105.

NOTE ON A NEW CONTINUED FRACTION APPLICABLE TO THE QUADRATURE OF THE CIRCLE.

[*Philosophical Magazine*, XXXVII. (1869), pp. 373—375.]

In a recent note [p. 689, above] inserted by the author in the *Philosophical Magazine* it was virtually shown, and indeed becomes almost self-evident *as soon as stated*, that the equation $u_{x+1} = \frac{u_x}{x} + u_{x-1}$ possesses two particular integrals, α_x, β_x, which are the products of x terms of the respective progressions

$$[1,\ 1,\ \tfrac{3}{2},\ 1,\ \tfrac{5}{4},\ 1,\ \tfrac{7}{6}, \ldots];$$
$$[1,\ \tfrac{2}{1},\ 1,\ \tfrac{4}{3},\ 1,\ \tfrac{6}{5},\ 1, \ldots].$$

Now any continued fraction whose partial quotients are

$$\frac{1}{k},\ \frac{1}{k+1}, \ldots, \frac{1}{x}$$

will be equal to the ratio of some two particular values of u_x in the above equation, that is, of two linear functions of α_x, β_x; and in especial when $k=1$ it will be found very easily that this fraction is $\frac{\beta_x - \alpha_x}{\alpha_x}$.

But, on supposing x infinite, $\frac{\beta_x}{\alpha_x}$ becomes equal to the well-known factorial expression for $\frac{\pi}{2}$, viz. $\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \ldots$. Hence we may deduce the following value for $\frac{\pi}{2}$ under the form of a continued fraction, namely,

$$\frac{\pi}{2} = 1 + \frac{1}{1+} \frac{1}{2^{-1}+} \frac{1}{3^{-1}+} \frac{1}{4^{-1}} \ \textit{ad infinitum}.$$

Reverting to pure integers, the above equality may be written as follows,

$$\frac{\pi}{2} = 1 + \frac{1}{1 + \frac{2}{1 + \frac{6}{1 + \frac{12}{1 + \frac{20}{1}}}}} \textit{ ad infinitum},$$

the denominators of the partial fractions being all units, and the numerators (after the first) the doubles of the natural series of triangular numbers 1, 3, 6, 10 .... This is obviously the simplest form of continued fraction for π that can be given, and yet, strange to say, has not, I believe, before been observed. Truly wonders never cease!

At first sight it might seem as if the above-stated continued fraction were incapable of teaching anything that cannot be got direct out of the Wallisian representation itself that has become transformed into it. Thus, for example, the convergent

$$1 + \frac{1}{1+}\,\frac{2}{1+}\,\frac{6}{1+}\,\frac{12}{1}, \text{ that is } \frac{64}{45},$$

is identical with the corresponding factorial product $\frac{2.2.4.4}{1.3.3.5}$. But I think a substantial difference does arise in favour of the continued fraction form, inasmuch as it indicates a certain obvious correction to be applied in order that the convergence may become more exact. For if we call

$$\frac{n(n+1)}{1+}\,\frac{(n+1)(n+2)}{1+} \ldots \textit{ ad infinitum} = u_n,$$

we have $u_n = \frac{n^2+n}{1+u_{n+1}}$. This shows that u_n cannot remain finite when n becomes infinite; for then u_{n+1} would also be finite, and consequently u_n would be a finite fraction of infinity, which is a contradiction in terms.

Hence ultimately

$$u_n \,.\, u_{n+1} = n^2 + n, \text{ that is } u_n = n,$$

or, in other words,

$$\frac{1}{n^{-1}+}\,\frac{1}{(n+1)^{-1}+}\,\frac{1}{(n+2)^{-1}+} \ldots \textit{ ad infinitum},$$

converges (and, it may be shown, always in an *ascending* direction) towards unity as its limit when n converges towards infinity. Thus we may write when n is very great,

$$\frac{\pi}{2} = 1 + \frac{1}{1+}\,\frac{2}{1+}\,\frac{6}{1+} \ldots + \frac{n^2-n}{1+n}{*}.$$

* This comes to the same thing as saying that for the purposes of calculation the continued fraction should be always considered as ending with a numerator, 1, and not with a denominator,

For example, when $n=4$, $\frac{\pi}{2}$ approximately equals

$$1+\frac{1}{1+}\,\frac{2}{1+}\,\frac{6}{1+}\,\frac{12}{1+4}, \text{ that is } =\frac{128}{81}, \text{ or } 1\cdot5802,$$

and, when $n=5$, will be found to be $\frac{352}{225}$ or 1·5644. The uncorrected convergent corresponding to the former of these is, as we have seen, $\frac{64}{45}$, or 1·4222; and the next is $\frac{384}{225}$, or 1·7056, the true value of $\frac{\pi}{2}$ being 1·5708. The errors given by the uncorrected factorial values are ·1486 and ·1348 respectively (of course with opposite signs), whereas the errors corresponding to the corrected values are only ·0094 and ·0064; the approximation being thus more than fifteen and twenty-one times bettered for the fourth and fifth convergents respectively by aid of the correction.

$\frac{1}{k}$. For example, $1+\frac{1}{1+1}$, that is, $\frac{3}{2}$ is a good deal nearer to $\frac{\pi}{2}$ than $1+\frac{1}{1+2}$, that is, $\frac{4}{3}$, is; and so $1+\frac{1}{1}+\frac{1}{\frac{1}{2}+1}$, or $\frac{8}{5}$, is much nearer to it than $1+\frac{1}{1}+\frac{1}{\frac{1}{2}+3}$, that is, $\frac{16}{9}$, is.

By taking the mean between two such consecutive corrected convergents, or, still better, the mean between two such consecutive means, and so on, a few terms will serve to give a very close approximation indeed to the limit $\frac{\pi}{2}$.