110.

ON THE THEOREM THAT AN ARITHMETICAL PROGRESSION WHICH CONTAINS MORE THAN ONE, CONTAINS AN INFINITE NUMBER OF PRIME NUMBERS.

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THIS celebrated theorem is one of those which no one would think of doubting, but which are of extreme difficulty to prove. A pretended proof had been given by Legendre a good part of a century ago, and occupies a whole chapter in the *Théorie des Nombres*; but the first real demonstration was accomplished by Lejeune Dirichlet in his great memoir published in the Berlin *Transactions* for the year 1837.

The present communication is limited to the case of Arithmetical Progressions, proceeding according to the common difference 4 or 6. The fundamental theorem employed is an identical equation, on the one side of which are algebraical fractions of the form $\frac{x^p}{1-x^{2p}}$, where p represents any combination of the simple powers of any system of primes taken with the positive or negative sign, according as p contains an even or an odd number of factors, and on the other side simple powers of x, whose indices are all the odd numbers not containing any one of the given system of primes as a factor. In the case of progressions with the common difference 4, all the primes of the form $4q+3$ and their primary combinations figure as indices on the first side of the equation, and consequently the powers of x on the other side have for their indices combinations of factors of the form $4q+1$. By writing in place of x the square root of negative unity into x, it is shown instantaneously that if the number of primes of the form $4q+3$ were finite, a finite series of fractions converging to an infinite value as x approaches to unity would be equal to another such series which would remain finite, which is of course absurd. The proof applicable to the case of progressions of the form $4q+1$ is not quite so simple: it depends on showing that, if their number were only i in number, it would be possible to have a rational integer function of the degree $2i$ in the logarithm of n greater than a finite multiple of n for a value of n unlimitedly great, which is known to be absurd.

A process precisely similar applies, *mutatis mutandis*, to the case of progressions of the form $6q+1$, $6q+5$, the sole difference being that, instead of substituting for x, x multiplied by the square root of negative unity, we must now substitute for it x multiplied successively by the two prime sixth-roots of unity, and subtract the results from one another. The method here successfully employed in the treatment of these elementary cases appears to differ fundamentally from Dirichlet's method, in regard of the circumstance that it deals with an infinitesimal variation in the value of the variable, whereas in Dirichlet's method the infinitesimal variation takes place in the index of the power of the variable.