational equality

$$a(u_\varepsilon(\zeta), v-u_\varepsilon(\zeta))+(j_\varepsilon(u(\zeta)), v-u_\varepsilon(\zeta)) = (f_3+B\zeta, v-u_\varepsilon(\zeta)), \quad \forall v \in V, \quad \zeta \in U_{ad}, \tag{4.3}$$

where

$$j_\varepsilon(\xi) = \Phi'_\varepsilon(\xi) \quad \text{and} \quad D_{\Phi\varepsilon} = V. \tag{4.4}$$

Then the following proposition can be proved:

Proposition 2: As $\varepsilon \to 0$ the solution $u_\varepsilon(\zeta)$ of (4.2) or (4.3) tends to the solution $u(\zeta)$ of (3.2).

Proof: Following the same argument as for the existence proof, we can take the limits with respect to ε in Eq. (4.2) and accordingly the proposition is proved. q.e.d.

Define further as ζ_ε the solution of the problem:

$$\text{Find } I_\varepsilon(\zeta_\varepsilon) = \inf_{z \in U_{ad}} I_\varepsilon(z), \tag{4.5}$$

where

$$I_\varepsilon(z) = ||Mu_\varepsilon(z)-h||_H^2+(Nz, z)_U. \tag{4.6}$$

Then the following proposition is valid:

Proposition 3: There exists at least one solution $\zeta_\varepsilon, u_\varepsilon$ of (4.3), (4.5), and as $\varepsilon \to 0$ there exists a subsequence denoted $\{\zeta_\varepsilon\}$ such that

$$\zeta_\varepsilon \to \zeta \quad \text{in} \quad U_{ad} \quad \text{(strongly)}, \tag{4.7}$$

$$u_\varepsilon(\zeta_\varepsilon) \to u(\zeta) \quad \text{in} \quad V \cap D_\Phi \quad \text{(strongly)}, \tag{4.8}$$

$$I_\varepsilon(\zeta_\varepsilon) \to I(\zeta). \tag{4.9}$$

Proof: The existence of the solution $\zeta_\varepsilon, u_\varepsilon$ results easily from the given existence proof. From the preceding proposition we have

$$I_\varepsilon(z) \to I(z), \quad \forall z \in U. \tag{4.10}$$

Hence

$$I_\varepsilon(\zeta_\varepsilon) \leqslant I_\varepsilon(\zeta) \to I(\zeta) = \inf I(z) \tag{4.11}$$

and thus

$$\limsup I_\varepsilon(\zeta_\varepsilon) \leqslant I(\zeta). \tag{4.12}$$

But

$$I_\varepsilon(\zeta_\varepsilon) \geqslant C||\zeta_\varepsilon||_U^2, \tag{4.13}$$

and thus $||\zeta_\varepsilon||_U$ is bounded. Accordingly we may extract a subsequence denoted $\{\zeta_\varepsilon\}$, such that $\zeta_\varepsilon \to \zeta_0$ weakly in U_{ad} because U_{ad} is weakly closed. It can be proved by taking the limit in the variational inequality that

$$u_\varepsilon(\zeta_\varepsilon) \to u(\zeta_0) \quad \text{weakly in} \quad V \cap D_\Phi \tag{4.14}$$

and thus

$$\lim\inf I_\varepsilon(\zeta_\varepsilon) \geqslant I(\zeta_0), \tag{4.15}$$

which together with Eq. (4.12) gives $\zeta = \zeta_0$ and accordingly Eq. (4.9) has been proved. We have for $\varepsilon \to 0$ that

$$\lim I_\varepsilon(\zeta_\varepsilon) = \lim\big(Mu_\varepsilon(\zeta_\varepsilon),\ Mu_\varepsilon(\zeta_\varepsilon)\big)_H + (N\zeta_\varepsilon, \zeta_\varepsilon)_U - 2(Mu_\varepsilon(\zeta_\varepsilon), h)_H + ||h||_H^2 = I(\zeta) \tag{4.16}$$

and from Eq. (4.10) that

$$\lim I_\varepsilon(0) = I(0). \tag{4.17}$$

Moreover, there exist constants $C_1 > 0$, $C_2 > 0$, such

$$C_2||\zeta||_U^2 \leqslant \big(M(u(\zeta)-u(0)),\ M\big(u(\zeta)-u(0)\big)_H + (N\zeta, \zeta)_U \leqslant C_1||\zeta||_U^2, \qquad \forall \zeta \in U, \tag{4.18}$$

which means that $\big(M(u(\zeta)-u(0)),\ M\big(u(\zeta)-u(0)\big)\big)_H + (N\zeta, \zeta)_U$ is a norm equivalent to $||\zeta||_U$. Accordingly, from Eqs. (4.16), (4.17) and (4.18) it results that the already obtained weak convergence in Eqs (4.7) and (4.8) is indeed strong convergence, q.e.d.

Accordingly, we can overcome the non-differentiability of the mapping Φ by considering the sequence of the regularized problems, whose numerical computation does not present any insuperable numerical difficulty.

5. Numerical approach by means of the dual problem — decomposition technique

It is known that minimization problems under some conditions can be formulated as inf. sup.—problems or as saddle point problems [1]. The minimization problem relative to the variational inequality (3.2) has the form

$$\inf_{v \in V \cap D_\Phi} \left\{\frac{1}{2}a(v, v) + \Phi(v) - (f_3 + Bz, v)\right\}, \quad \forall z \in U_{ad}, \tag{5.1}$$

and its dual problem will be formulated.

Let us introduce the Banach space P and P' its dual space and denote by $<\ ,\ >$ the duality pairing between these two spaces. In the case of the first and second problems considered $P \equiv L^2(\Gamma)$, in the case of the third problem $P \equiv L^2(\Omega)$. Further, let us inrotduce a closed convex bounded set QCP' containing the origin and an operator $S: V \to P$ such that

$$\Phi(v) = \sup_{q \in Q} < q, S(v) >. \tag{5.2}$$

Then the problem (5.1) can be written as the inf. sup. — problem

$$\inf_{v \in V \cap D_\Phi,} \sup_{q \in Q} \left\{\pi(v, q; z) = \frac{1}{2}a(v, v) + < q, S(v) > - (f_3 + Bz, v)\right\}, \tag{5.3}$$

whose dual formulation exists and is

$$\sup_{q\in Q}\inf_{v\in V\cap D_\Phi}\left\{\tilde{\pi}(v,q;z)=\frac{1}{2}a(v,v)+<q,S(v)>-(f_3+Bz,v)\right\}. \tag{5.4}$$

According to [8], [1] if the mapping

$$v\to<q,S(v)>\qquad \forall q\in Q \tag{5.5}$$

is convex and w. l. s.c, there exists a saddle point u, r of $\tilde{\pi}(v,q;z)$, such that

$$\tilde{\pi}(u,q;z)\leqslant\tilde{\pi}(u,r;z)\leqslant\tilde{\pi}(v,r;z),\quad \forall v\in V\cap D_\Phi,\quad q\in Q,\quad z\in U_{ad} \tag{5.6}$$

and, accordingly, a necessary and sufficient condition for r to solve the dual problem is

$$<r-q,S\big(u(r)\big)>\geqslant 0,\quad \forall q\in Q, \tag{5.7}$$

if S is continuous from V-weak into P'-weak$_*$ (star topology). It follows from the duality theory that if r solves the dual problem (5.4), $u(r)$ will solve the primal problem (5.3). By means of the inf sup-formulation of the variational inequality (3.2) the optimal control problem takes the form

a) To find $r(z)$ and $u\big(r(z)\big)$ such that to solve the problem

$$\sup_{q\in Q,v\in V\cap D_\Phi}\inf\ \tilde{\pi}(v,q;z),\quad \forall z\in U_{ad}. \tag{5.8}$$

b) To find $\zeta\in U_{ad}$ such that

$$I(\zeta)=\inf I(z),\quad \forall z\in U_{ad}. \tag{5.9}$$

For the numerical calculation of problem (5.8), (5.9) a multilevel decomposition technique [2] will be used. The decomposition is similar to the "non-feasible gradient controller technique". The problem (5.8), (5.9) is decomposed into two levels. The first level problem has the form:

Find, for q fixed, the solution $u(q;z)$ of

$$\inf_{v\in V\cap D_\Phi}\ \tilde{\pi}(v,q;z),\quad \forall z\in U_{ad} \tag{5.10}$$

and the solution $\zeta(q)$ of

$$I\big(\zeta(q)\big)=\inf I(z;q),\quad \forall z\in U_{ad}. \tag{5.11}$$

Variational inequality (5.7) constitutes the second level problem. Its task is to determine a value of q and to supply it to the first level problem. The algorithm starts with an estimated value of q, say q_0. From the first level problem, the $\zeta_1=\zeta(q_0)$ and $u_1=u\big(q_0,\zeta(q_0)\big)$ are obtained. They are transmitted to the second level problem, i.e., to the variational inequality (5.7), which gives for $u(r)=u_1$ a new value of q say q_1, which in its turn is transmitted to the first level problem and so on.

The following proposition is valid:

Proposition 4: Under the given functional assumptions of the preceding proposition and for operator S, there exists at least one solution of the decomposed problem (5.10), (5.11).

Proof: By introducing finite dimensional subspaces (Galerkin's bases) of the V, U, R-spaces, the problem becomes finite dimensional. For the finite dimensional problem the existence of a solution is obvious. Then it is easy to pass to the limit in the variational inequality formulations of Eqs. (5.10) and (5.11) by using the same technique as we did in proposition 1; q.e.d.

6. Numerical computation. Application

In order to obtain the numerical solution of the optimal control problems defined for the thin plates with unilateral boundary conditions, both the "regularization-penalization approach" and the "decomposition approach" can be used. In the second case the variational inequality (5.7) will be solved by the "projection method" for the solution of variational inequalities [1].

Moreover, the continuous problem must be approximated by any discrete scheme, either by the "finite difference method" or the "finite element method". Generally speaking we have to operate in finite dimensional subspaces of the used spaces. Let us denote by h the dimension of the used subspaces; h is destined to tend to inifinity and u_h, ζ_h to the solution of the discretized problem. If the "regularization-penalization" technique is used, the approximate solution $u_{\varepsilon h}$, $\zeta_{\varepsilon h}$ can easily be obtained by an optimization algorithm. Moreover, the strong convergence of the solution $u_{\varepsilon h}$, $\zeta_{\varepsilon h}$ to the solution u, ζ of the initial problem can be proved by following the same procedure as in the proof of proposition 3. The only difference is that we have to take the limit for h for $h \to \infty$, $\varepsilon \to 0$. The convergence question of the solution of the discretized decomposed problem (5.10), (5.11) to the solution of the continuous problem (5.8), (5.9) is still an open problem.

The developed theory is applied to the solution of the following optimal control problem: consider a simply supported square thin plate and let $\Omega = (-1, 1) \times (-1, 1)$ in the R^2-plane (Fig. 3a). On the part Γ_U of the boundary the static boundary condition $M_\tau = 0$ holds, whereas in the remaining part Γ_S the "friction boundary" condition:

$$\text{(6.1)} \qquad \text{if} \quad |M_\tau| < 1 \Rightarrow \frac{\partial u}{\partial \eta} = 0,$$

$$\text{(6.2)} \qquad \text{if} \quad |M_\tau| = 1 \Rightarrow \exists \lambda \geqslant 0 \quad \text{such that} \quad \frac{\partial u}{\partial \eta} = \lambda M_\tau .$$

From Eqs. (6.1), (6.2) we can easily obtain the relation $\Phi(v) = \int\limits_{\Gamma_S} \left| \frac{\partial v}{\partial \eta} \right| d\Gamma$ and $D_\Phi = V$, where $V = \{v | v \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega)\}$. The considered optimal control problem has the following form: Find $u \in V$ and $\zeta \in U_{ad} \subset U = L^2(\Omega)$, such that

$$\text{(6.3)} \qquad D \int\limits_\Omega \Delta u \Delta(v-u)\, d\Omega - \int\limits_\Omega (f_3 + z)(v-u)\, d\Omega + \int\limits_{\Gamma_S} \left| \frac{\partial v}{\partial \eta} \right| d\Gamma - \int\limits_{\Gamma_S} \left| \frac{\partial u}{\partial \eta} \right| d\Gamma \geqslant 0,$$

$$\forall z \in U_{ad}, \quad \forall v \in V,$$

where

$$U_{ad} = \left\{ z | z \in L^2(\Omega), 0 \leqslant z \leqslant 50, \int\limits_{\Omega} z^2 d\Omega = 1000 \right\} \tag{6.4}$$

and $I(\zeta) = \inf I(z)$, $\forall z \in U_{ad}$, where

$$I(z) = \int\limits_{\Gamma_S} \left| \frac{\partial v(z)}{\partial \eta} - h \right|^2 d\Gamma. \tag{6.5}$$

Both the "regularization-penalization approach" and the "decomposition approach" have been used. In the second case, $P = P' = L^2(\Gamma)$ where the restriction to Γ_S is taken,

$$Q = \{q | q \in P', -1 \leqslant q \leqslant 1 \quad \text{on} \quad \Gamma_S\}, \tag{6.6}$$

and

$$\tilde{\pi}(v, q; z) = \frac{1}{2} D \int\limits_{\Omega} (\Delta v)^2 d\Omega - \int\limits_{\Omega} f_3 v \, d\Omega + \int\limits_{\Gamma_S} q \frac{\partial v}{\partial \eta} d\Gamma. \tag{6.7}$$

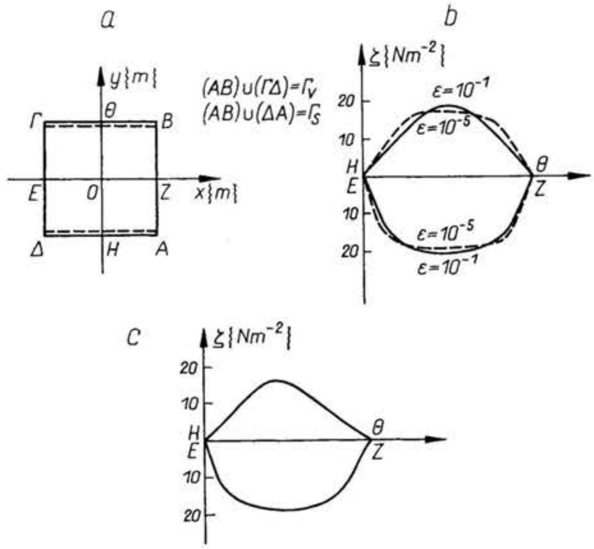

FIG. 3.

Following the "projection technique" [1] we define the projection operator θ of $L^2(\Gamma_S)$ on the convex set Q and we obtain that inequality (5.7) is equivalent to the relation

$$r = \theta \left(r + \sigma \frac{\partial u(r)}{\partial \eta} \right), \quad \forall \sigma \geqslant 0, \quad \text{where} \quad \theta \cdot q = \frac{q}{\sup(1, |q|)}, \quad \forall q \in L^2(\Gamma_S). \tag{6.8}$$

In the present example we have taken $D = 1\text{Nm}$, $h = 0.10$, $f = -100\text{Nm}^{-2}$. The classical finite difference method has been used for the numerical computation. In Fig. 3b the optimal control distribution obtained with the regularization-penalization technique is given for different values of the parameter ε. After discretization, the problem constitutes a non-linear programming problem for which the SUMT code originated by FIACCO and McCORMICK [5] has been used. For a constant step size in the finite difference method equal to 1/10 the computing time in a C.D.C. 6400 computer was about 15 minutes.

The same problem has been solved by the decomposition technique (Fig. 3c). In this case $q_0 = 0$ initially and we compute the value q_i from the value q_{i-1} by means of the recurrence criterion

$$q_i = \theta\cdot(q_{i-1}) + \sigma\,\frac{\partial u_i(q_{i-1})}{\partial \eta}, \quad \sigma \geqslant 0, \tag{6.9}$$

where σ is a parameter chosen properly to assure a better convergence. In the present example σ ranges from 1.0 to 7.0. The value of u_i, ζ_i is obtained by the relations (5.10), (5.11), written in their operator form [11]

$$\Delta^2 u_i = f_3 + \zeta_i, \quad u \in V, \tag{6.10}$$

$$u_i = 0 \quad \text{on} \quad \Gamma. \tag{6.11}$$

$$\Delta u_i = -q_{i-1} \quad \text{on} \quad \Gamma_S, \quad M_{\tau_i}\ (\text{see Eq. (2.3)}) = 0 \quad \text{on} \quad \Gamma_V, \tag{6.12}$$

$$\inf \int_{\Gamma_S} \left|\frac{\partial u_i}{\partial \eta} - h\right|^2 d\Gamma, \quad \zeta_i \in U_{ad}. \tag{6.13}$$

Again the resulting non-linear programming problem was calculated by means of the SUMT code and the time of computation was about 12 minutes in a C.D.C. 6400 computer.

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