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Gravitational instability of a composite rotating plasma

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THE GRAVITATIONAL instability of a finitely conducting hydromagnetic composite rotating plasma is considered to include the effects due to finite Larmor radius, Hall currents and collisions with neutrals. It is found that Jeans' criterion still determines the gravitational instability in the presence of rotation, finite Larmor radius, Hall currents and collisions with neutrals. The effect of rotation consists in decreasing the Larmor radius by an amount depending on the wave number of perturbation.

Rozważono problem grawitacyjnej niestateczności hydromagnetycznej złożonej plazmy wirującej o skończonej przewodności, uwzględniając efekty wynikające ze skończonego promienia Larmora, prądów Halla i zderzeń z cząstkami neutralnymi. Stwierdzono, że kryterium Jeansa obejmuje zjawisko niestateczności grawitacyjnej w przypadku rotacji, skończonego promienia Larmora, prądów Halla i zderzeń z cząstkami neutralnymi. Wpływ rotacji polega na zmniejszeniu promienia Larmora o wielkość zależną od liczby falowej perturbacji.

Рассмотрена проблема гравитационной, гидродинамической неустойчивости сложной, вращающейся плазмы с конечной проводимостью, учитывая эффекты, вытекающие из конечного радиуса Лармора, токов Холла и столкновений с нейтральными молекулами. Констатировано, что критерий Джинса охватывает явление гравитационной неустойчивости в случае вращения, конечного радиуса Лармора, токов Холла и столкновений с нейтральными молекулами. Влияние вращения заключается в уменьшении радиуса Лармора на величину зависящую от волнового числа возмущения.

List of notations

ϱ — density of fluid, ϱ_d — density of neutral gas, $\mathbf{\Omega}(0, 0, \Omega)$ — rotation velocity, $\mathbf{H}(0, 0, H)$ — vertical magnetic field, $\mathbf{g}(0, 0, -g)$ — gravity force, η — resistivity, N — electron number density, e — charge of an electron, c — velocity of sound in the medium, $\mathbf{q}_d$ — velocity of neutral gas, ν_c — mutual frictional (collisional) effects between the two components of the composite plasma and G — gravitational constant.

Introduction

THE GRAVITATIONAL instability of an infinite homogeneous self-gravitating medium has been discussed by CHANDRASEKHAR (1961) and Jeans' criterion for instability has been found to remain unaffected by the presence, separately or simultaneously, of rotation and magnetic field. The stabilizing influence of finite Larmor radius, which exhibits itself in the form of a magnetic viscosity in the fluid equations, on plasma instabilities has been pointed out by ROBERTS and TAYLOR (1962). SHARMA (1974) studied the gravitational instability of a rotating plasma with the inclusion of finite Larmor radius effects. SHARMA and PRAKASH (1974) studied the gravitational instability of a finitely conducting plasma to include the effects due to rotation, finite Larmor radius and Hall currents.

In all the above studies, a fully ionized plasma has been considered.

Quite frequently it happens that the plasma is not fully ionized and may be permeated with neutral atoms. As a reasonably simple approximation, it may be taken as an idealized composite mixture of a hydromagnetic (ionized) component and a neutral component, the two components interacting through mutual collisional effects. BHATIA and GUPTA (1973) considered the finite Larmor radius effects on gravitational instability of a composite plasma. BHATIA and STEINER (1972) studied the collisional and finite Larmor radius effects on the stability of superimposed media and found that the collisions as well as the finite Larmor radius were stabilizing on composite media.

The object of the present paper is to study the combined influence of collisions with neutrals, finite Larmor radius, Hall currents and rotation on the gravitational instability of a composite medium.

Here we consider an infinite homogeneous composite medium consisting of a finitely conducting hydromagnetic fluid of density ϱ and a neutral gas of density ϱ_d, which is uniformly rotating with velocity $\mathbf{\Omega}$ and acted upon by a uniform vertical magnetic field $\mathbf{H}$ and gravity force $\mathbf{g}$. We make the assumptions that both the ionized fluid and the neutral gas behave like continuum fluids and the effects on neutral component resulting from the presence of a magnetic field and the fields of gravity and pressure are neglected.

2. Perturbation equations and dispersion relation

The linearized perturbation equations appropriate to the problem are:

$$(2.1)\qquad \varrho\frac{\partial \mathbf{q}}{\partial t} = -\nabla\delta p - \nabla\mathbf{P} + \frac{1}{4\pi}(\nabla\times\mathbf{h})\times\mathbf{H} + \varrho\nabla\delta U + 2\varrho(\mathbf{q}\times\mathbf{\Omega}) + \varrho_d\nu_c(\mathbf{q}_d-\mathbf{q}),$$

$$(2.2)\qquad \frac{\partial \mathbf{q}_d}{\partial t} = -\nu_c(\mathbf{q}_d-\mathbf{q}),$$

$$(2.3)\qquad \frac{\partial}{\partial t}\delta\varrho = -\varrho\nabla\cdot\mathbf{q},$$

$$(2.4)\qquad \delta p = c^2\delta\varrho,$$

$$(2.5)\qquad \nabla^2\delta U = -4\pi G\delta\varrho,$$

$$(2.6)\qquad \frac{\partial \dot{h}}{\partial t} = \nabla\times(\mathbf{q}\times\mathbf{H}) + \eta\nabla^2\mathbf{h} - \frac{c}{4\pi Ne}\nabla\times[(\nabla\times\mathbf{h})\times\mathbf{H}],$$

$$(2.7)\qquad \nabla\cdot\mathbf{h} = 0.$$

Here $c\left(=\sqrt{\frac{rp}{\varrho}}\right)$ and δp, $\delta\varrho$, δU, $\mathbf{q}(u, v, w)$, $\mathbf{h}(h_x, h_y, h_z)$ denote the perturbations in pressure, density, gravitational potential, velocity and magnetic field, respectively.

For the magnetic field along the z-axis, the components of pressure tensor $\mathbf{P}$, taking into account the finite ion gyration radius, are (ROBERTS and TAYLOR 1962):

$$(2.8)\qquad P_{xx} = -\varrho\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\quad P_{yy} = \varrho\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$

(2.8) [cont.]
$$P_{zz} = 0, \quad P_{xy} = P_{yx} = \varrho\nu\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right),$$

$$P_{xz} = P_{zx} = -2\varrho\nu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad P_{yz} = P_{zy} = 2\varrho\nu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right).$$

Here $\varrho\nu = \dfrac{NT}{4\omega_H}$, where N and T are number density and ion temperature of the hydromagnetic component, and ω_H is ion-gyration frequency.

Analyzing in terms of normal modes, we seek solutions whose dependence on the space and time coordinates is of the form

$$\exp\, i\,(k_x x + k_x z + \sigma t), \tag{2.9}$$

where k_x, k_z are the wave numbers of the perturbation along the x- and z-axes, and σ is the growth rate of perturbation.

Equations (2.1)—(2.8) with the help of expression (2.9) give

$$\sigma' u = -\left(\frac{k_x}{k^2}\right)\Omega_\tau^2 \cdot s + [i\nu(k_x^2 + 2k_z^2) - 2i\Omega]v + \frac{H}{4\pi\varrho}(k_z h_x - k_x h_z), \tag{2.10}$$

$$\sigma' v = [2i\Omega - i\nu(k_x^2 + 2k_z^2)]u - 2i\nu k_x k_z w + \frac{H}{4\pi\varrho}k_z h_y, \tag{2.11}$$

$$\sigma' w = -\left(\frac{k_z}{k^2}\right)\Omega_\tau^2 s + 2i\nu k_x k_z v, \tag{2.12}$$

$$(i\sigma + \eta k^2)h_x = ik_z Hu - \left(\frac{cH}{4\pi Ne}\right)k_z^2 h_y, \tag{2.13}$$

$$(i\sigma + \eta k^2)h_y = ik_z Hv - \left(\frac{cH}{4\pi Ne}\right)[-k_z^2 h_x + k_x k_z h_z], \tag{2.14}$$

$$(i\sigma + \eta k^2)h_z = -ik_x Hu + \left(\frac{cH}{4\pi Ne}\right)k_x k_z h_y. \tag{2.15}$$

Here $s\left(= \dfrac{\delta\varrho}{\varrho}\right)$ denotes the condensation in the medium, $k^2 = k_x^2 + k_z^2$, $\Omega_\tau^2 = c^2k^2 - 4\pi G\varrho$, $\sigma' = \sigma\left(1 + \dfrac{\alpha_0 \nu_c}{i\sigma + \nu_c}\right)$ and $\alpha_0 = \varrho_d/\varrho$.

Equations (2.10)—(2.15) can be written in the determinantal form

$$|X|\,|Y| = 0, \tag{2.16}$$

where $|X|$ is a sixth order determinant and $|Y|$ is a single column vector, whose elements are u, v, w, h_x, h_y, s.

The vanishing of the determinant $|X|$ gives us the dispersion relation and we will discuss the problem for transverse ($k_z = 0$) and longitudinal ($k_x = 0$) modes of wave propagation separately.

Parallel propagation $(k_z = k, k_x = 0)$

For perturbation along the direction of the magnetic field, the determinant $|X|$ gives (2.17)

$$\begin{vmatrix} 2i\Omega-2i\nu k^2 & -\sigma' & 0 & 0 & \frac{H}{4\pi\varrho}k & 0 \\ 0 & 0 & -\sigma' & 0 & 0 & \frac{-\Omega_\tau^2}{k} \\ ikH & 0 & 0 & -(i\sigma+\eta k^2) & -\left(\frac{cH}{4\pi Ne}\right)k^2 & 0 \\ 0 & (i\sigma+\eta k^2)ikH & 0 & (i\sigma+\eta k^2)\frac{cH}{4\pi Ne}k^2 & -(i\sigma+\eta k^2)^2 & 0 \\ \sigma' & 2i\Omega-2i\nu k^2 & 0 & -\frac{kH}{4\pi\varrho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\sigma\sigma'-\Omega_\tau^2) \end{vmatrix} = 0.$$

Equations (2.17) can be simplified to give

(2.18) $$\sigma'(\sigma\sigma'-\Omega_\tau^2)(i\sigma+\eta k^2)[W^6-AW^5+BW^4-CW^3+DW^2-EW+F] = 0,$$

where we have written $iw = \sigma$ and

$$A = 2(\eta k^2+\overline{1+\alpha_0}\nu_c),$$

$$B = 4(-\Omega+\nu k^2)^2+2k^2V^2+(\overline{1+\alpha}_0^2\nu_c^2+4\eta k^2\nu_c\overline{1+\alpha}_0+\eta^2k^4)+\left(\frac{cH}{4\pi Ne}\right)^2k^4,$$

$$C = 8(-\Omega+\nu k^2)^2(\eta k^2+\nu_c)+2k^2V^2(\eta k^2+\overline{2+\alpha}_0\nu_c)$$
$$+2\eta k^2\nu_c(1+\alpha_0)(\overline{1+\alpha}_0\nu_c+\eta k^2)+2\nu_c(1+\alpha_0)\left(\frac{cH}{4\pi Ne}\right)^2k^4,$$

$$D = 4(-\Omega+\nu k^2)^2(\eta^2k^4+\eta k^2\nu_c+\nu_c^2)+k^4\left[V^2-(-2\Omega+2\nu k^2)\frac{cH}{4\pi Ne}\right]^2$$
$$+2k^2V^2(2\eta k^2\nu_c+\nu_c^2+\overline{\eta k^2+\nu_c}\alpha_0\nu_c)+\eta^2k^4\nu_c^2(1+\alpha_0)^2+\left(\frac{cH}{4\pi Ne}\right)^2(1+\alpha_0)^2k^4\nu_c^2,$$

$$E = 8(-\Omega+\nu k^2)^2(\eta k^2+\nu_c)\eta k^2\nu_c+2(1+\alpha_0)k^4V^2\nu_c\eta$$
$$+2k^4\left[V^2-(-2\Omega+2\nu k^2)\frac{cH}{4\pi Ne}\right]^2\nu_c,$$

$$F = 4(-\Omega+\nu k^2)^2\eta^2k^4\nu_c+k^4\left[[V^2-(-2\Omega+2\nu k^2)\frac{cH}{4\pi Ne}\right]^2\nu_c^2.$$

The first factor of eq. (2.18) can be written as

(2.19) $$\sigma[-\sigma+i\nu_c(1+\alpha_0)] = 0,$$

i.e. either $\sigma = 0$ or $\sigma = i\nu_c(1+\alpha_0)$.

The first value of σ in Eq. (2.19) gives a mode of neutral stability while the second one corresponds to a viscous type of damped mode modified by frictional effects with neutrals.

The second factor of Eq. (2.18) can be written as

$$\sigma^2\left(1+\frac{\alpha_0\nu_c}{i\sigma+\nu_c}\right) = \Omega_\tau^2,$$

or, replacing σ by iW, as

$$W^3-\nu_c(\alpha_0+1)W^2+\Omega_\tau^2 W-\Omega_\tau^2\nu_c = 0;$$

when $\Omega_\tau^2 < 0$, at least one root of the above equation is negative, what means that the plasma is unstable. The Jeans' criterion thus determines the gravitational instability.

The third factor of Eq. (2.18) gives

$$\sigma = i\eta k^2$$

what corresponds to a viscous type of a damped mode modified by finite conductivity.

The last factor of Eq. (2.18) has alternately positive and negative coefficients. The real part of W is therefore positive and the composite plasma is stable.

Perpendicular propagation $(k_x = k, k_z = 0)$

For this mode of propagation the determinant $|x|$ gives

$$(2.20)\quad \begin{vmatrix} 2i\Omega-i\nu k^2 & -\sigma' & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma' & 0 & 0 & 0 \\ 0 & 0 & 0 & -(i\sigma+\eta k^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & -(i\sigma+\eta k^2) & 0 \\ \left(\sigma'-\dfrac{ik^2V^2}{i\sigma+\eta k^2}\right) & 2i\Omega-i\nu k^2 & 0 & 0 & 0 & \dfrac{\Omega_\tau^2}{k} \\ i\dfrac{k^3V^2}{i\sigma+\eta k^2} & ik(\nu k^2-2\Omega) & 0 & 0 & 0 & (\sigma\sigma'-\Omega_\tau^2) \end{vmatrix} = 0.$$

Equation (2.20) gives the dispersion relation

$$(2.21)\qquad \sigma'^2(i\sigma+\eta k^2)^2[W^5-A'W^4+B'W^3-C'W^2+D'W-E'] = 0,$$

where we have written $iW = \sigma$ and

$$A' = (\eta k^2+2\overline{1+\alpha_0}\,\nu_c),$$

$$B' = (-2\Omega+\nu k^2)^2+(1+\alpha_0)(1+\alpha_0+2\eta k^2)\nu_c+\Omega_\tau^2+k^2V^2,$$

$$C' = (-2\Omega+\nu k^2)^2(\eta k^2+2\nu_c)+(1+\alpha_0)^2\nu_c\eta k^2+\Omega_\tau^2\,\overline{(2+\alpha_0}\,\nu_c+\eta k^2)+k^2V^2(2+\alpha_0)\nu_c,$$

$$D' = (-2\Omega+\nu k^2)^2(2\eta^2+\nu_c)\nu_c+k^2V^2(1+\alpha_0)\nu_c^2+\Omega_\tau^2\,\overline{(1+\alpha_0}\,\nu_c+\overline{2+\alpha_0}\,\eta k^2)\nu_c,$$

$$E' = \eta k^2\nu_c^2[\Omega_\tau^2(1+\alpha_0)+(-2\Omega+\nu k^2)^2].$$

The first factor of Eq. (2.21) can be written as

$$(2.22)\qquad \sigma^2[-\sigma+i\nu_c(1+\alpha_0)]^2 = 0,$$

i.e. either $\sigma = 0$ or $\sigma = i\nu_c(1+\alpha_0)$.

The first value of σ in Eq. (2.22), gives a mode of neutral stability while the second one corresponds to a viscous type of a damped mode modified by frictional effects with neutrals.

The second factor of Eq. (2.21) gives

$$\sigma = i\eta k^2$$

what corresponds to a viscous type of a damped mode modified by finite conductivity.

Considering now the last factor of Eq. (2.21), we find that for

$$\Omega_\tau^2(1+\alpha_0)+(-2\Omega+\nu k^2)^2 > 0 \tag{2.23}$$

the roots of W are either all real and positive or there is a positive real root (or three positive real roots) and the remaining ones are complex. The real roots are responsible for stable modes. In the case of complex roots, Re W is positive, since the equation has alternately positive and negative coefficients. The composite plasma is stable.

If

$$\Omega_\tau^2(1+\alpha_0)+(\nu k^2-2\Omega)^2 < 0,$$

i.e.

$$\nu^2 k^4+[c^2(1+\alpha_0)-4\nu\Omega]k^2+4[\Omega^2-\pi G\varrho(1+\alpha_0)] < 0, \tag{2.24}$$

at least one root of the last factor of Eq. (2.21) is negative real and so the plasma is unstable when inequality (2.24) holds true.

The critical wave number k^* is given by

$$k^{*2} = \frac{4\nu\Omega-c^2(1+\alpha_0)\pm\sqrt{[c^2(1+\alpha_0)-4\nu\Omega]^2+16\nu^2[\pi G\varrho(1+\alpha_0)-\Omega^2]}}{2\nu^2}. \tag{2.25}$$

The plasma is unstable for the wave number range $k < k^*$. It is clear from Eq. (2.25) that the critical wave number is independent of finite conductivity and Hall currents.

Hence we conclude that the Jeans' criterion for gravitational instability holds true even if the effects due to rotation, collisions with neutrals, finite Larmor radius and Hall currents are included. The effect of rotation consists in decreasing the Larmor radius by on amount depending on the wave number of perturbation.

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