Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 2, pp. 197-204, Warszawa 1977

Two-dimensional drop in the presence of an electric field

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THIS PAPER is concerned with the behaviour of a two-dimensional drop immersed in a dielectric fluid in the presence of a uniform electric field. Assuming that the influence of the electric stresses on the fluid is included with no reciprocal effect of the motion on the fields, the equations of electrohydrodynamics are solved under the Stokes approximation. The circulation of fluid in and round a two-dimensional drop is compared with that of a spherical drop and the differences between them are examined. The surface-force density required to retain the shape of a two-dimensional drop is calculated and it is shown that the equilibrium geometry does not depend on the ratio of the viscosities of fluids in and out of the drop.

Praca poświęcona jest zagadnieniu zachowania się dwuwymiarowej kropli zanurzonej w płynnym dielektryku i poddanej działaniu jednorodnego pola elektrycznego. Uwzględniając wpływ napięć elektrycznych na płyn, lecz pomijając oddziaływania odwrotne ruchu na pole, rozwiązuje się równania elektrohydrodynamiki w przybliżeniu Stokesa. Cyrkulacje płynu wewnątrz i wokół dwuwymiarowej kropli porównuje się z odpowiednimi wynikami dotyczącymi kropli sferycznej, zwracając uwagę na pojawiające się tu różnice. Obliczono gęstość sił powierzchniowych potrzebnych do utrzymania kształtu kropli dwuwymiarowej; wykazano, że geometria stanu równowagi nie zależy od stosunku lepkości płynów wewnątrz kropli i w ośrodku otaczającym.

Работа посвящена задачи поведения двухмерной капли, погруженной в жидком диэлектрике, и подвергнутой действию однородного электрического поля. Учитывая влияние электрических натяжений на жидкость, но пренебрегая обратным воздействием движения на поле, решаются уравнения электрогидродинамики в приближении Стокса. Циркуляции жидкости внутри и вокруг двухмерной капли сравниваются с соответствующими результатами, касающимися сферической капли, обращая внимание на появляющиеся здесь разницы. Вычисляется плотность поверхностных сил необходимых для удержания формы двухмерной капли; показано, что геометрия состояния равновесия не зависит от отношения вязкости жидкостей внутри капли и в окружающей среде.

1. Introduction

THE STUDY of a flow system in which the electric field and the velocity field affect each other has been termed electrohydrodynamics. The applications of electrohydrodynamics are numerous: cryogenic fluid management in the zero-gravity environment of space, formation and coelesence of solid and liquid particles, electrogasdynamic high voltage and power generation, insulation research, physicochemical hydrodynamics, heat, mass and momentum transfer fluid mechanics, electrofluid dynamics of biological systems, and atmospheric and cloud physics [1]. In some applications explicit knowledge of the flow due to a single drop is required. Experimentally and theoretically it has been shown [2] that a circulation can occur in the drop and its surroundings in the presence of a uniform electric field. The equilibrium geometry of the drop was examined and the force required to retain the spherical shape was calculated. The equilibrium geometry varies between oblate and prolate ellipsoids, depending on the ratios of viscosities, dielectrics and conductivities of fluids in and out of the drop. In the limiting case in which the drop is highly conducting compared to the surrounding fluid, the electric field acts as normal with regard to the interface and hence the viscosity ratio does not play any particular role [3].

The behaviour of a two-dimensional drop immersed in fluid has been examined ([4, 5]) and it has been found that the two-dimensional solutions obtained have many features in common with the observed behaviour of three-dimensional drops. In this paper, the behaviour of a two-dimensional dielectric fluid drop immersed in another dielectric fluid in the presence of a uniform electric field is considered. The general view of the circulation of fluid in and round the drop is similar to that of a three-dimensional drop. A remarkable difference between the two-dimensional case and the three-dimensional one is that the equilibrium geometry of a two-dimensional drop does not depend on the viscosities of fluids in and out of the drop. However, when a drop is highly conducting as compared to the surrounding fluid, the viscosity ratio does not play any role because the electric field acts as normal with regard to the interface. The two-dimensional case and three-dimensional one may show a similar situation. Thus, considering the two-dimensional case it is possible to obtain some results about the three-dimensional case.

2. Governing equations

The magnetic induction in the fluid in and out of the drop is negligible because of dynamic currents is small enough. It is assumed that the influence of the electrical stresses on the fluid is included in the model, but there is no reciprocal effect of the motion on the fields. Therefore, the appropriate laws of electrodynamics are essentially those of electrostatics. Under the conditions considered here the governing equations of electrohydrodynamics are [3]

$$\nabla \times \mathbf{E} = 0, \tag{2.1}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{2.2}$$

$$\mathbf{I} = \sigma \mathbf{E}, \tag{2.3}$$

$$\nabla p = \mu \nabla^2 \mathbf{u}, \tag{2.4}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.5}$$

where $\mathbf{E}$ is the electric field intensity, $\mathbf{I}$ the electric current density, σ the electric conductivity, $\mathbf{u}$ the velocity, p the pressure, μ the viscosity; throughout the paper MKS units are used.

The boundary conditions to be applied at the interface of a drop in an electric field are [3]

$$\mathbf{n} \times \{\mathbf{E}\} = 0, \tag{2.6}$$

$$\mathbf{n} \cdot \{\sigma \mathbf{E}\} = 0, \tag{2.7}$$

$$\mathbf{n} \cdot \{\mathbf{u}\} = 0, \tag{2.8}$$

$$\mathbf{n} \times \{\mathbf{u}\} = 0, \tag{2.9}$$

$$\mathbf{n} \times \{\mathbf{\Sigma} + \mathbf{t}\} = 0, \tag{2.10}$$

$$\mathbf{n} \cdot \{\mathbf{\Sigma} + \mathbf{t}\} + T\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 0, \tag{2.11}$$

where $\mathbf{\Sigma}$ is the viscous stress which is given by

$$\mathbf{\Sigma} = -p\mathbf{n} + \mu[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\cdot\mathbf{n}, \tag{2.12}$$

and $\mathbf{t}$ is the electric stress which is given by

$$\mathbf{t} = -\frac{1}{2}\varepsilon\mathbf{E}\cdot\mathbf{E}\mathbf{n} + \varepsilon\mathbf{E}(\mathbf{E}\cdot\mathbf{n}) \tag{2.13}$$

and $\{A\}$ denotes the jump of A across the interface. T is the surface tension, and R_1 and R_2 are the radii of curvature of the surface; these radii are reckoned as positive when the corresponding centre of curvature lies on the side of the interface to which $\mathbf{n}$ points.

Under the conditions considered here, the electric field $\mathbf{E}$ and the velocity field $\mathbf{u}$ can be determined independently by Eqs. (2.1)–(2.5) and then, they can be related by the boundary conditions (2.6)–(2.11).

3. Electric field

We consider a drop or bubble, assuming that its shape is cylindrical with radius a. Electrodes lie at a distance of many radii from the drop and then the electric field is uniform far from the drop. Appropriate cylindrical polar coordinates are defined as originating

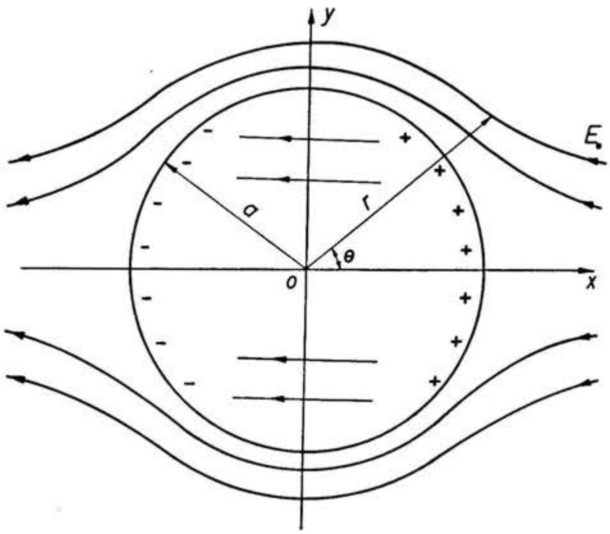

FIG. 1. The two-dimensional drop of radius a is immersed in a liquid in the presence of a uniform electric field of magnitude E_0 far from the drop.

at the centre of the drop; the x-axis is in the opposite direction of the applied electric field and the y-axis is normal to it (see Fig. 1). There are four boundary conditions for the electric field intensity: (i) $\mathbf{E}$ is finite inside the drop; (ii) the tangential component of the electric field is continuous across the surface of the drop; (iii) there is no surface current;

and (iv) $\mathbf{E}$ tends to $\mathbf{E}_0$ as $|\mathbf{x}|$ tends to infinity. Subject to these boundary conditions, Eqs. (2.1)–(2.3), (2.6) and (2.7) give that outside the drop

$$\mathbf{E} = \frac{2}{1+\alpha}[\mathbf{E}_0-(1-\alpha)(\mathbf{E}_0\cdot\mathbf{n})\mathbf{n}], \tag{3.1}$$

and inside the drop

$$\bar{\mathbf{E}} = \frac{2}{1+\alpha}\mathbf{E}_0. \tag{3.2}$$

The expression in Eq. (3.2) shows that the electric field inside the drop is uniform.

The circulation in and round the drop is responsible for electric force density which is related to the Maxwell stress tensor. We need the expressions of $\mathbf{t}$ over the surface of the drop. The tangential and normal component differences of $\mathbf{t}$ across the surface of the drop are

$$\mathbf{n}\times\{\mathbf{t}\} = \frac{4\bar{\varepsilon}}{(1+\alpha)^2}(\alpha\beta-1)(\mathbf{n}\cdot\mathbf{E}_0)(\mathbf{n}\times\mathbf{E}_0), \tag{3.3}$$

$$\mathbf{n}\cdot\{\mathbf{t}\} = \frac{2\bar{\varepsilon}}{(1+\alpha)^2}[(1-\beta)\mathbf{E}_0\cdot\mathbf{E}_0+(\beta\alpha^2+\beta-2)(\mathbf{E}_0\cdot\mathbf{n})^2], \tag{3.4}$$

where $\beta = \varepsilon/\bar{\varepsilon}$ is the ratio of the permittivities.

4. Velocity field

The flow considered in this paper is governed by Eqs. (2.4) and (2.5). The boundary conditions for the velocity are: (i) $\mathbf{u}$ is finite inside the drop and tends to zero as $|\mathbf{x}|$ tends to infinity; (ii) $\mathbf{u}\cdot\mathbf{n} = 0$ and $\bar{\mathbf{u}}\cdot\mathbf{n} = 0$ at the interface; (iii) the tangential component of the velocity across the drop is continuous; (iv) tangential electric stress and tangential viscous stress are in balance at the interface.

Following the general arguments given in [6] and [7] we write the pressure and the velocity in the fluid outside the drop

$$\frac{p-p_\infty}{\mu} = A\left(\frac{b_{jj}}{r^2}-\frac{2b_{kj}x_kx_j}{r^4}\right), \tag{4.1}$$

$$u_i = b_{jj}x_if(r)+b_{ij}x_jg(r)+b_{kj}x_kx_jx_ih(r), \tag{4.2}$$

where

$$b_{ij} = E_{0i}E_{0j}, \quad b_{ij} = b_{ji}, \quad b_{ii} = E_0^2,$$

and A is a constant. Using the same reasoning as for outside the drop, we write the pressure and the velocity in the fluid inside the drop

$$\frac{\bar{p}-\bar{p}_0}{\bar{\mu}} = C(b_{jj}x_ix_i-2b_{kj}x_kx_j), \tag{4.3}$$

$$\bar{u}_i = b_{jj}x_i\bar{f}(r)+b_{ij}x_j\bar{g}(r)+b_{kj}x_kx_jx_i\bar{h}(r), \tag{4.4}$$

where C is a constant.

Inserting Eqs. (4.1)–(4.4) into Eqs. (2.4) and (2.5) and then using the boundary conditions, we obtain, after some lengthy calculation,

$$A = \frac{2a}{E_0^2}U, \quad C = \frac{6}{a^3E_0^2}U, \quad U = -\frac{\bar{\varepsilon}E_0^2 a(\alpha\beta-1)}{2\bar{\mu}(1+\alpha)^2(1+\gamma)},$$

$$f = \frac{U}{aE_0^2}\left(\frac{a^2}{r^2}-\frac{a^4}{r^4}\right), \quad g = -\frac{2U}{aE_0^2}\frac{a^4}{r^4}, \quad h = \frac{2U}{a^3E_0^2}\left(\frac{2a^6}{r^6}-\frac{a^4}{r^4}\right),$$

$$\bar{f} = \frac{U}{aE_0^2}\left(\frac{r^2}{a^2}-1\right), \quad \bar{g} = \frac{2U}{aE_0^2}\left(1-\frac{2r^2}{a^2}\right), \quad \bar{h} = \frac{2U}{a^3E_0^2},$$

where U is a velocity (its meaning will be given in the next section) and $\gamma = \mu/\bar{\mu}$.

For a later use it is convenient to give the expression of the normal component of $\mathbf{\Sigma}$ across the interface. It can be written as

$$\mathbf{n}\cdot\{\mathbf{\Sigma}\} = \bar{p}_0 - p_\infty - \frac{\bar{\varepsilon}(\alpha\beta-1)}{(1+\alpha)^2}[\mathbf{E}_0\cdot\mathbf{E}_0 - 2(\mathbf{E}_0\cdot\mathbf{n})^2]. \tag{4.5}$$

It is a remarkable fact that the normal stress difference does not depend on the viscosities in and out of the drop. This situation may occur when the electric field acts as normal with regard to the interface [3].

We use the appropriate cylindrical polar coordinates defined in Fig. 1. Outside the drop, using $\mathbf{x} = r\mathbf{n}$ in Eq. (4.2), we have

$$\mathbf{u} = [\mathbf{E}_0\cdot\mathbf{E}_0\, rf + (\mathbf{E}_0\cdot\mathbf{n})^2 r^3 h]\mathbf{n} + (\mathbf{E}_0\cdot\mathbf{n})rg\mathbf{E}_0.$$

Considering that

$$\mathbf{E}_0\cdot\mathbf{n} = -E_0\cos\theta, \quad \mathbf{E}_0\cdot\mathbf{e}_\theta = E_0\sin\theta,$$

where $\mathbf{n} = \mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors in cylindrical polar coordinates, we obtain the u_r and u_θ components of the velocity in the form

$$u_r = -U\left(\frac{a}{r}-\frac{a^3}{r^3}\right)\cos 2\theta,$$

$$u_\theta = U\frac{a^3}{r^3}\sin 2\theta,$$

and similarly inside the drop we have

$$\bar{u}_r = -U\left(\frac{r^3}{a^3}-\frac{r}{a}\right)\cos 2\theta,$$

$$\bar{u}_\theta = -U\left(\frac{r}{a}-2\frac{r^3}{a^3}\right)\sin 2\theta.$$

It is possible to define a stream function which is related to the u_r and u_θ components of the velocity by the relation

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{\partial\psi}{\partial r}.$$

An integration gives out the drop

$$\frac{\psi}{Ua} = -\frac{1}{2}\left(1-\frac{a^2}{r^2}\right)\sin 2\theta,$$

and in the drop

$$\frac{\bar{\psi}}{Ua} = -\frac{1}{2}\left(\frac{r^4}{a^4}-\frac{r^2}{a^2}\right)\sin 2\theta .$$

It is very interesting to note that the stream lines are exactly of the same form in the present case and in the case of a bubble in a pure straining motion in the absence of an electric field.

There are nine stagnation points, four of them at the surface of the drop which are located at $(a, 0)$, $\left(a, \frac{\pi}{2}\right)$, (a, π), $\left(a, 3\frac{\pi}{2}\right)$, one at the centre of the drop, four inside the drop and located at $\left(\frac{\sqrt{2}}{2}a, \frac{\pi}{4}\right)$, $\left(\frac{\sqrt{2}}{2}a, \frac{3\pi}{4}\right)$, $\left(\frac{\sqrt{2}}{2}a, \frac{5\pi}{4}\right)$, $\left(\frac{\sqrt{2}}{2}a, \frac{7\pi}{4}\right)$. The location of the stagnation points in the drop is symmetrical. For a spherical drop the stagnation points in the drop are located at $\left(\sqrt{\frac{3}{5}}a, 54^\circ\right)$, $\left(\sqrt{\frac{3}{5}}a, 126^\circ\right)$, $\left(\sqrt{\frac{3}{5}}a, -126^\circ\right)$, $\left(\sqrt{\frac{3}{5}}a, -54^\circ\right)$. The stagnation points in the cylindrical drop are closer to the centre of

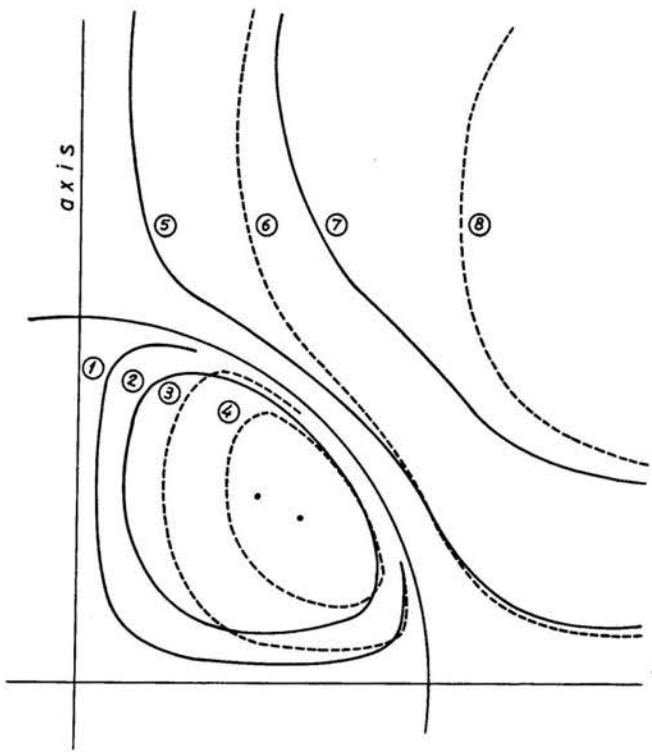

FIG. 2. Streamlines, drawn in a one fourth plane alone, in the present case and Ref. [2] by G. I. Taylor; ———, the present case; ①:0.02, ②:0.05, ⑤:-0.05, ⑦:-0.2; the numbers are values of ψ/Ua. - - - -, Taylor's case: ③:0.02, ④:0.05, ⑥:-0.05, ⑧:-0.2; the numbers are values of ψ/Ua^2. ● and ○ show the stagnation points in the present case and Taylor's case, respectively.

the drop than to that of the spherical drop. As illustrated in Fig. 2, the ratio of the location of the stagnation points in the cylindrical drop to that of a spherical drop having the same radius as the cylindrical drop is $0.71/0.77 = 0.92$. Figure 2 gives a comparison between the two-dimensional case and three-dimensional one. It shows that there is a fairly good agreement between streamlines in both cases.

Although the two-dimensional drops considered might be thought to be of little relevance to the three-dimensional drops encountered in practice, the solutions derived show remarkable similarities with the observed behaviour of the latter. As the discussion points out it is possible to have some ideas about the three-dimensional case when the two-dimensional one is examined.

Now we consider the velocity at the surface of the drop, and then we write

$$u_\theta = U\sin 2\theta \quad \text{at} \quad r = a.$$

This shows that

$$|(u_\theta)_{\max}| = U \quad \text{for} \quad \theta = \pm\frac{\pi}{4} \quad \text{and} \quad \theta = \pm 3\frac{\pi}{4};$$

thus, U is the maximum velocity. If the drop is insulating as compared to the surrounding fluid, $\beta\alpha$ is less than unity and then U becomes greater than zero. When $U > 0$, if $0 < \theta < \frac{\pi}{2}$, u_θ becomes positive, and if $0 < \theta < \pi$, u_θ becomes negative.

5. The equilibrium geometry of the drop

The balance of the normal stresses on the interface of the drop is given by Eq. (2.11). Since we assume that the interface of the drop is to be cylindrical and of circular cross-section we replace the last term in Eq. (2.11) by $-T/a$. Equations (2.11), (3.4) and (4.5) give

$$p_\infty - \bar{p}_0 = \frac{\bar{\varepsilon}E_0^2}{(1+\alpha)^2}[3-\beta(2+\alpha)] - \frac{T}{a}, \tag{5.1}$$

$$\frac{2\bar{\varepsilon}E_0^2}{(1+\alpha)^2}[\beta(\alpha^2+\alpha+2)-3] = 0, \tag{5.2}$$

where $p_\infty - \bar{p}_0$ gives the relative hydrostatic pressure.

When Eq. (5.2) is satisfied the drop has a circular shape. In order to find out whether the drop will become oblate or prolate under conditions where Eq. (2.11) is not quite satisfied, we employ Taylor's technique [2] and assume that a stress $F_0\cos^2\theta \left(= F_0\frac{(\mathbf{E}_0\cdot\mathbf{n})^2}{E_0^2}\right)$ applied normally to the surface of the drop is necessary to keep it circular. If we replace T in the modified form of Eq. (2.11) by $T+F_0\cos^2\theta$ and equate the coefficients of $\cos^2\theta$ we find

$$F_0 = \frac{2\bar{\varepsilon}E_0^2}{(1+\alpha)^2}\Phi, \tag{5.3}$$

where

$$\Phi = \beta(\alpha^2+\alpha+1)-3.$$

The equilibrium geometry depends on Φ, namely the functional relation which is given by α and β. It is very remarkable that the equilibrium geometry does not depend on γ, namely the ratio of the viscosities. This may be so due to the electric field which acts as normal with regard to the interface. If $\Phi = 0$, the drop is in steady-state equilibrium and if $\Phi < 0$, in the absence of F_0, the shape of the drop will decrease its extent in the direction of the applied electric field. If $\Phi > 0$, similar reasoning indicates that the drop would elongate in the direction of the applied electric field.

Acknowledgment

The comments of the referee regarding an earlier version of this paper are deeply appreciated. Computer calculations were performed at the Technical University of Istanbul Computer Center. The author expresses appreciation to Dr. O. E. PEREMECI who assisted with the numerical computations.

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Received May 21, 1975.