Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 2, pp. 299-312, Warszawa 1977

Certain analytical results in the die-swell theory of viscoelastic fluids

S. ZAHORSKI (WARSZAWA)

TAKING into account various numerical solutions of the die-swell problem (cf. [15, 16, 24]) and our theoretical proposals to the entry-flow problem (cf. [25, 26]), certain approximate results are presented for a viscoelastic simple fluid emerging from a long tube at very small Reynolds numbers. These theoretical predictions, e.g. the relations for the die-swell degree, for the angle tangential to the jet surface etc., are compared with experimental data on concentrated polyisobutylene solutions.

Na podstawie różnych numerycznych rozwiązań zagadnienia rozszerzania strugi (por. [15, 16, 24]) oraz naszych teoretycznych propozycji dla przepływów w obszarach wejściowych (por. [25, 26]), przedstawiono niektóre przybliżone wyniki dla lepkosprężystej cieczy prostej wypływającej z długiej rury przy bardzo małych liczbach Reynoldsa. Teoretyczne przewidywania, np. zależności dla stopnia rozszerzenia, dla kąta stycznego do powierzchni strugi itp., porównano z danymi doświadczalnymi dla skoncentrowanych roztworów poliizobutylenu.

На основе разных численных решений задачи расширения струи (ср. [15, 16, 24]), а также наших теоретических предложений для течений во входных областях (ср. [25, 26]), представлены некоторые приближенные результаты для вязко-упругой простой жидкости, истекающей из длинной трубы при очень малых числах Рейнольдса. Теоретические предвидения, например зависимости для степени расширения, для угла касательного к поверхности струи и т. п., сравнены с экспериментальными данными для концентрированных растворов полиизобутилена.

1. Introduction

THE DIE-SWELL phenomenon observed for various viscoelastic fluids emerging from capillary tubes or slits, also called the BARUS effect [1] or the MERRINGTON effect [2], has been studied experimentally and theoretically by numerous authors (e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11]). This phenomenon plays an essential role in polymer processing and, in particular, in artificial fibre formation from polymer melts and solutions (cf. [12]). The usually observed degrees of swell, i.e. the ratios of the expanded jet diameter to the internal tube diameter, reach values from 1 to 2.5 and, under extreme conditions, even up to 8 (cf. [5, 12]). Such degrees of swell cannot be explained only by means of purely volumetric changes and the majority of authors takes into consideration various viscoelastic properties of fluids, including normal stress effects, stress relaxation, elastic recovery etc. Effects of the geometry of a duct, gravitational, inertial and frictional forces as well as surface or interface tension forces are also taken into account.

Because of serious difficulties connected with the mixed boundary value problem, the unknown shape of an expanded jet and the singularities in stresses appearing at the tube edge (cf. MICHAEL [13]), the most frequently used theoretical approach was base on the momentum balance for some characteristic volumes and cross-sections of a tube and jet. This method was applied in early papers by GASKINS and PHILIPPOFF [3] and by METZNER et al. [4]. It was also assumed, among other simplifications occurring in many papers, that fully-developed viscometric flows continued up to the tube exit (cf. [3, 4, 6, 7]). Frequently, certain additional and unfounded assumptions were made on the pressure distribution at the exit cross-section (cf. the discussion by DAVIES et al. [14]). The results obtained on the basis of the momentum balance method not always correspond to experimental data in a quantitatively good manner, even for relatively high Reynolds numbers, i.e. for dominating inertial effects. For strictly creeping (inertialess) flows the momentum balance method is not adequate, since the resulting equations involve neither final jet diameter nor final velocity and, therefore, no bounds on the die-swell can be found. In such cases the boundary value problem must be solved directly (cf. [15, 16]).

The momentum balance method gives a 13% contraction in the jet diameter of purely viscous Newtonian fluids; this contraction may be less for fluids described by a power-law equation.(cf. [3, 4, 17]). Other simplified considerations, e.g. COLEMAN et al. [18], cannot explain the considerable die-swell observed experimentally for Newtonian fluids at very small Reynolds numbers. MIDDLEMAN and GAVIS [19] were probably the first who found the die-swell of order 13% for Newtonian fluids at Reynolds numbers less than 16. Next, GAVIS [20], GOREN and WRONSKI [21], GAVIS and MODAN [22], in their papers devoted to surface tension effects and related corrections, confirmed the degree of swell about 1.13, for simultaneously small surface tensions and Reynolds numbers about 2. Similar values resulted from unpublished data obtained by TANNER (cf. [16]) for certain silicone liquids emerging from a tube at very small Reynolds numbers of order 10^{-3}. In Tanner's experiments the gravitational effects were entirely eliminated and the interfacial tension, inertial and external viscous forces were highly reduced by means of specially matched surrounding baths. The recent experiments of BATCHELOR and HORSFALL [23] seem to prove the existence of a 13.5% limit in the die-swell of Newtonian fluids, at least for Reynolds numbers and surface tension forces tending to zero.

Many earlier theoretical attempts to solve the problem of die-swell for Newtonian fluids in creeping and slow flows can be found in the available literature (cf. ZIDAN [6], RICHARDSON [8, 9], JOSEPH [11]). It must be stressed, however, that credit for the first complete numerical solution goes to TANNER and his collaborators [15, 16]. The stress and velocity distributions in the exit region as well as in the swelling jet region were determined by means of the specially worked out finite element method. In the next paper, TANNER et al. [24] presented certain particular results for second-order fluids and for Newtonian fluids in flows with finite values of Reynolds numbers.

In this paper, taking into account some of Tanner's numerical results (cf. [16, 24]) and our own proposals for viscoelastic flows in the entry regions of channels and tubes (cf. [25, 26]), we present certain approximate analytical results for viscoelastic fluids emerging from a long tube at very small Reynolds numbers (creeping flow). We assume that the constitutive equations describing fluid behaviour are those for an incompressible simple fluid (cf. [27]) and that surface tension forces as well as frictional effects can be disregarded. The results obtained may be useful for certain comparisons of theoretical predictions with experimental data.

2. Equilibrium and boundary conditions

Let us consider the tube and the emerging free jet presented in Fig. 1, where l_e and l_s denote the corresponding lengths of the exit region inside the tube with diameter d and in the jet region where maximum diameter D is reached. These lengths are purely conventional; for Newtonian fluids (cf. [15, 24]) they can be considered as $l_e \simeq l_s \simeq d$ or $l = l_e + l_s \simeq 2d$. For viscoelastic fluids similar or slightly longer regions are reported for sufficiently long tubes (cf. [24, 28]).

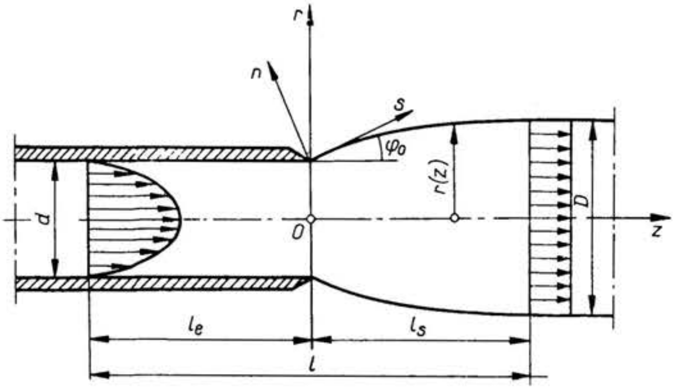


FIG. 1.

In the system of cylindrical coordinates (r, θ, z), the origin of which corresponds to the centre of exit cross-section, for $z \leqslant -l_e$ we have a fully-developed viscometric flow of the Poiseuille type. Physical components of the stress tensor satisfying the equations of equilibrium for an incompressible simple fluid are in the form (cf. [26]):

$$
(2.1) \qquad
\begin{aligned}
T^{\langle rz\rangle} &= -\frac{1}{2} fr, \\
T^{\langle rr\rangle} &= -h - k(r) + \varrho\psi + fz, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
T^{\langle zz\rangle} &= T^{\langle rr\rangle} + \hat{\sigma}_1(S) - \hat{\sigma}_2(S),
\end{aligned}
$$

where h is a constant depending on the hydrostatic pressure, f denotes the specific driving force (pressure gradient), ψ — potential of conservative body forces, ϱ — density of a fluid, and

$$
(2.2) \qquad \frac{dk}{dr} = \frac{1}{r}\left(T^{\langle rr\rangle} - T^{\langle\theta\theta\rangle}\right), \qquad k(r) = \int_0^r \frac{1}{\xi}\hat{\sigma}_2\left(\frac{f\xi}{2}\right) d\xi.
$$

The modified functions of normal stress differences (cf. [18]) are defined as follows:

$$
(2.3) \qquad \hat{\sigma}_1(S) = T^{\langle zz\rangle} - T^{\langle\theta\theta\rangle}, \qquad \hat{\sigma}_2(S) = T^{\langle rr\rangle} - T^{\langle\theta\theta\rangle},
$$

where $S = \frac{1}{2} fr$.

For $z \geqslant l_s$ in a free jet without gravitational forces, we have

$$
(2.4) \qquad T^{\langle rz\rangle} = T^{\langle rr\rangle} = T^{\langle zz\rangle} = 0.
$$

If the jet is loaded with its own weight, then $T^{\langle zz\rangle} = T - \bar{g}z$, where T denotes an additional extensional stress at the tube exit (for $z = 0$) and $\bar{g}$ is the gravity acceleration.

In the central region $-l_e < z < l_s$ the flow is highly non-viscometric and the stress components just at the exit lip, i.e. for $r = \frac{d}{2}$, $z = 0$, may exhibit essential singularities (cf. [8, 11]). It can be concluded, on the basis of Tanner's results (cf. [15, 16, 24]), that the shear stress $T_w^{\langle rz \rangle}$ at the tube wall changes along the exit region from $-\frac{1}{4} fd$ to higher values when approaching the tube lip; outside the tube this shear stress vanishes at the free surface. The axial stress $T_w^{\langle zz \rangle}$ at the tube wall changes its negative value into a positive one in the neighbourhood of $z = 0$, and the stress distribution across the tube exit must be such that the total axial load is properly equilibrated. For Newtonian fluids, for which the total axial load amounts to zero, the axial stress for $r = z = 0$ is of a negative sign. A change of sign is also observed for the radial stress component, although this component does not exhibit any remarkable discontinuity. The pressure at the exit centre, i.e. for $r = z = 0$, is usually positive and any extrapolation to the tube wall, i.e. for $r = d/2$, $z = 0$, does not lead to zero values. The stress singularity at the tube lip disappears very rapidly for Newtonian as well as for other viscoelastic fluids described by simple models (cf. TANNER et al. [24]).

Physical components of the stress tensor in the exit and swelling jet regions must satisfy the following equilibrium equations:

$$(2.5)\qquad \begin{aligned} &\partial_r T^{\langle rr \rangle} + \partial_z T^{\langle rz \rangle} + \frac{1}{r}(T^{\langle rr \rangle} - T^{\langle \theta\theta \rangle}) - \varrho \partial_r \psi = 0, \\ &\partial_r T^{\langle rz \rangle} + \frac{1}{r} T^{\langle rz \rangle} + \partial_z T^{\langle zz \rangle} - \varrho \partial_z \psi = 0, \end{aligned}$$

and the appropriate boundary conditions (see Sect. 3), in particular, on the free surface of a jet.

Let us introduce an auxilliary system of orthonormal coordinates (s, n) connected with the free surface (Fig. 1). If the axis s forms an angle φ with the axis z, we shall denote: $\varphi = \varphi_0$ for $z = 0^+$ and $\varphi = 0$ for $z = l_s$. The free surface of jet corresponds to a stream surface; this implies that the normal velocity component satisfies the following conditions: $v_n = 0$ and $\partial v_n / \partial s = 0$. It is also necessary that normal and tangential components of the stress tensor vanish at the free surface, i.e. $\overline{T}^{\langle nn \rangle} = \overline{T}^{\langle ns \rangle} = 0$, if only the pressure of the surroundings is taken as equal to zero.

On transforming the stress tensor $\mathbf{T}$ as follows:

$$(2.6)\qquad \overline{\mathbf{T}} = \mathbf{Q}\mathbf{T}\mathbf{Q}^t, \quad [\mathbf{Q}] = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix},$$

where superscript t denotes the transpose, we arrive at

$$(2.7)\qquad \begin{aligned} &\overline{T}^{\langle ns \rangle} = T^{\langle rz \rangle}\cos 2\varphi - \frac{1}{2}(T^{\langle zz \rangle} - T^{\langle rr \rangle})\sin 2\varphi = 0, \\ &\overline{T}^{\langle nn \rangle} = T^{\langle rr \rangle}\cos^2\varphi + T^{\langle zz \rangle}\sin^2\varphi - T^{\langle rz \rangle}\sin 2\varphi = 0. \end{aligned}$$

It is seen from the above relations that under the assumption of the positively defined normal stress difference $T^{\langle zz\rangle}-T^{\langle rr\rangle}>0$, the angle φ is positive if $T^{\langle rz\rangle}>0$ in the neighbourhood of the jet surface, and negative if $T^{\langle rz\rangle}<0$. For very small positive angles φ, or more exactly for $\varphi\to 0$, it results from the relation

$$T^{\langle rz\rangle}\sin 2\varphi = T^{\langle zz\rangle}-(T^{\langle zz\rangle}-T^{\langle rr\rangle})\cos^2\varphi, \tag{2.8}$$

that the radial stress is also positive. It may happen, however, that $T^{\langle rr\rangle}$ is negative for some small but finite values of the angle φ; then

$$\mathrm{tg}\varphi > \left(-\frac{T^{\langle rr\rangle}}{T^{\langle zz\rangle}}\right)^{\frac{1}{2}} \quad \text{if} \quad T^{\langle zz\rangle}>0. \tag{2.9}$$

Therefore, the pressure along the flow axis and the boundary (the tube wall together with the jet surface) changes from positive to negative values. These changes of signs are realized earlier at the boundary than at the axis. In the jet where the maximum diameter is reached ($z=l_s$), the pressure $-T^{\langle rr\rangle}$ tends to zero, retaining positive values in the whole cross-section.

The above discussion shows an important role played in the stress analysis by the angle $\varphi\neq 0$. For small values of φ_0, i.e. the angle tangential to the jet surface for $z=0^+$, it results from Eq. $(2.7)_1$ that

$$\mathrm{tg}\varphi_0 = \frac{T^{\langle rz\rangle}\left(\frac{d}{2},0^+\right)}{T^{\langle zz\rangle}\left(\frac{d}{2},0^+\right)-T^{\langle rr\rangle}\left(\frac{d}{2},0^+\right)}. \tag{2.10}$$

The above relation is used in Sect. 3, where certain approximate expressions for stresses are specified.

All the above considerations are also valid for plane jets emerging from long channels or slits. Then, instead of Eqs. (2.1) we use (cf. [25])

$$\begin{aligned} T^{\langle xy\rangle} &= -fy, \qquad T^{\langle yy\rangle} = -h+\varrho\psi+fx,\\ T^{\langle xx\rangle} &= T^{\langle yy\rangle}+\sigma_1(\varkappa)-\sigma_2(\varkappa), \end{aligned} \tag{2.11}$$

where $\varkappa$ denotes the shear gradient and $\sigma_i(\varkappa)$ $(i=1,2)$ are the normal stress differences. Other relations remain the same after changing formally r into y and z into x.

3. Stress distributions in the exit and swelling jet regions

In our previous papers [25, 26] on viscoelastic flows in the entry regions of plane channels and tubes we proposed approximate stress distributions satisfying the equations of equilibrium and the corresponding boundary conditions. To this end we expanded the stresses into series with respect to $\varepsilon = d/l$, where d denoted the tube diameter (or the channel height) and l — the total length of the entry region. The series contained three arbitrary functions of z (or x) and their derivatives with respect to z (or x), the values of which at various cross-sections resulted from the boundary conditions. Such an approach made it possible to treat any type of continuous entry flow as "an almost parallel flow" (cf. [29]), in which stress gradients in the axial direction were by one order of ε less than those in the transverse direction. The effective, self-consistent solutions could be obtained for second order approximations, i.e. retaining terms proportional to ε^2.

Trying to apply a similar procedure to the case under consideration, we must assume that:

1) the length of the exit region plus the length of the swelling jet region ($l = l_e + l_s$) is sufficiently greater than the tube diameter d, what justifies treating ε as a small parameter;

2) singularity in stresses at the tube lip can be expressed by certain discontinuous functions at the exit cross-section ($z = 0$).

Under the above assumptions physical components of the stress tensor up to terms proportional to ε^3 can be written as follows (cf. [25, 26]):

$$T^{\langle rz\rangle} = \frac{1}{2}(c+g'(z)+N'(z))r+\frac{1}{12}(g'''(z)+M'''(z)+N'''(z))r^3,$$

(3.1)
$$T^{\langle rr\rangle} = -h-k(r)+\varrho\psi-cz-g(z)-M(z)-\frac{1}{4}(g''(z)+N''(z))r^2,$$

$$T^{\langle zz\rangle} = -h-k(r)+\varrho\psi-cz-g(z)-N(z)-\frac{1}{3}(g''(z)+M''(z)+N''(z))r^2,$$

where c, h are constants and $g(z), M(z), N(z)$ are arbitrary generalized functions with possible jumps for $z = 0$.

If the functions g, M and N, having the dimension of fd (or $\hat{\sigma}_i$), are of order 1, the terms

(3.2)
$$g^{(n)}(z)r^n, \quad M^{(n)}(z)r^n, \quad N^{(n)}(z)r^n,$$

having the dimension of $fd(d/l)^n$, are assumed to be of order ε^n. Thus the terms proportional to ε^3 in Eqs. (3.1) are included into the shear stress, while those proportional to ε^2 are present in the expressions for normal stresses. It is easy to verify that the equations of equilibrium (2.5) are satisfied identically if the term $\frac{1}{12}(g^{IV}(z)+M^{IV}(z)+N^{IV}(z))r^3$, having the dimension of $f(d/l)^4$, can be disregarded as proportional to ε^4.

It can be shown that the functions g, M and N satisfy the appropriate boundary and continuity conditions for $z = -l_e$ and $z = l_s$, leading to self-consistent relations.

At the upstream end of the exit region, we have successively ((cf.2.1)):

(3.2)
$$T^{\langle rz\rangle}(r,-l_e) = -\frac{1}{2}fr \rightarrow \begin{cases} c+g'(-l_e)+N'(-l_e) = -f, \\ g'''(-l_e)+M'''(-l_e)+N'''(-l_e) = 0, \end{cases}$$

(3.4)
$$\partial_z T^{\langle rz\rangle}(r,-l_e) = 0 \rightarrow g''(-l_e)+N''(-l_e) = 0,$$

(3.5)
$$T^{\langle rr\rangle}(r,-l_e) = -h-k(r)+\varrho\psi-fl_e \rightarrow \begin{cases} cl_e-g(-l_e) = -fl_e, \\ M(-l_e) = 0, \end{cases}$$

(3.6)
$$\partial_z T^{\langle rr\rangle}(r,-l_e) = \partial_z T^{\langle zz\rangle}(r,-l_e) = \varrho\partial_z\psi+f \rightarrow$$
$$\rightarrow \begin{cases} -c-g'(-l_e) = f, \; M'(-l_e) = N'(-l_e), \\ g'''(-l_e)+M'''(-l_e)+N'''(-l_e) = 0, \end{cases}$$

(3.7)
$$T^{\langle zz\rangle}(0,-l_e)-T^{\langle rr\rangle}(0,-l_e) = 0 \rightarrow M(-l_e) = N(-l_e),$$

and, consequently, for the normal stress difference

$$(3.8)\qquad T^{\langle zz\rangle}(r,-l_e)-T^{\langle rr\rangle}(r,-l_e) = \hat{\sigma}_1(S)-\hat{\sigma}_2(S) = -\frac{1}{12}M''(-l_e)r^2.$$

Outside the tube, at the cross-section corresponding to the maximum jet diameter, we also have (cf. (2.4)):

$$(3.9)\qquad T^{\langle rz\rangle}(r,l_s) = 0 \to \begin{cases} c+g'(l_s)+N'(l_s) = 0, \\ g'''(l_s)+M'''(l_s)+N'''(l_s) = 0, \end{cases}$$

$$(3.10)\qquad T^{\langle zz\rangle}(r,l_s) = T^{\langle rr\rangle}(r,l_s) = 0 \to \begin{cases} -h-cl_s-g(l_s)-N(l_s) = 0, \\ g''(l_s)+N''(l_s) = M''(l_s) = 0, \\ M(l_s) = N(l_s), \end{cases}$$

and consequently

$$(3.11)\qquad T^{\langle zz\rangle}(r,l_s)-T^{\langle rr\rangle}(r,l_s) = 0.$$

At the beginning of a free jet, i.e. for $z = 0^+$, we see that

$$(3.12)\qquad T^{\langle rz\rangle}(r,0^+) = A(0^+)r+B(0^+)r^3,$$

$$(3.13)\qquad T^{\langle zz\rangle}(r,0^+)-T^{\langle rr\rangle}(r,0^+) = M(0^+)-N(0^+)-\frac{1}{12}M''(0^+)r^2,$$

where $A(z)$, $B(z)$ denote generalized functions resulting from Eqs. (3.1).

Therefore, the shear stress distribution for $z = 0^+$ is determined by a polynomial of third order in r, while the normal stress dependence on r is rather parabolic. The relations (3.3) to (3.11) are by no means contradictory and can be simultaneously satisfied by properly chosen functions g, M and N, e.g. in the form of higher order polynomials involving certain jump functions and their derivatives. It follows from Eqs. (3.1) that the stress tensor exhibits certain discontinuities in the components $T^{\langle rz\rangle}$ and $T^{\langle zz\rangle}$ if only the derivatives M'' and M''' have some jump properties for $z = 0$. Since the shape of a jet is not known a priori, such an approach may lead to essential difficulties in satisfying all the relations (3.3) to (3.11) and the conditions (2.7) defined on the free surface. What follows, bearing in mind at least a theoretical possibility of such reasonable solutions, we seek some simplified relations between the angle φ_0 and the degree of swell $s = D/d$.

For small angles φ_0, after substituting Eqs. (3.12) and (3.13) into Eq. (2.10), we can write

$$(3.14)\qquad \mathrm{tg}\,\varphi_0 \equiv r'(0^+) = \frac{A(0^+)r+B(0^+)r^3}{M(0^+)-N(0^+)-\frac{1}{12}M''(0^+)r^2},$$

where $r = r(z)$ formally denotes the equation of jet surface in the coordinates (r, θ, z). It is evident, of course, that $r(0^+) = d/2$ and $r(l_s) = D/2$.

Taking into account the order (with respect to ε) of terms appearing in the numerator and denominator of Eq. (3.14), we can conclude that the shear stress contains terms proportional to ε and ε^3, while the normal stress difference — those proportional to 1 and ε^2. On the other hand, it follows from the boundary conditions for $z = -l_e$ that the functions g, M and N involve terms either proportional to the shear stress at the wall

$S_w = \frac{1}{4} fd$ in the upstream Poiseuille flow or proportional to the first normal stress difference $N_w = \hat{\sigma}_1(S_w) - \hat{\sigma}_2(S_w)$ in the same type of flow. The quantities S_w and N_w are not entirely independent; they depend in some way on the rate of shear.

The above remarks lead to the relation:

$$\text{tg}\,\varphi_0 = \frac{(a_1\varepsilon + a_3\varepsilon^3)S_w + (c_1\varepsilon + c_3\varepsilon^3)N_w}{(b_0 + b_2\varepsilon^2)S_w + (d_0 + d_2\varepsilon^2)N_w} = \frac{\bar{a} + \bar{c}\nu}{\bar{b} + \bar{d}\nu}, \tag{3.15}$$

where $\nu = N_w/2S_w$. If the parameter ε can be taken as a quantity independent of S_w, N_w or ν, the terms denoted by $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are also constant. This assumption seems to be arbitrary and does not result from any physical premises. It is very likely that the exit length as well as the swelling jet length might be related to viscometric flow characteristics far inside the tube. On the other hand, it turns out from available experiments (cf. [23, 16, 24, 28]) that in many Newtonian creeping flows, $l = l_e + l_s$ is of an order of two diameters ($\varepsilon \simeq 1/2$) and increases rather slightly for viscoelastic fluids $\left(\varepsilon \simeq \frac{1}{3} - \frac{1}{2}\right)$. This important matter cannot be settled only in an experimental way, since l_e as well as l_s are not measurable. On the contrary, the angle φ_0 can easily be determined from sufficiently enlarged photographs of a jet.

For very slow flows in which the model of an incompressible second-order fluid is relevant (cf. [27]), the ratio $\nu = N_w/2S_w$ is roughly proportional to shear stress at the wall S_w. In this case Eq. (3.15) can be written in the form

$$\text{tg}\,\varphi_0 = \frac{\bar{A} + \bar{C}S_w}{\bar{B} + \bar{D}S_w}, \tag{3.16}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ are new constants independent of the Poiseuille flow kinematics; they depend on three material constants characterizing properties of a second-order fluid. For free Newtonian ($\nu = 0$) jets without gravitational forces, the ratio $\bar{a}/\bar{b}$ in Eq. (3.15) must be equal to $\text{tg}\,\varphi_0$. According to the data of TANNER [15] and BATCHELOR and HORSFALL [23], we obtain $\bar{a}/\bar{b} \simeq 0.2$, what corresponds to $\varphi_0 \simeq 11.3°$ (then $s \simeq 1.13$). Although $\text{tg}\,\varphi_0$ results from Eq. (3.15), Eq. (3.16) for $S_w = 0$ is undetermined (for $S_w = 0$ there exists no flow), the lower limit value $\bar{A}/\bar{B} \simeq 0.2$ must be valid. On the other hand, for jets subjected to gravitational forces, the value $\bar{a}/\bar{b}$ or $\bar{A}/\bar{B}$ and the corresponding values of $\text{tg}\,\varphi_0$ may be much less (see Sect. 5).

4. Relations for the jet expansion

On the basis of Eqs. (2.7) and (3.1) an approximate differential equation of the jet contour can be derived, the integration of which leads to further simple relations between the die-swell degree $s = D/d$ and the angle φ_0 tangential to the jet surface at the exit.

For small values of $\varphi(\text{tg}\,\varphi \ll 1)$, we have from Eq. $(2.7)_1$

$$\text{tg}\,\varphi \equiv r'(z) = \frac{T^{\langle rz\rangle}(r, z)}{T^{\langle zz\rangle}(r, z) - T^{\langle rr\rangle}(r, z)}, \qquad 0 < z \leqslant l_s, \tag{4.1}$$

where $r = r(z)$ describes the jet free surface. Substituting for $T^{\langle rz\rangle}$, $T^{\langle zz\rangle}$, $T^{\langle rr\rangle}$ the corresponding expressions (3.1), we arrive at (cf. (3.14))

$$r'(z) = \frac{A(z)r+B(z)r^3}{M(z)-N(z)-\frac{1}{12}M''(z)r^2}. \tag{4.2}$$

Assuming that for sufficiently small angles φ the normal stress difference $T^{\langle zz\rangle}-T^{\langle rr\rangle}$ on the jet surface, appearing in the denominator of Eq. (4.2), does not differ significantly from the value for $r = d/2$[1], Eq. (4.1) may be written in the approximate form:

$$r'(z) \simeq f_1(z)r+f_3(z)r^3. \tag{4.3}$$

Introducing the auxilliary non-dimensional quantities:

$$R = \frac{2r}{d}, \quad Z = \frac{2z}{d}, \quad L_s = \frac{2l_s}{d}, \quad \psi_1(Z) = \frac{1}{\operatorname{tg}\varphi_0}f_1(z)\frac{d}{2}, \tag{4.4}$$
$$\psi_3(Z) = \frac{1}{\operatorname{tg}\varphi_0}f_3(z)\left(\frac{d}{2}\right)^3$$

we arrive at the equation

$$R'(Z) = \operatorname{tg}\varphi_0(\psi_1(Z)R+\psi_3(Z)R^3) \tag{4.5}$$

and the boundary conditions: $R(0^+) = 1$, $R(L_s) = s$. Since, moreover,

$$R'(0^+) = \operatorname{tg}\varphi_0, \quad R'(L_s) = 0, \tag{4.6}$$

the functions ψ_1 and ψ_3 are not arbitrary but must satisfy the following relations:

$$\psi_1(0^+)+\psi_3(0^+) = 1, \quad \psi_1(L_s) = \psi_3(L_s) = 0. \tag{4.7}$$

Equation (4.5) is the Bernoulli type equation, the general solution of which can be presented in the form:

$$R(Z) = \exp\Big(\int_{0^+}^{Z}\operatorname{tg}\varphi_0\psi_1(\Lambda)d\Lambda\Big)\Big[1-2\int_{0^+}^{Z}\operatorname{tg}\varphi_0\psi_3(\Omega)\exp\Big(2\int_{0^+}^{\Omega}\operatorname{tg}\varphi_0\psi_1(\Lambda)d\Lambda\Big)d\Omega\Big]^{-\frac{1}{2}}. \tag{4.8}$$

On expanding all exponential expressions into convergent Taylor series for $\operatorname{tg}\varphi_0$ close to zero, retaining only terms proportional to $\operatorname{tg}\varphi_0$ and additionally taking into account that in the denominator

$$(1-2x)^{\frac{1}{2}} \simeq 1-x, \tag{4.9}$$

we finally arrive at

$$s = \frac{1+m\operatorname{tg}\varphi_0}{1+n\operatorname{tg}\varphi_0}, \quad s = \frac{D}{d} = R(L_s), \tag{4.10}$$

where

$$\int_{0^+}^{L_s}\psi_1(\Lambda)d\Lambda = m, \quad \int_{0^+}^{L_s}\psi_3(\Lambda)d\Lambda = n. \tag{4.11}$$

[1] This difference is really very small for small jet expansions and tends to zero rapidly for $z \to l_s$.

The formula (4.10) gives an approximate relation between the degree of die-swell s and the small angle φ_0 tangential to the jet surface at the exit cross-section. It is seen that $s = 1$ for $\mathrm{tg}\varphi_0 = 0$, and $s > 1$ for $\mathrm{tg}\varphi_0 > 0$. Since for free Newtonian jets $s \simeq 1.13$ and $\mathrm{tg}\varphi \simeq 0.2$, we may postulate that

$$1.54\, m - 1.77\, n = 1. \tag{4.12}$$

Substituting formally Eqs. (3.15) or (3.16) into Eqs. (4.10), we obtain the following alternative relations:

$$s = \frac{\bar{A}_1 + \bar{C}_1 \nu}{\bar{B}_1 + \bar{D}_1 \nu} \quad \text{or} \quad s = \frac{\bar{A}_2 + \bar{C}_2 S_w}{\bar{B}_2 + \bar{D}_2 S_w}, \tag{4.13}$$

where $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, $\bar{D}_i$ $(i = 1, 2)$ denote certain new constants.

5. Comparisons with experimental data

We believe that the formulae (3.15) or (3.16) for the angle tangential to the jet surface, and the formulae (4.10) or (4.13) for the jet expansions may describe with sufficient accuracy the die-swell phenomenon of many viscoelastic fluids in creeping flows. Any direct comparisons with experimental data seem, however, pretty difficult because of very few available measurements, especially for the angle φ_0.

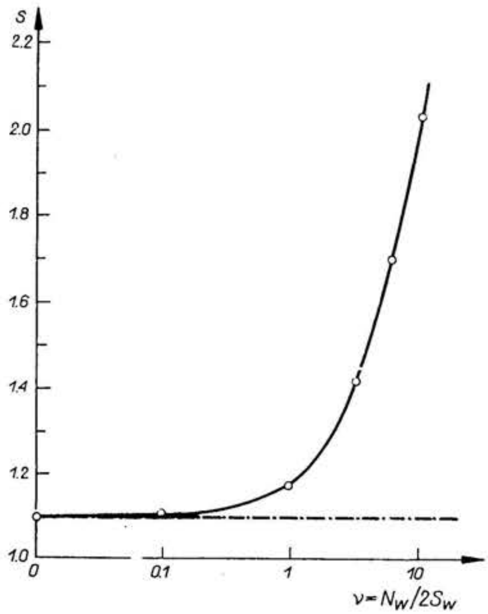


FIG. 2.

By way of illustration, we show in Fig. 2 the relation between the normal to shear stress ratio ν and the degree of swell s (see $(4.13)_1$). The experimental points are taken from the graphs presented by TANNER [7] and the constants in Eq. $(4.13)_1$ calculated for three values of ν, e.g. 0, 1 and 10. The horizontal line denotes $s = 1.1$.

Further comparisons are made using the experimental results obtained by CLERMONT et al. [28] for a 50% solution of polyisobutylene (Vistanex LMMS) in the Bayol 80 oil (Esso), the viscosity of which was of order 10^3 poises. These experiments were performed for vertical jets loaded with their own weights; surface tension effects as well as frictional effects were not considered[(1)].

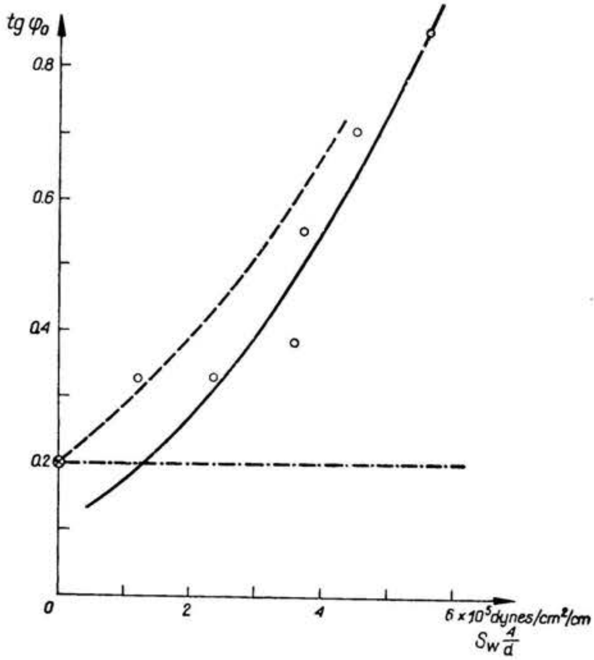
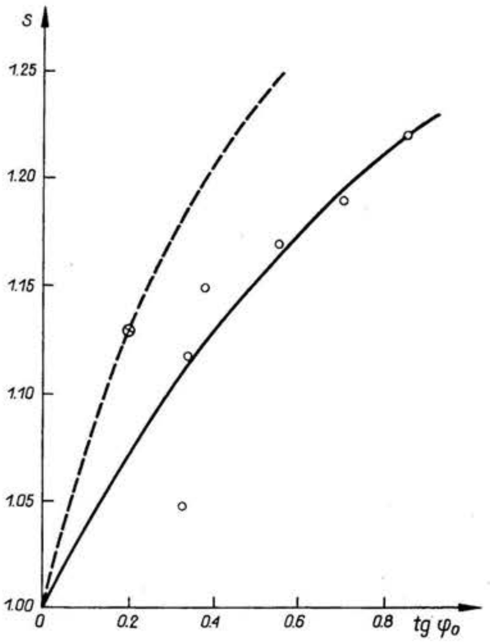


FIG. 3.

In Fig. 3 we show the experimental points and the relation $\text{tg}\varphi_0$ versus S_w resulting from Eq. (3.16). The particularly chosen constants are: $\bar{A} = 10$, $\bar{B} = 100$, $\bar{C} = 3.79 \cdot 10^6$, $\bar{D} = -4.43 \cdot 10^6$. For jets without gravitational forces an approximate character of the curve is marked with the broken line; the horizontal line $\text{tg}\varphi_0 \simeq 0.2$ corresponds to the case of Newtonian fluid. Next, Fig. 4 demonstrates a dependence of the die-swell degree s on the angle φ_0. The constants m and n are chosen as 0.87 and 0.50, respectively. The broken line with the point ⊗ refers to anticipated fluid behaviour without gravitational effects. Finally, Fig. 5 presents the curve drawn according to Eq. $(4.13)_2$, i.e. the relation between the shear stress at the wall S_w and the die-swell degree s. The constants $\bar{A}_2 = 104$, $\bar{B}_2 = 100$, $\bar{C}_2 = -1.08 \cdot 10^6$ and $\bar{D}_2 = -2.42 \cdot 10^6$ are calculated for previously chosen values of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ and m, n, after substituting Eq. (3.16) into Eq. (4.10).

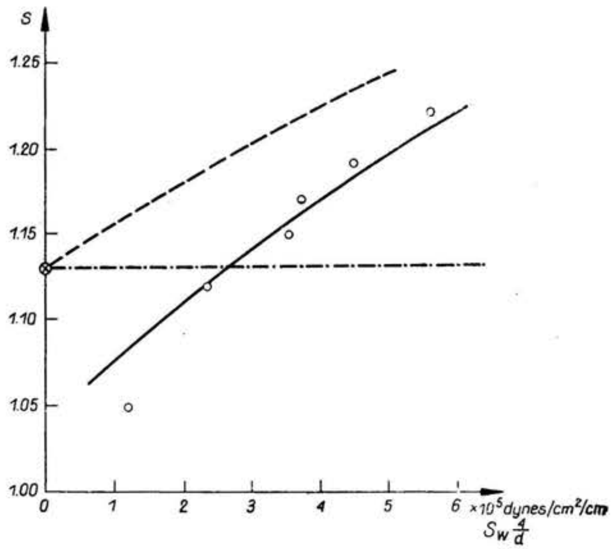
A general correspondence between the curves shown graphically and the available experimental data seems to be fairly good. Some discrepancies can be seen only for the

(1) The paper [28] was rather concerned with the determination of velocity profiles, using the laser anemometry method. Values of φ_0 were not published; for various shear stress levels they were measured afterwards from the photographs taken at the Institute of Mechanics in Grenoble.



s
1.25
1.20
1.15
1.10
1.05
1.00
0
0.2
0.4
0.6
0.8
tg φo

FIG. 4.



s
1.25
1.20
1.15
1.10
1.05
1.00
0
1
2
3
4
5
6
×10^5 dynes/cm²/cm
Sw 4/d

FIG. 5.

first experimental points characterized by smaller values of the shear stress S_w or the angle φ_0. Much more experimental evidence is required before any final conclusions can be drawn out.

Acknowledgment

The author is indebted to Drs. P. LE ROY, J. M. PIERRARD and J. R. CLERMONT from the Institute of Mechanics, University of Grenoble, for certain experimental data used for comparisons.

References

1. C. BARUS, *Isothermals, isopiestics and isometrics relative to viscosity*, Amer. J. Sci., **45**, 87, 1893.
2. A. C. MERRINGTON, *Measurements of anomalous viscosity by the capillary tube method*, Nature, **152**, 214, 1943.
3. F. H. GASKINS, W. PHILIPPOFF, *The behavior of jets of viscoelastic fluids*, Trans. Soc. Rheol., **3**, 181, 1959.
4. A. B. METZNER, W. T. HOUGHTON, R. A. SAILOR, J. L. WHITE, *A method for the measurement of normal stresses in simple shearing flow*, Trans. Soc. Rheol., **5**, 133, 1961.
5. A. S. LODGE, *Elastic liquids*, Academic Press, London-New York 1964.
6. M. ZIDAN, *Zur Rheologie des Spinnprozesses*, Rheol. Acta, **8**, 89, 1969.
7. R. I. TANNER, *A theory of die-swell*, J. Polymer Sci., A-2, **8**, 2067, 1970.
8. S. RICHARDSON, *A "stick-slip" problem related to the motion of a free jet of a low Reynolds number*, Proc. Camb. Phil. Soc., **67**, 477, 1969.
9. S. RICHARDSON, *The die-swell phenomenon*, Rheol. Acta, **9**, 193, 1970.
10. W. BÜRGER, *Kontraktion und Expansion von Flüssigkeitsstrahlen*, Deutsche Rheologen Tagung, 1974.
11. D. D. JOSEPH, *Slow motion and viscometric motion; stability and bifurcation of the rest state of a simple fluid*, Arch. Rational Mech. Analys., **56**, 99, 1974.
12. A. ZIABICKI, *Fizyka procesów formowania włókien*, WNT, Warszawa 1970.
13. D. H. MICHAEL, *The separation of a viscous liquid at a straight edge*, Mathematica, **5**, 82, 1958.
14. J. M. DAVIES, J. F. HUTTON, K. WALTERS, *Theory for normal stresses in slits and capillaries*, J. Phys. D: Appl. Phys., **6**, 2259, 1973.
15. R. I. TANNER, *Die-swell reconsidered: some numerical solutions using a finite element program*, Appl. Polymer Symp., 20, 201, 1973.
16. R. E. NICKELL, R. I. TANNER, B. CASWELL, *The solution of viscous incompressible jet and free-surface flows using finite-element methods* J. Fluid Mech., **65**, 189, 1974.
17. S. MIDDLEMAN, *The flow of high polymers*, Interscience, New York-London-Sydney-Toronto1968.
18. B. D. COLEMAN, H. MARKOVITZ, W. NOLL, *Viscometric flows of non-Newtonian fluids*, Springer, Berlin-Heidelberg-New York 1966.
19. S. MIDDLEMAN, J. GAVIS, *Expansion and contraction of capillary jets of Newtonian liquids*, Phys. Fluids, **4**, 355, 1961.
20. J. GAVIS, *Contributions of surface tension of expansion and contraction of capillary jets*, Phys. Fluids, **7**, 1097, 1964.
21. S. L. GOREN, S. WRONSKI, *The shape of low speed capillary jets of Newtonian liquids*, J. Fluid Mech., **25**, 185, 1966.
22. J. GAVIS, M. MODAN, *Expansion and contraction of jets of Newtonian liquids in air: effect of the tube length*, Phys. Fluids, **10**, 487, 1967.
23. J. BATCHELOR, F. HORSFALL, *Die-swell in elastic and viscous fluids*, Rubber and Plastics Res. Assoc. of Great Britain, Res. Rep. 189, 1971.

24. R. I. TANNER, R. E. NICKELL, R. W. BILGER, *Finite element methods for the solution of some incompressible non-Newtonian fluid mechanics problem with free surfaces*, Comp. Methods in Appl. Mech. and Engin., **6**, 155, 1975.
25. S. ZAHORSKI, *Plane entry flows of viscoelastic fluids*, In "Theoretical Rheology", Edit. J. F. HUTTON, J. R. A. PEARSON and K. WALTERS, App. Sci. Publishers, Barking 1975, p. 40.
26. S. ZAHORSKI, *Slow flows of viscoelastic fluids at the entries to tubes and channels* [in Polish], Mech. Teoret. Stos., **12**, 561, 1974.
27. C. TRUESDELL, W. NOLL, *The non-linear field theories of mechanics*, In "Handbuch der Physik", Edit. S. FLÜGGE, vol. III/3, Springer, Berlin-Heidelberg-New York 1965.
28. J. R. CLERMONT, J. M. PIERRARD, O. SCRIVENER, *Experimental study of a non-viscometric flow: kinematics of a viscoelastic fluid at the exit of a cylindrical tube*, Colloquium Euromech 72, Toulouse, Juillet 1975; J. Non-Newtonian Fluid Mech., **1**, 173, 1976.
29. R. I. TANNER, *Some almost-parallel viscoelastic flows*, Trans. Soc. Rheol., **11**, 39, 1967.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received April 28, 1976.