103.

OUTLINE TRACE OF THE THEORY OF REDUCIBLE CYCLODES*, THAT IS A PARTICULAR FAMILY OF SUCCESSIVE INVOLUTES TO A CIRCLE WHOSE DETERMINATION DEPENDS ON THE SOLUTION OF AN ALGEBRAICO-DIOPHANTINE EQUATION, AND OF THE NUMBER AND CLASSIFICATION OF THE FORMS OF SUCH FAMILY FOR ANY GIVEN ORDER OF SUCCESSION†.

La conquête de la vérité exige le concours harmonique de toutes les facultés humaines.

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(1) A CYCLODE is the continued [nth] involute of a circle. The centre of the circle is the pole of the cyclode.

(2) If x, y are the *inclination* and *polar distance* of the tangent to a cyclode, $y = (x, 1)^n$ is its equation, where n is the number of unwindings, or,

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* See *Philosophical Magazine* (October and December, 1868) [pp. 630, 641, above] for preliminary notions on involutes in general and on the successive involutes to a circle. Attached to the second of these notes is a valuable plate containing beautifully drawn specimens of cyclodes of the first three orders, executed by Mr Spottiswoode.

† It may be right to mention that the results arrived at in the denumerational portion of this paper have been obtained almost exclusively by processes of observation, comparison, experiment, and induction, satisfying the sense of the perfect and beautiful; and consequently the theory must be regarded as essentially non-mathematical, it having been authoritatively laid down by a great naturalist and eminent public instructor in the *Fortnightly Review* for June 1869, that Mathematics is "that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing" [does any science know much?] "of causation." Lagrange has expressed himself pointedly (in a passage which I cannot at the moment recall) in the very opposite sense to Professor Huxley; so certainly would Fermat, Euler, Gauss, Jacobi, Abel, Cauchy, Eisenstein, Kronecker, Plücker, Riemann, and the other great masters of our art, had their opinion on the subject been appealed to. In a subjective sense this paper owes its origin to my attention having been drawn by the Rev. James White, of Trinity Church, Woolwich, to Capt. Moncrieff's self-reversing gun-carriage, the rack in which for steadying and regulating the motion is the curve which would be traced on the plane of a wheel rolling on a rail by a point fixed on above or below the rail. When the point is level with the centre of the wheel, the curve traced is Archimedes' Spiral, when level with the rail the first Involute of the circle, and when midway between the rail and the centre a *Square root* (in the quaternion sense) of that Second

so to say, the order of the involution. When the radius of the circle is unity and the axis of reference is the line joining the pole to the cusp of the $(n-1)$th evolute of the cyclode (that is the first involute of the generating circle), the coefficients of x^n, x^{n-1} become 1 and 0 respectively. In what follows, it is to be understood that in general this simplified form is the one referred to.

(3) If r is radius, ρ radius of curvature, s length of arc reckoned from any origin on the curve,

$$r^2 = y^2 + \left(\frac{dy}{dx}\right)^2,$$

$$s = \int \rho dx = \int y dx + \frac{dy}{dx}.$$

Hence every cyclode is rectifiable, s and r^2 being unicursally related.

(4) The arco-radial equation is of the form $F(s, r^2) = 0$. When $y^2 + y'^2 = \square$, s and r become unicursally related, so that

$$F(s, r^2) = \Phi(s, r)\,\Phi(s, -r),$$

and the cyclode is said to be *reducible**.

(5) Order is the degree of the parabolic function $(x, 1)^n$. Class is the number of distinct factors in $(x, 1)^n$, that is the number of *distinct* tangents that can be drawn to a cyclode from its pole; for in the case of such tangents $y = 0$.

(6) Reducible cyclodes form a *family*: the *class* and the *order* of every reducible cyclode must both be *even*. They are of two *kinds*, symmetrical and unsymmetrical. The simplified form for the symmetrical kind is $(x^2, 1)^r$.

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involute to the circle whose apse is midway between the centre of the circle and the cusp of the first involute, to which second involute or cyclode I have given the name of the Norwich Spiral, to mark the fact of its birth at the Meeting of the British Association held last year in that city. The theory has since grown upwards and outwards faster than I have been able to climb after it, like the *bean-stalk* of profound significance in the well-known child's story. A very elegant and simple (because philosophically conceived) solution of the problem of finding the form of the rack (which I call a *Convolute* to the circle, and which the Arsenal people mistakenly but naturally enough supposed to be a Cycloidal curve) was given by Mr White in the *Educational Times* (*vide* Reprint for latter half of 1868). The problem was originally proposed many years ago by Mr Earnshaw, as a Senate-House problem, suggested to him by seeing a boy trundle a wheel against a brick wall in the streets of Cambridge.

* σ (the length of the arc of the pedal in respect of the pole) is $\int dx \cdot r$; so that s, σ, x, y, r are all unicursally related in the case of *reducible* cyclodes. The equation in the text, $y^2 + \left(\frac{dy}{dx}\right)^2$ = a perfect square, is the algebraico-diophantine equation referred to in the title. A more general equation, towards the solution of which I have made a few steps, is the following—

$$y^2 + y'^2 = \eta^2 + \lambda\eta'^2;$$

but I do not at present perceive its geometrical bearing.

A symmetrical cyclode is divided into two equal and similar parts by the line joining the pole to the cusp of its $(n-1)$th evolute. Non-symmetrical reducible cyclodes are left- and right-handed; they appear in pairs, the equations to the two of a pair being $y = Fx$, $y = F(-x)$ respectively.

(7) Let ν be divided in all possible ways (say ω ways) into a given number μ of parts; let the *partitionments* be $P_1, P_2, \ldots P_\omega$. Then $(P_1, P_2, \ldots P_\omega)^2$ developed contains terms of the form $P_j \cdot P_j$, and of the form $P_j P_k$, $P_k P_j$. Every such binary combination may be called a *Diptych* of a group of reducible cyclodes of order 2ν and class 2μ; and the total of such groups will be the total of reducible cyclodes of order 2ν and class 2μ. The sum of these totals for all classes from 2 to 2ν will be the *ensemble* of reducible cyclodes of order 2ν.

(8) The *alæ* of a diptych are the two sets of terms which form its two sides. These *alæ* will sometimes be represented as folded together, and sometimes expanded, as convenience may dictate. Under these *alæ* each diptych, as will be seen, shelters a group or brood of cyclodes. The relation between the *alæ* and the parabolic functions $(x, 1)^n$ of the reducible cyclodes proper to the *diptych* is as follows. Suppose that

$$i, j, k, \ldots \mid i_1, j_1, k_1, \ldots$$

is the diptych. Then by hypothesis

$$i + j + k \ldots = i_1 + j_1 + k_1 \ldots = \nu,$$

the semi-order.

$i, j, k \ldots$ are the same in number (that number being μ the semi-class) as $i_1, j_1, k_1, \ldots$

$y = U U_1{}^*$ is the equation to every cyclode proper to the given diptych,

where
$$U = (x + a)^i\,(x + b)^j\,(x + c)^k \ldots$$
$$U_1 = (x + a_1)^{i_1}(x + b_1)^{j_1}(x + c_1)^{k_1} \ldots$$

To find $a, b, c, \ldots$; $a_1, b_1, c_1, \ldots$ write

$$V = (\lambda + a)(\lambda + b)(\lambda + c) \ldots, \qquad V_1 = (\lambda + a_1)(\lambda + b_1)(\lambda + c_1) \ldots,$$

$$E = \left(2i \frac{d}{da} + 2j \frac{d}{db} + 2k \frac{d}{dc} \ldots \right), \qquad E_1 = \left(2i_1 \frac{d}{da_1} + 2j_1 \frac{d}{db_1} + 2k_1 \frac{d}{dc_1} \ldots \right),$$

and let $EV = E_1 V_1 = V - V_1$ for all values of λ. This implies

$$2\Sigma i = 2\Sigma i_1 = \Sigma a - \Sigma a_1,$$
$$\Sigma 2i\,(b + c \ldots) = \Sigma 2i_1\,(b_1 + c_1 + \ldots) = \Sigma ab - \Sigma a_1 b_1,$$
$$\Sigma 2i\,(bc \ldots) = \Sigma 2i_1\,(b_1 c_1 \ldots) = \Sigma abc - \Sigma a_1 b_1 c_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

* The U and U_1 may be called the two alar constituents, or more simply the segments of the parabolic function appertaining to any reducible cyclode; and we see that the radial tangents (or tangent radii from the centre) of every such cyclode constitute two distinct groups corresponding to these segments.

The equality $\Sigma i = \Sigma i_1$ has been presupposed between the elements of the diptych. Remain $2\mu - 1$ equations between the 2μ quantities a, b, c, ... a_1, b_1, c_1, ...

Hence the quantities themselves are, as they should be, indeterminate; but their differences will be determinate (as is easily proved)*. If we employ the simplified form, then in addition to the $2\mu - 1$ equations above given, there will be the equation

$$\Sigma a + \Sigma a_1 = 0,$$

which will make the system perfectly determinate.

(9) If $i, j, k, \ldots i_1, j_1, k_1$ are all unequal, the number of solutions will be

$$1 \times 1 . 2 \times 2 . 3 \ldots \times (\mu - 1) . \mu,$$

that is $\Pi(\mu - 1)\Pi(\mu)$.

(10) The effect of interchanging the *alæ* of the diptych (when unsymmetrical) is to convert every a into $-a$, and every a_1 into $-a_1$; but when the *alæ are alike* this change leaves the solutions unaltered. The solutions proper to a conformable diptych (that is one in which the *alæ* agree term for term) resolve themselves into two groups:—one group in which

$$a = -a_1, \quad b = -b_1, \quad c = -c_1, \ldots,$$

which is the group of symmetrical reducible cyclodes; and a second group, in which the solutions appear in *pairs* of the form

$$a, \quad b, \quad c, \quad \ldots \quad a_1, \quad b_1, \quad c_1, \ldots$$
$$-a_1, \quad -b_1, \quad -c_1, \quad \ldots -a, \quad -b, \quad -c, \ldots$$

Thus non-symmetrical reducible cyclodes are of two sorts:—congeminate, where the left- and right-handed curves belong to the same system of equations; and contrageminate, where the left- and right-handed curves belong to opposite systems. The former are associated with the symmetrical species, inasmuch as they belong to diptychs which engender along with them the symmetrical species; the latter belong to unconformable diptychs, that is with dissimilar *alæ*†. We now see more clearly the exactitude of the representation before stated of the diptychs of a given class by $(P_1, P_2, \ldots P_\omega)^2$, and the meaning of the coefficient 2 which affects the product $P_j . P_k$.

(11) The equation-system for unsymmetrical reducible cyclodes may be

* The property of *reducibility* is of course unaffected by the rotation of a cyclode round the pole; hence, not the roots of the parabolic function, but their differences ought to determine the existence of this property.

† The equation-system belonging to a conformable diptych may be resolved into two distinct systems containing respectively the symmetrical and the non-symmetrical solutions.

said to be *dualistic*.   For symmetrical cyclodes it is *monistic**, and takes the form

$$\nu = \Sigma a,$$
$$\Sigma i\,(b + c + d \dots) = 0,$$
$$\Sigma i\,(bc + bd + cd \dots) = \Sigma abc,$$
$$\Sigma i\,(bcd \dots) = 0,$$
$$\Sigma i\,(bcde \dots) = \Sigma abcde,$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots$$

Hence if $i, j, k, \dots$ are all unlike, the number of solutions for a monistic system belonging to a diptych of the μth class† is the product of μ terms of the fluctuating progression

$$1 \,.\, 1 \,.\, 3 \,.\, 3 \,.\, 5 \dots$$

(12) The number of solutions of a determinate algebraic system of equations may be called its *denumerant*. The denumerant may be algebraical or arithmetical.

In estimating the former, all solutions count, whether or not deducible from one another by interchange between the unknowns.

In estimating the latter, solutions which become identical by permuting the unknowns are regarded as one and the same solution. Thus, for example, the algebraical denumerant of $[x^2 + y^2 = c,\ x^4 + y^4 = e]$ is 8, its arithmetical denumerant is 4.

In the theory under consideration, it is always the arithmetical‡ denumerant which is intended, unless the contrary is expressed. Thus we see

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* The *dualistic* system for a conformable diptych resolves itself into the *monistic* system above given, which contains the symmetrical solutions, and another system (with which I do not think it desirable to encumber this sketch, but) which I have easily succeeded in isolating, and which contains the congeminate solutions. The congeminate denumerants do not, however, appear to lend themselves to a distinct law of summation apart from the general mass as the symmetrical ones do.

† It will be convenient to refer to the diptych of a cyclode of the order 2ν and class 2μ as being itself of the order ν and class μ.

‡ It is a singular and advantageous fact, quite contrary to what might have been expected from analogy, that it is these denumerants (corresponding to the actual number of distinct geometrical forms) and not the algebraical denumerants (the ordinary measures of the so-called orders of the various systems of equations) which lend themselves to summation. In the common theory of partitions *per contra*, if we wish to express their number algebraically, we must reckon permuted as distinct arrangements; this is as if we should substitute algebraical for arithmetical denumeration in the theory under consideration. This fact is partly to be accounted for by considering that the diophantine equation to be solved, namely, $f^2 + f'^2 = \phi^2$, may be attacked *exoscopically*, the coefficients in f and ϕ being posited as the unknowns, in which scheme of procedure permutations of equal factors in f would not present themselves at all : my original tentative efforts were made in this the seemingly most natural direction to follow ;

that one cause of the *Reduction* of the denumerant of a diptychical system of equations is the *consolidation* of several solutions into a single one. A second cause of reduction, arising from roots passing off to infinity, may be termed *evaporation*. For example, in the equation-system $[x - ky = c, x^3 - y^3 = e]$ the denumerant is 3; but when $k = 1$, one root passes off to infinity and the denumerant becomes 2.

Consolidation and evaporation may go on simultaneously: thus, for example, in the system $[x + ky = c,\ x^3 + y^3 = e]$, when $k = 1$, the denumerant is reduced first by evaporation to 2, and then by consolidation to 1.

(13)　The problem of which I shall indicate a complete method of solution is that of obtaining the denumerant to any given diptych whatever.

The original problem proposed for solution was to find the *total* number of reducible cyclodes of any given order; not only this, but the number of the same of any given *class*, as well as *order*, admits of easy *statement*.

If $D(n, m)$ be used to denote the total number of reducible cyclodes of the order $2n$ and class $2m$ (obtainable from a given circle), then

$$D(r + s, r) = D(r + s, s) = \frac{\Pi(r + s - 1)\,\Pi(r + s - 2)}{\Pi r\,\Pi(r - 1)\,\Pi s\,\Pi(s - 1)}.$$

Calling $D(r + s, r) = Q_{r, s}$, the annexed Table exhibits the values of $Q_{r, s} = Q_{s, r}$* for different values of r and s.

r \ Q	s 1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	3	6	10	15	21
3	1	6	20	50	105	196
4	1	10	50	175	498	1176
5	1	15	105	496	1764	5292
6	1	21	196	1176	5292	19404

but how futile it would have been to have persevered in this course, and how all but impossible to disentangle, by aid of it, the classes and genera and diptychical groups, is apparent from the very nature of the results here brought to light. Of all the analytical questions left unsolved by my predecessors or contemporaries which I have yet grappled with (and they are neither few nor facile), this presents beyond all comparison the hardest knot I have ever succeeded in untying. But what will not yield to patient contemplation! Nature, though coy, is kind, and does not predetermine to mock the pursuit of her worshippers.

* That $Q_{r, s} = Q_{s, r}$ implies $D(n, r) = D(n, n - r)$,—a truly remarkable theorem, which exists also when Δ replaces D.

$D(m, n)$ will consist of a certain number of symmetrical cyclodes, say $\Delta_{m, n}$, and $\dfrac{D(m, n) - \Delta(m, n)}{2}$ pairs of unsymmetrical cyclodes, giving a total of $\dfrac{D(m, n) + \Delta(m, n)}{2}$ absolutely distinct forms when left- and right-handed similar curves are treated as identical.

(14)   Let $\qquad\qquad \Delta(r + s, r) = q_{r, s}$,

then $q_{r, s} (= q_{s, r})$ is the quantity in the rth line and sth column of the Table B subjoined, deduced by zigzag multiplication from the adjacent Table A*.

TABLE A.

1	1	1	1	1	1
1	2	3	4	5	6
1	2	3	4	5	6
1	3	6	10	15	21
1	3	6	10	15	21
1	4	10	20	35	56

TABLE B.

				s			
r	$q\dagger$	1	2	3	4	5	6
1		1	1	1	1	1	1
2		1	1	2	2	3	3
3		1	2	4	6	9	12
4		1	2	6	9	18	24
5		1	3	9	18	36	60
6		1	3	12	24	60	100

(15) $D(n)$, $\Delta(n)$ being used to denote the absolute number of reducible cyclodes and the number of symmetrical reducible cyclodes, respectively, of the order $2n$ of all classes, we have

$$D(n) = \frac{\Pi(2n - 2)}{\Pi n \cdot \Pi(n - 1)},$$

$$\Delta(2k + 1) = \frac{\Pi(2k)}{(\Pi k)^2}, \quad \Delta(2k) = \tfrac{1}{2}\Delta(2k + 1).$$

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* Calling
$$1, 1, \quad 1, \quad 1, \quad 1, \ldots = L$$
$$1, 2, \quad 3, \quad 4, \quad 5, \ldots = L_1$$
$$1, 3, \quad 6, \quad 10, \quad 15, \ldots = L_2$$
$$1, 4, \quad 10, \quad 20, \quad 35, \ldots = L_3$$
....

the Table A itself is the product of zigzag multiplication of
$$\begin{Bmatrix} L & L_1 & L_2 & L_3 & L_4 \ldots \\ 1 & 1 & 1 & 1 & 1 \ldots \end{Bmatrix}.$$

† It is easily seen that when

$$r = 2h + 1, \quad s = 2k + 1; \quad q_{r, s} = \left(\frac{\Pi(h + k)}{\Pi h \Pi k} \right)^2,$$

$$r = 2h, \qquad s = 2k + 1; \quad q_{r, s} = \frac{\Pi(h + k)\,\Pi(h + k - 1)}{\Pi h \Pi(h - 1)(\Pi k)^2},$$

$$r = 2h, \qquad s = 2k; \qquad q_{r, s} = \frac{\{\Pi(h + k - 1)\}^2}{\Pi(h - 1)\,\Pi h\,\Pi(k - 1)\,\Pi(k)}.$$

The total number of reducible cyclodes and of reducible symmetrical cyclodes for all orders from 2 to 20 inclusive is exhibited in the Table below, where n is the semi-order.

n	Δ	D
1	1	1
2	1	1
3	2	2
4	3	5
5	6	14
6	10	42
7	20	132
8	35	429
9	70	1430
10	125	5236

The ratio of the successive values of Δ alternates between 2 and a variable quantity which converges ascendingly towards 2; that of the successive values of D converges (also ascendingly) towards 4. It will be observed that D and Δ agree, until $2n = 8$, showing that the non-symmetrical reducible cyclodes only begin to make their appearance with the eighth order of involution, that is after the generating tight string has been unwound 8 times in succession.

(16) The *number of distinct species of diptychs of any given* class *is limited,* being independent of the order ; *and every diptych of such species will have the same denumerant.* The doctrine of *genus and species* is founded on the notion of *parallel equalities**.

A parallel equality is one existing between the sum of any number of elements in one *ala* of a diptych, and the sum of a like number of elements in

* The foundation of the *conception* of the doctrine of parallel equalities lies in the easily demonstrable fact that the leading coefficient in the final resolving equation of the *monistic* system of equations proper to any conformable diptych $(a, b, c \dots l)^2$ is made up exclusively of powers of linear factors of the form $a - b, a + b - c - d, a + b + c - d - e - f$, &c., and that the same for the *dualistic* system proper to a general diptych $[a, b, c \dots l \mid a, \beta, \gamma \dots \lambda]$ is made up exclusively of powers of linear factors of the form $a - a, a + b - a - \beta, a + b + c - a - \beta - \gamma$, &c.: this fact is sufficient for proving that evaporation can only commence to take place by virtue of the existence of one or more parallel equalities, but is not sufficient to demonstrate the truth of the grand law that not only the *existence*, but the *amount* of evaporation is governed solely and exclusively by the nature of the parallel equality or system of parallel equalities (when several coexist either separately or mutually superimposed) in any given diptych. For a proof of what is stated in this note in respect of the leading coefficient of the monistic system, see *Phil. Mag.* for May 1869 [p. 689, below]; I am also in possession of the corresponding proof and theory for the dualistic system.

the opposite *ala*; that is, it is an inter-alar equinomial relation of equality to which it will sometimes be convenient to refer to as a *parallelism*. Two antagonistic parties of creditors to a bankrupt's estate, equal in number and claiming for equal amounts, gives an image of the parallelism in question. By the definition of a diptych there is necessarily one such parallel equality, namely between all the elements in one *ala* and all in the other*.

Any two diptychs in which the entire system of parallel equations is the same for each are said to belong to the same *genus*, and their *algebraical* denumerants will be the same. When in two diptychs belonging to the same genus all the equalities between the elements of each *ala* taken alone in the one are matched by like equalities between the analogous elements of the corresponding *ala* in the other, they are said to belong to the same *species*, and their *arithmetical* denumerants will be identical†.

(17) Now let it be proposed to determine the denumerants, for greater simplicity, say the total denumerants (including non-symmetrical as well as symmetrical cyclodes), of the several species belonging to any given class. There are as many unknowns, say, d_1, d_2, ... d_s, as there are species (s) included in such class: the total number of diptychs of that class belonging to any given order increases with the order, being represented by $(P_1, P_2, ... P_\omega)^2$, where P_1, P_2, ... P_ω denote the partitionments of the number n into m parts. For every assumed value of n we shall be able to obtain the value of $D(P_1, P_2, ... P_\omega)^2$ as a known linear function of d_1, d_2, ... d_s; but $D(P_1, P_2, ... P_\omega)^2$ is a known function of n and m‡; hence we may obtain the values of a sufficient number of linear functions of d_1, d_2, ... d_s to obtain each of them separately; and we know *à priori* that each is integer and positive.

Thus, regarding the denumerants as gas in relation to the equation-systems viewed as the gross matter from which this gas is extracted (a simile suggested by Mr Clifford), we witness in this theory the interesting phenomenon of this gas becoming again condensed, as it were (like solidified carbonic acid gas), into a sort of crystallized matter finer than that from

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* Consequently any partial inter-alar parallel equality implies a second complementary one; in other words, all the parallel equalities, excepting the implied total one, are conjugate and make their appearance in pairs.

† Thus we see that the elements of the diptych being all increased or diminished by the same quantity, or, more generally, being replaced respectively by the same linear function of each of them, will not affect the genus or species, as such substitution will leave the identities and parallel equalities unaltered; so that we may say that the species of a Reducible Cyclode is, as it were, a semi-invariant of the indices of the factors of its parabolic function.

‡ See Art. 13, where $r = m$, $s = n - m$, and $D(r + s, r)$ means the same as $D(P_1, P_2, ... P_\omega)^2$ in the text above.

which it was originally generated, the denumerants of the original equation-system becoming the unknowns of a new (linear) equation-system.

[Chasles's theory (with Zeuthen's, Cayley's, and Salmon's extensions) of conics which satisfy given conditions, the new theory of geometric Unicursal or successive Quadric Transformation, and this theory of Cyclodes, all belong to a peculiar branch of algebra to which might be given the name of pneumatic analysis; and they have many features in common.]

In the same way it may obviously be shown that the denumerants of the *symmetrical* cyclodes proper to the distinct species of conformable diptychs of any given class may be deduced from the solution of a system of linear equations as many in number as such species. In fact, however, only as many linear functions (in either case) need to be formed as there are distinct *genera*; for the denumerants (arithmetical) of the several *species* of the same genus are known, *à priori*, in terms of each other, since their algebraical denumerants are identical.

(18)　A most important and wonderful relation exists between $D(r+s, r)$ or $\Delta(r+s, r)$ and the corresponding denumerants of the conformable diptych $[a^r, b^s]^2$, where a^r means a, r times repeated, and b^s means b, s times repeated. [Observe that any such diptych whose class $r+s$ is given belongs to a single genus; for we cannot have

$$\rho a + \sigma b = \rho' a + \sigma' b \quad \text{and} \quad \rho + \sigma = \rho' + \sigma',$$

unless $a = b$, which is of course excluded.] Namely *the total denumerant of this diptych is equal to* $D(r+s, r)$, *and its special denumerant is equal to* $\Delta(r+s, r)$.

(19)　A very good example of *evaporation* is afforded by the diptych $[a^r]^2$; for the algebraical denumerant without evaporation would be $\Pi r \cdot \Pi (r-1)$; but the actual algebraical denumerant must contain $(\Pi r)^2$. Hence the evaporation is *total*, that is there is no reducible cyclode of any order or class whose parabolic function is made up of distinct factors raised all to the same power (and, as a particular case, made up exclusively of distinct simple factors), except when the cyclode is of the second degree, for $\Pi r \cdot \Pi (r-1)$ contains $(\Pi r)^2$ when $r = 1$; and in fact the equation of the Norwich spiral, which gave birth to this theory, is in its simplified form $y = (x+1)(x-1)$. This is analogous to the multiplicity (in Hirst's and Cremona's theory) of the *principal points* of any nucleus (as such system is termed by Professor Cayley) when, and not before, the degree of the transformation transcends the second. In the present theory we see that, except for the Norwich spiral, in all reducible cyclodes a certain number of the points, the tangents to which pass through the pole, must be cusps of a more or less elevated order.

(20)   As an example of the method of classification, grounded on parallel equalities, I subjoin the complete system of genera and species of diptychs of the third (corresponding to cyclodes of the 6th) class, with their corresponding denumerants*.

Genus (1)	Denumerant	Genus (2)	Denumerant	Genus (5)	Denumerant
a b c a β γ	12	a b c a β γ	10	a b c a a γ	4
a b c a β β	6	a b c a β β	5	a b b a a γ	2
a b c a a a	2	**Genus (3)†** a b c a b c	7	**Genus (6)** a b c a a a	
a b b a a β	3	**Genus (4)**		**Genus (7)**	
a b b a a a	1	a b b a b b	1	a a a a a a	0

Of the above 7 genera, the 3rd and 4th alone give rise to symmetrical cyclodes, the Δ (that is the symmetrical) denumerant for the 3rd being 3 and the 4th 1.

(21)   All species of conformable diptychs necessarily belong to distinct genera; that is the genera of conformable diptychs each contain only a single species. The conformable species of the fourth‡ class are as follows: [the numbers under Δ signify the number of *symmetrical* cyclodes proper to each genus, and under E the amount of evaporation].

* The *algebraical* denumerant, it will be noticed, is the same for all the species of the same genus. Thus, for example, in genus (1) for the first species this is 12, for the second $\Pi 2 . 6$, for the third $\Pi 3 . 2$, for the fourth $\Pi 2 . \Pi 2 . 3$, and for the last $\Pi 2 . \Pi 3 . 1$. The first genus, it will be seen, contains four, the second and fifth two, the third, fourth, sixth, and seventh one *species* respectively.

† The *special* denumerant means the number of *symmetrical* cyclodes contained by a diptych. The diptych $(a, b, c)^2$ having the special denumerant 3 and the total denumerant 7, we see that the number of congeminate pairs is $\dfrac{7-3}{2}$, that is, 2. So the number of congeminate *pairs* to $(a, b, c, d)^2$ is 36; and I have almost conclusive grounds for believing that in general the value of $D - \Delta$ for $(a_1, a_2, \ldots a_i)^2$ is $(i-2) \{\Pi (i-1)\}$; so that the number of right- and left-handed (congeminate) pairs contained in such diptych is

$$\frac{(i-2) \{\Pi (i-1)\}^2}{2}.$$

‡ There are upwards of 40 distinct *genera* of the 4th class when unconformable diptychs are taken into account. It is a problem of great interest to determine the number of genera and species contained in any given class.

Species	Δ	E†
$(a\ b\ c\ d)^2$	9	0
$(a\ b\ b\ d)^2$	4	1
$(a\ a\ c\ c)^2$	1	5
$(a\ b\ b\ b)^2$	1	3
$(a\ a\ a\ a)^2$	0	9
$(a\ b\ c\ b+c-a)^2$	7	2
$(a\ b\ b\ 2b-a)^2$	3	3

(22) The conformable species of the fifth class will be most easily represented concretely by means of actual numbers, so as to avoid the necessity of employing equations to denote the parallel equalities‡.

Types of species	Δ	E
$(1\ 2\ 3\ 5\ 8)^2$	45	0
$(1\ 2\ 3\ 5\ 5)^2$	21	3
$(1\ 2\ 3\ 3\ 3)^2$	6	9
$(1\ \dot{2}\ \dot{2}\ \dot{2}\ \dot{3})^2$	5	15
$(1\ 2\ 2\ 5\ 5)^2$	9	9
$(1\ 1\ 2\ 2\ 2)^2$	2	21
$(1\ 1\ 1\ 1\ 2)^2$	1	21
$(1\ 1\ 1\ 1\ 1)^2$	0	45
$(1\ 2\ 3\ 4\ 8)^2$	43	2
$(1\ \dot{2}\ \dot{4}\ \dot{5}\ \dot{7})^2$	41	4
$(1\ \dot{2}\ \dot{3}\ \dot{4}\ \dot{5})^2$	39	6
$(1\ 3\ 4\ 4\ 5)^2$	20	5
$(1\ \dot{2}\ 3\ 4\ 4)^2$	19	7
$(1\ \dot{2}\ \dot{2}\ \dot{3}\ 4)^2$	18	9
$(1\ \dot{2}\ \dot{2}\ 3\ 3)^2$	8	13

† It should be stated that the E column is deduced from the Δ column, and not *vice versâ*. It might have been imagined, *à priori*, that the amount of evaporation could be determined independently, in the first instance, and the reduction of the denumerants inferred therefrom; but it appears as a datum of experience that the formulæ of denumeration are always simpler than those of evaporation, although, of course, theoretically speaking, the evaporating solutions ought to be obtainable by calculating the solutions for which the new homogenizing variable takes the value zero, or, which means the same thing, for which 1=0. In a word, although in the order of nature the evaporation is the cause of the reduction, in the order of our knowledge the reduced denumerant, coming first, plays the part of a subtractor, and the evaporation that of a remainder.

‡ Since the above was sent to press, I have hit upon a very simple expedient for representing the parallels of any form of diptych by means of marks of connotation. This method is used in

(23)   Call the cyclodic order 2Ω, and the class 2χ.

Then for $\Omega = 1, 2, 3, 4, 5, 6, 7$ the diptychs grouped in classes, omitting the *vacuous* ones [that is those for which the evaporation is *total*], will be as follows :—

$$\Omega = 1 \quad \chi = 1 \quad (1)^2$$

$$\Omega = 2 \quad \chi = 1 \quad (2)^2$$

$$\Omega = 3 \quad \chi = 1 \quad (3)^2$$
$$\chi = 2 \quad (1\ 2)^2$$

$$\Omega = 4 \quad \chi = 1 \quad (4)^2$$
$$\chi = 2 \quad (1\ 3,\ 2\ 2)^2$$
$$\chi = 3 \quad (1\ 1\ 2)^2$$

$$\Omega = 5 \quad \chi = 1 \quad (5)^2$$
$$\chi = 2 \quad (1\ 4,\ 2\ 3)^2$$
$$\chi = 3 \quad (1\ 1\ 3,\ 1\ 2\ 2)^2$$
$$\chi = 4 \quad (1\ 1\ 1\ 2)^2$$

$$\Omega = 6 \quad \chi = 1 \quad (6)^2$$
$$\chi = 2 \quad (1\ 5,\ 2\ 4,\ 3\ 3)^2$$
$$\chi = 3 \quad (1\ 1\ 4,\ 1\ 2\ 3,\ 2\ 2\ 2)^2$$
$$\chi = 4 \quad (1\ 1\ 1\ 3,\ 1\ 1\ 2\ 2)^2$$
$$\chi = 5 \quad (1\ 1\ 1\ 1\ 2)^2$$

$$\Omega = 7 \quad \chi = 1 \quad (7)^2$$
$$\chi = 2 \quad (1\ 6,\ 2\ 5,\ 3\ 4)^2$$
$$\chi = 3 \quad (1\ 1\ 5,\ 1\ 2\ 4,\ 1\ 3\ 3,\ 2\ 2\ 3)^2$$
$$\chi = 4 \quad (1\ 1\ 1\ 4,\ 1\ 1\ 2\ 3,\ 1\ 2\ 2\ 2)^2$$
$$\chi = 5 \quad (1\ 1\ 1\ 1\ 3,\ 1\ 1\ 1\ 2\ 2)^2$$
$$\chi = 6 \quad (1\ 1\ 1\ 1\ 1\ 2)^2.$$

the sequel ; but I have not thought it worth while to disturb the text in this place otherwise than by introducing marks of connotation to distinguish the terms of the parallel equalities. It ought to be noticed that when any element of a parallel equality occurs more than once, the mark of connotation is to be placed indifferently over any (but only one) of the group of quantities thus repeated.

I may take advantage of this opportunity to refer, although a little out of order, to Mr Clifford's extremely ingenious idea of making the proof of the existence of a root to an algebraical function fx of degree n depend on the same for the degree $\dfrac{n(n-1)}{2}$, and consequently in the last resort on the same for a function of an odd degree, for which case it flows direct from the principle of continuity. Equating to zero each coefficient in the linear remainder of fx divided by $x^2 + px + q$, we obtain two simultaneous equations of known degrees (dependent on the value of n) in p and q. Now the pinch of the required proof consists in our being able to show that the p, q system satisfying these equations has exactly $\dfrac{n(n-1)}{2}$ values, neither more nor less ; so that it is requisite to establish, *à priori*, a certain definite amount of evaporation, which I believe no one has yet quite succeeded in doing. This example affords an

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Expanding we obtain the following set of diptychs, the numbers opposite to which are their symmetrical and total denumerants respectively. The Table extends from $\Omega = 1$ to $\Omega = 7$ inclusive.

Diptych	Δ	D
$(1)^2$	1	1
$(2)^2$	1	1
$(3)^2$	1	1
$(1\,2)^2$	1	1
$(4)^2$	1	1
$(1\,3)^2$	1	1
$1\,3\,.\,2\,2$	0	1
$(2\,2)^2$	0	0
$(1\,1\,2)^2$	1	1
$(5)^2$	1	1
$(1\,4)^2$	1	1
$(1\,4\,.\,2\,3)$	0	2
$(2\,3)^2$	1	1
$(1\,1\,3)^2$	1	1
$(1\,1\,3\,.\,1\,2\,2)$	0	2
$(1\,2\,2)^2$	1	1
$(1\,1\,1\,2)^2$	1	1

Diptych	Δ	D
$(6)^2$	1	1
$(1\,5)^2$	1	1
$(1\,5\,.\,2\,4)$	0	2
$(2\,4)^2$	1	1
$(1\,5\,.\,3\,3)$	0	1
$(2\,4\,.\,3\,3)$	0	1
$(3\,3)^2$	0	0
$(1\,1\,4)^2$	1	1
$(1\,1\,4\,.\,1\,2\,3)$	0	4
$(1\,2\,3)^2$	3	7
$1\,1\,4\,.\,2\,2\,2$	0	1
$1\,2\,3\,.\,2\,2\,2$	0	1
$(2\,2\,2)^2$	0	0
$(1\,1\,1\,3)^2$	1	1
$(1\,1\,1\,3\,.\,1\,1\,2\,2)$	0	4
$(1\,1\,2\,2)^2$	1	1
$(1\,1\,1\,1\,2)^2$	1	1*
$(7)^2$	1	1
$(1\,6)^2$	1	1
$(1\,6\,.\,2\,5)$	0	2
$(2\,5)^2$	1	1
$(1\,6)\,(3\,4)$	0	2
$(2\,5\,.\,3\,4)$	0	2
$(3\,4)^2$	1	1

instructive corroboration of the remark in the last preceding foot-note, concerning evaporation coming second in the order of our knowledge, although first in the order of nature. The arrangement of the diptychical denumerants for a given order and class in the text which follows is columnar. The more natural arrangement (and one so likely to be fruitful in consequences, that I much regret not thinking of it in time) is obviously that of a square, of which each side will contain as many places as there are partitionments corresponding to the order and class. In that diagonal which separates the square into two symmetrical sets of figures (and nowhere else) there will be double entries, corresponding to the D and Δ denumerants respectively of the conformable diptychs.

* Thus for the 12th order and 4th class the number of symmetrical reducible cyclodes is $1+1$, that is, 2, and the total number is $1+2\,.\,2+1+2\,.\,1+2\,.\,1+0=10$; so for the 12th order

Diptych	Δ	D
$(1\ 1\ 5)^2$	1	1
$(1\ 1\ 5\ .\ 1\ 2\ 4)$	0	4
$(1\ 2\ 4)^2$	3	7
$(1\ 1\ 5\ .\ 1\ 3\ 3)$	0	2
$(1\ 2\ 4\ .\ 1\ 3\ 3)$	0	5
$(1\ 3\ 3)^2$	1	1
$(1\ 1\ 5\ .\ 2\ 2\ 3)$	0	3
$(1\ 2\ 4\ .\ 2\ 2\ 3)$	0	4
$(1\ 3\ 3\ .\ 2\ 2\ 3)$	0	2
$(2\ 2\ 3)^2$	1	1
$(1\ 1\ 1\ 4)^2$	1	1
$(1\ 1\ 1\ 4\ .\ 1\ 1\ 2\ 3)$	0	6
$(1\ 1\ 2\ 3)^2$	4	16
$(1\ 1\ 1\ 4\ .\ 1\ 2\ 2\ 2)$	0	3
$(1\ 1\ 2\ 3\ .\ 1\ 2\ 2\ 2)$	0	7
$(1\ 2\ 2\ 2)^2$	1	1
$(1\ 1\ 1\ 1\ 3)^2$	1	1
$(1\ 1\ 1\ 1\ 3\ .\ 1\ 1\ 1\ 2\ 2)$	0	4
$(1\ 1\ 1\ 2\ 2)^2$	2	6
$(1\ 1\ 1\ 1\ 1\ 2)^2$	1	1*

The above Table gives a complete conspectus of the groups of reducible cyclodes of all the even (the only admissible) orders up to the 14th inclusive.

(24) The diophantine equation $y^2 + y'^2 = \square$ may be transformed into

$$1 + \left(\frac{y'}{y}\right)^2 = \square.$$

---

and 8th class these numbers are $1+1$ and $1+2\ .\ 4+1$, that is, 2 and 10 as before, as should be the case according to article (13). The equations to be resolved are of the degree D for unconformable, and of the degrees Δ and $\dfrac{D-\Delta}{2}$ for conformable diptychs; hence up to the 12th order of reducible cyclodes, inclusive, we have not to deal with equations of a higher degree than the 4th, and consequently all the reducible cyclodes of order inferior to 14 admit of explicit algebraical representation.

* Looking at the 3rd and 4th classes, the sums of the numbers in the D column will be found to be 30 and 34 respectively; but on doubling the numbers printed opposite to the unconformable diptychs (which represent not single but paired cyclodes), the number is the same (as it ought to be) for each class, namely, 50, that being the number in the 3rd column and 4th line (or *vice versâ*) of the Q Table, Art. 13. Similarly the corresponding number for the 2nd and 5th classes, doubling as before, becomes 15 instead of 9 for the one, and 11 for the other. It is a noticeable fact that the number of pairs is not the same in two conjugate classes.

In this equation we may suppose

$$y = (x + a)^\alpha (x + b)^\beta (x + c)^\gamma \dots,$$

$\alpha, \beta, \gamma \dots$ being now *general* quantities and no longer necessarily integers.

If　　　　　　　$\eta = \log y = \Sigma \alpha \log (x + a),$

x, η are the coordinates of a compound logarithmic wave.

If the number of terms in Σ is even, and the quantities $a, b, c \dots$ be divided arbitrarily into two sets containing a like number of terms, as, say, $a, b, c \dots, a', b', c' \dots, \alpha, \beta, \gamma, \dots, \alpha', \beta', \gamma'$ being corresponding values of the coefficients,

$$1 + \left(\frac{d\eta}{dx}\right)^2 = \square,$$

provided that

$$2\alpha = \frac{(a - a')(a - b')(a - c') \dots}{(a - b)(a - c) \dots}, \quad -2\alpha' = \frac{(a' - a)(a' - b)(a' - c) \dots}{(a' - b')(a' - c') \dots} *,$$

and similarly for $\beta, \gamma \dots; \beta', \gamma' \dots$.

From these equations it follows that $\Sigma \alpha = \Sigma \alpha'$ as before.

(25)　Conversely, provided

$$\alpha + \beta + \gamma \dots = \alpha' + \beta' + \gamma' \dots,$$

$\alpha, \beta, \gamma \dots, \alpha', \beta', \gamma' \dots$ may be treated as a diptych, and $a, b, c \dots, a', b', c' \dots$ be deduced by equations of the same form as before †: this explains a seeming paradox, that when $y = x^n + cx^{n-2} + \dots$, so that it contains only $(n - 1)$ disposable constants, we can satisfy the equation

$$y^2 + y'^2 = (x^n + hx^{n-1} + kx^{n-2} + \dots)^2,$$

which involves the satisfaction of $2n$ equations by $(2n - 1)$ unknowns.

(26)　It is well worthy of notice that if s is the length of the arc of the compound logarithmic wave

$$\eta = \Sigma \alpha \log (x + a) + \Sigma \alpha' \log (x + a'),$$

---

* Thus by substitution and multiplication we obtain the important equality

$$\frac{\zeta(abc\dots)}{\zeta(a'b'c'\dots)} = \pm \sqrt{\left(\frac{\alpha'\beta'\gamma' \dots}{\alpha\beta\gamma \dots}\right)},$$

from which it follows that, although the *maximum denumerant* of a diptych of the class μ is $\Pi(\mu - 1)\Pi(\mu)$ (see Art. 9), the degree of the resolving equation need never exceed $\dfrac{\Pi(\mu - 1)\Pi(\mu)}{2}$.

[ζ I use here, as of old elsewhere, to denote the product of the differences of the quantities which it precedes taken in some certain prescribed order of succession.]

† I mean that the equation-system above written may be *transformed* into the equation-system of article (8).

where the α and α' quantities form a diptych, if s be the length of the arc reckoned from a certain point in the curve

$$s - x = \Sigma\alpha \log(x + a) - \Sigma\alpha' \log(x + a').$$

Such a curve is called a rectifiable compound logarithmic wave; and if the a and a' series and the α, $-\alpha'$ series agree term for term, it is said to be symmetrical.

(27)   From what has been stated in article (18), it follows that the total number of reducible cyclodes of the order $2n$ is the same as the total number of rectifiable compound logarithmic waves that can be formed with $2n$ non-coincident wavelets of any two given species; and this remarkable equality also subsists when only the symmetrical cyclodes and the symmetrical waves form the subject of comparison. If the class $2m$ be given as well as the order $2n$ of the cyclodes, both the above equalities continue to apply when $2m$ of the wavelets are limited to be of one, and $2n - 2m$ of the other species.

(28)   The diptychical equations may be applied to the solution of the diophantine equation $y^2 + y'^2 = \square$ when y is any rational function of x of which the numerator and denominator are each of a given order.

The distinction of class, genus, and species will be still applicable as before, but there will be a fall in the number of solutions when the degree of y in x is zero, that is when the given orders of numerator and denominator are the same, so that the rational fraction of the degree is zero: apart from this cause of difference, the number of solutions corresponding to any given diptych will be governed in all cases solely by the principle of parallel equalities.

(29)   It is obvious that the theorems giving the value of the D and Δ* denumerants for any given class, combined with the theorems giving their sum for any given class and order, amount to independent theorems in the theory of partitions; and as regards the Δ denumerants these theorems belong to the subject of partitions of a comparatively simple character. For instance, the table of Δ for the fourth class enables us to affirm that understanding x, y, z, t to be all distinct quantities, and meaning by $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6$ the denumerants (that is the number of positive-integer solutions, *zeros* excluded) of the indeterminate equations or equation pairs below written

$$x + y + z + t = n; \quad x + y + 2z = n; \quad x + y = \frac{n}{2}; \quad x + 3y = n;$$

$$\begin{bmatrix} x + y + z + t = n \\ x + y = z + t \end{bmatrix}; \quad \begin{bmatrix} x + y + 2z = n \\ x + y = 2z \end{bmatrix};$$

---

* I shall henceforth call D the total and Δ the special denumerant, the *total* referring to solutions of either kind indifferently, the *special* to symmetrical solutions exclusively.

we have

$$9\,(\delta_1 - \delta_5) + 4\,(\delta_2 - \delta_6) + \delta_3 + \delta_4 + 7\delta_5 + 3\delta_6 = \frac{(k+1)^2 k}{2} \text{ when } n = 2k,$$

$$\text{or} = \frac{(k+1)\,k^2}{2} \text{ when } n = 2k - 1\,^*.$$

Moreover, throughout the whole theory, as, for example, in certain marvellous formulæ of reduction which I have obtained concerning the maximum Δ denumerants, say of the diptych $[a^r . b^s . c^t \dots]^2$, problems of partition of numbers occur and recur again and again, some of a known, others of a totally new kind. In this particular we see another point of resemblance between the theory of reducible cyclodes and that of geometric transformations.

(30) Annexed is a Table of the genera of diptychs† (the vacuous one not excepted) of the fourth class. I give it *subject to correction*. The calculations require great care to exclude repetitions or incompatible equalities and to guard against omissions.

It will be recollected that the sum of the terms is always the same for each *ala*; so that if the *alæ* agree in three places they must agree also in the fourth. When there exist binomial parallelisms, one only of the conjugates is exhibited in the Table, and in all cases only such (and in their simplest form) as are sufficient to define the genus. The mononomial parallel equalities are of course represented by the use of the same letter for the equal terms. The binomial parallelisms are indicated by affixing the same mark to all the terms which enter into each side of the equation constituting the expression of such parallelism.

The lines of division in the Table are introduced to give greater facility for exhaustive verification; the artificial groups of genera thus formed are

---

* The left-hand side of this equation, that is, $9\delta_1 + 4\delta_2 + \delta_3 + \delta_4 - 2\delta_5 - \delta_6$ it will easily be seen may be expressed in terms of $\dfrac{n-4}{1:1:1:1:}$, $\dfrac{n-4}{1:1:2:}$, $\dfrac{n-4}{2:2:}$, $\dfrac{n-4}{1:3:}$, $\dfrac{n-4}{4:}$, where $\dfrac{n}{p:q:r:\&c.}$ means the number of ways of making up the number n with p, q, r, &c.; but the expression will not be a linear function of these quantities, inasmuch as δ_5, δ_6 introduce combinations of the second order. The term (-4) in the symbolical numerators arises from the fact of zero values of x, y, z, t being inadmissible.

† If we choose to regard the union of two molecules (alæ), each of like weight (amount) and containing a like number of atoms (elements), as a chemical duad, we see that it is possible for such a duad to be resolved in one or more, and sometimes in a great variety of ways, into two or more subordinate duads, satisfying the same law of number and weight; and it is really quite within the bounds of possibility that in establishing and determining diptychical genera and species, we are at the same time solving a problem of the Chemistry of the Future. The combination of two similar ("conformable") molecules (alæ) into a duad (diptych) conveys a sort of image of Brodie's idea of the combination of a substance such as oxygen with itself; and in fact the whole theory of diptychical partition and combination forcibly reminds one of that writer's profound morphological views of the nature of chemical being, action, and passion.

those in which respectively four, two, one, or no element contained (singly or repeatedly) in one *ala* are or is contained also in the other *ala*.

DIPTYCHS.   DIPTYCHS.

a	b	c	d		a	b	c	d
---	---	---	---		---	---	---	---
a	b	c	c		a	b	c	c
a	a	c	c		a	a	c	c
a	a	a	d		a	a	a	d
a	a	a	a		a	a	a	a
a	b	c	d		a	b	c	d
a	b	c	c		a	b	c	c

a	b	c	d		a	β	c	d
---	---	---	---		---	---	---	---
a	b	c	c		a	β	c	c
a	b	c	d		a	β	c	d
a	b	c	d		a	β	c	d
a	b	c	d		a	c	c	d
a	b	c	d		c	c	c	d
a	b	c	d		c	d	c	d

a	b	c	c		a	β	a	β
---	---	---	---		---	---	---	---
a	b	c	c		a	a	a	δ
a	b	a	d		a	β	a	δ
a	b	a	d		a	β	γ	β
a	b	a	d		a	β	a	β
a	b	a	d		a	a	a	δ
a	b	a	b		a	β	a	β
a	b	a	b		a	a	a	a

Right column of DIPTYCHS:

a	b	c	d		a	β	γ	d
---	---	---	---		---	---	---	---
a	b	c	d		a	β	d	d
a	b	c	d		d	d	d	d
a	b	c	d		a	β	γ	d
a	a	c	d		a	β	γ	d
a	a	a	d		a	β	γ	d
a	a	c	d		a	a	γ	d

a	b	c	d		a	β	γ	δ
---	---	---	---		---	---	---	---
a	b	c	d		a	β	γ	δ
a	b	c	d		a	β	a	δ
a	b	c	d		a	β	a	β
a	b	c	d		a	a	a	δ
a	b	c	d		a	a	a	a
a	b	c	c		a	β	a	δ

a	a	a	d		a	δ	δ	δ
---	---	---	---		---	---	---	---
a	b	c	d		a	β	γ	δ
a	b	c	d		a	β	a	δ
a	b	c	d		a	β	a	δ
a	b	c	c		a	β	a	δ
a	b	c	c		a	β	a	β
a	b	a	d		a	β	a	δ
a	b	a	d		a	β	γ	β

In the last seven diptychs it will be observed double binomial parallelisms occur; thus, for example, in the last of all $a + b = \alpha + \beta$, $a + a = \alpha + \gamma$.

The total number above enumerated is 44, but not improbably one or more (most likely an odd number) have been overlooked. The first group

contains only one species to each genus; not so for the other groups. Thus, for example, $a\ b\ c\ d \mid \alpha\ \beta\ c\ d$ represents the two species,

$$
\begin{array}{cccc|cccc}
a & b & c & d & \alpha & \beta & c & d \\
a & b & c & d & a & a & c & d \\
\end{array}
$$

So, again, $a\ b\ c\ d \mid \alpha\ \beta\ \gamma\ d$ represents the several species,

$$
\begin{array}{cccc|cccc}
a & b & c & d & \alpha & \beta & \gamma & d \\
a & b & c & d & \alpha & \beta & \beta & d \\
a & b & c & d & a & a & a & d \\
a & a & c & d & \alpha & \beta & \beta & d \\
a & a & c & d & a & a & a & d \\
\end{array}
$$

and $a\ b\ c\ d \mid \alpha\ \beta\ \gamma\ \delta$ will represent, in like manner, the several species,

$$
\begin{array}{cccc|cccc}
a & b & c & d & \alpha & \beta & \gamma & \delta \\
a & b & c & d & \alpha & \beta & \gamma & \gamma \\
a & b & c & d & \alpha & \beta & \beta & \beta \\
a & b & c & d & a & a & a & a \\
a & b & c & c & \alpha & \beta & \gamma & \gamma \\
a & b & c & c & \alpha & \beta & \beta & \beta \\
a & b & c & c & a & a & a & a \\
a & b & b & b & \alpha & \beta & \beta & \beta \\
a & b & b & b & a & a & a & a \\
a & b & c & d & a & a & \gamma & \gamma \\
a & b & c & c & a & a & \gamma & \gamma \\
a & b & b & b & a & a & \gamma & \gamma \\
\end{array}
$$

But $a\ a\ c\ c \mid \alpha\ \alpha\ \gamma\ \gamma$ would belong to a different genus; for in such diptych we should have $2a + 2c = 2\alpha + 2\gamma$, and consequently $a + c = \alpha + \gamma$ a new parallel; and *à fortiori* $a\ a\ a\ a \mid \alpha\ \alpha\ \gamma\ \gamma$ belongs to a different genus. The algebraical denumerants will be the same for all the last-written twelve species, namely, $\Pi 3 . \Pi 4$, that is 144, the maximum value for the fourth class,—the arithmetical denumerants, on the other hand, being respectively 144, 72, 24, 6, 36, 12, 3, 4, 1, 36, 18, 6. Possibly it may turn out that the number of species may be given by as simple or a simpler function of the index of the class as or than that which expresses the number of genera. It seems very desirable to ascertain if either or both is or are amenable to algebraical or analytical quantification.

As Crystallography was born of a chance observation by Haüy of the cleavage-planes of a single fortunately fragile specimen, and the theory of Invariants owes its existence to a solitary individual accidentally encountered and put on record by Eisenstein, so out of the slender study of the Norwich Spiral has sprung the vast and interminable Calculus of Cyclodes, which strikes such far-spreading and tenacious roots into the profoundest strata of

denumeration, and, by this and the multitudinous and multifarious dependent theories which cluster around it, reminds one of the Scriptural comparison of the Kingdom of Heaven "to a grain of mustard-seed which a man took and cast into his garden, and it grew and waxed a great tree, and the fowls of the air lodged in the branches of it "*.

(31) The annexed computation of the denumerants of *diptychs* of the 3rd class and 8th and 9th orders respectively is given in illustration of articles (20) and (13).

The triadic partitionments of 8, it will be observed, are 1 1 6 : 1 2 5 : 1 3 4 : 2 2 4 : 2 3 3. Those of 9, 1 1 7 : 1 2 6 : 1 3 5 : 1 4 4 : 2 2 5 : 2 3 4 : 3 3 3. And of 7, 1 1 5 : 1 2 4 : 1 3 3 : 2 2 3.

In the D column the denumerants of the unconformable diptychs are *doubled,* in order to save the necessity of transposing the *alæ* in summing for the total denumerant [the $Q_{r,\,s}$ of Art. 13].

Diptychs		D
1 1 6	1 1 6	1
1 1 6	1 2 5	8
1 1 6	1 3 4	8
1 1 6	2 2 4	6
1 1 6	2 3 3	6
1 2 5	1 2 5	7
1 2 5	1 3 4	20
1 2 5	2 2 4	8
1 2 5	2 3 3	10
1 3 4	1 3 4	7
1 3 4	2 2 4	10
1 3 4	2 3 3	8
2 2 4	2 2 4	1
2 2 4	2 3 3	4
2 3 3	2 3 3	1
$\dfrac{\Pi 7\ \Pi 6}{\Pi 3\ \Pi 2\ \Pi 5\ \Pi 4} =$		105

* The three most remarkable and almost simultaneous births of time for the current year are, I think, Janssen's and Lockyer's hydrogenous solar chromosphere, Tyndall's indefinitely attenuated cometary matter, and the still more impalpable and shadowy product of cerebration embodied in diptychs with their *quasi*-chemical composition and parallels stretching between and connecting, as it were, with forces of affinity the atomic elements of the associated geminate molecules, in this sketch set forth. All three theories have originated alike in observation, followed up by processes of experiment and verification working with very different instruments, but having the same *primum mobile* in the human intelligence. On second thoughts I ought to tack on to this list of memorabilia, which must for ever make 1869 stand out in the Fasti of

Diptychs		D
1 1 5	1 1 5	1
1 1 5	1 2 4	8
1 1 5	1 3 3	4
1 1 5	2 2 3	6
1 2 4	1 2 4	7
1 2 4	1 3 3	10
1 2 4	2 2 3	8
1 3 3	1 3 3	1
1 3 3	2 2 3	4
2 2 3	2 2 3	1
$\dfrac{\Pi 6 \ \Pi 5}{\Pi 3 \ \Pi 2 \ \Pi 4 \ \Pi 3} =$		50*

Diptychs		D
1 1 7	1 1 7	1
1 1 7	1 2 6	8
1 1 7	1 3 5	8
1 1 7	1 4 4	4
1 1 7	2 2 5	6
1 1 7	2 3 4	12
1 1 7	3 3 3	2
1 2 6	1 2 6	7
1 2 6	1 3 5	20
1 2 6	1 4 4	10
1 2 6	2 2 5	8
1 2 6	2 3 4	20
1 2 6	3 3 3	4
1 3 5	1 3 5	7
1 3 5	1 4 4	10
1 3 5	2 2 5	10
1 3 5	2 3 4	20
1 3 5	3 3 3	2
1 4 4	1 4 4	1
1 4 4	2 2 5	6
1 4 4	2 3 4	8
1 4 4	3 3 3	2
2 2 5	2 2 5	1
2 2 5	2 3 4	8
2 2 5	3 3 3	2
2 3 4	2 3 4	7
2 3 4	3 3 3	2
3 3 3	3 3 3	0
$\dfrac{\Pi 8 \ \Pi 7}{\Pi 3 \ \Pi 2 \ \Pi 6 \ \Pi 5} =$		196

science, Capt. Andrew Noble's mechanical invention for measuring up to the millionth part of a second the rate of motion of a shot inside a cannon, and Dr Christian Wiener's wonderful realization in stereoscopic drawings of the Salmon-Cayley 27 lines on a cubic surface on the one hand, and on the other (Hermite's pupil, pupil worthy of his master) M. Camille Jordan's surprising discovery of their application to the trisection of Abelian functions. Surely with as good reason as had Archimedes to have the cylinder, cone, and sphere engraved on his tombstone might our distinguished countrymen leave testamentary directions for the cubic eikosi-heptagram to be engraved on theirs. Spirit of the Universe! whither are we drifting, and when, where, and how is all this to end?

* This Table has been already given, differently arranged, at page [677], where, however, the numbers corresponding to pairs in the *D* column are not, as here, doubled.

(32) The Tables following illustrate the values assigned to the special denumerants (Δ column) in the Tables of Arts. 21, 22. The diptychs being of the conformable kind, are, for greater simplicity, represented each by a single *ala*. The asterisk at the side distinguishes those diptychs in which one or more binomial parallel equalities exist.

Class = 8 Order = 28		Class = 10 Order = 30		†Class = 10 Order = 32	
Ala	Δ	Ala	Δ	Ala	Δ
1.1.1.11	1	1.1.1.1.11	1	1.1.1.1.12	1
1.1.2.10	4	1.1.1.2.10	6	1.1.1.2.11	6
1.1.3.9	4	1.1.1.3.9	6	1.1.1.3.10	6
1.1.4.8	4	1.1.1.4.8	6	1.1.1.4.9	6
1.1.5.7	4	1.1.1.5.7	6	1.1.1.5.8	6
1.1.6.6	1	1.1.1.6.6	2	1.1.1.6.7	6
1.2.2.9	4	1.1.2.2.9	9	1.1.2.2.10	9
1.2.3.8	9	1.1.2.3.8	21	1.1.2.3.9	21
1.2.4.7	9	1.1.2.4.7	21	1.1.2.4.8	21
*1.2.5.6	7	1.1.2.5.6	19	1.1.2.5.7	21
1.3.3.7	4	1.1.3.3.7	9	1.1.2.6.6	9
*1.3.4.6	7	*1.1.3.4.6	19	1.1.3.3.8	9
1.3.5.5	4	1.1.3.5.5	9	1.1.3.4.7	21
1.4.4.5	4	1.1.4.4.5	9	1.1.3.5.6	21
2.2.2.8	1	1.2.2.2.8	6	1.1.4.4.6	9
2.2.3.7	4	*1.2.2.3.7	20	1.1.4.5.5	9
2.2.4.6	4	1.2.2.4.6	21	1.2.2.2.9	6
2.2.5.5	1	1.2.2.5.5	9	*1.2.2.3.8	20
2.3.3.6	4	1.2.3.3.6	21	1.2.2.4.7	21
*2.3.4.5	7	*1.2.3.4.5	39	*1.2.2.5.6	19
2.4.4.4	1	1.2.4.4.4	6	1.2.3.3.7	21
3.3.3.5	1	*1.3.3.3.5	5	*1.2.3.4.6	41
3.3.4.4	1	1.3.3.4.4	9	1.2.3.5.5	21
		2.2.2.2.7	1	*1.2.4.4.5	19
		2.2.2.3.6	6	1.3.3.3.6	6
		2.2.2.4.5	6	*1.3.3.4.5	20
		2.2.3.3.5	9	1.3.4.4.4	6
		2.2.3.4.4	9	2.2.2.2.8	1
		*2.3.3.3.4	5	2.2.2.3.7	6
		3.3.3.3.3	0	2.2.2.4.6	6
				2.2.2.5.5	2
				2.2.3.3.6	9
				*2.2.3.4.5	19
				2.2.4.4.4	2
				2.3.3.3.5	6
				*2.3.3.4.4	8
				3.3.3.3.4	1
$\dfrac{\{\Pi(5+2-1)\}^2}{\Pi1\ \Pi2\ \Pi4\ \Pi5} =$	90	$\dfrac{\Pi6.\Pi7}{\Pi4.\Pi5\,(\Pi2)^2} =$	315	$\dfrac{(\Pi7)^2}{(\Pi5\ \Pi2)^2} =$	441

† That is, the cyclode is of such class and order, but the diptych is of class 5 and order 16, and so for the rest. I repeat that the diptychical class and order is the number of elements and their sum in *each* ala, and must be doubled to give the class and order of the corresponding cyclode.

Ala	Δ		Ala	Δ
Class =10 Order=34			Class =10 Order=34	
1.1.1.1.13	1		i.2.3.4.7	43
1.1.1.2.12	6		i.2.3.5.6	41
1.1.1.3.11	6		i.2.4.4.6	20
1.1.1.4.10	6		i.2.4.5.5	19
1.1.1.5.9	6		1.3.3.3.7	6
1.1.1.6.8	6		i.3.3.4.6	19
1.1.1.7.7	2		i.3.3.5.5	8
1.1.2.2.11	9		1.3.4.4.5	20
1.1.2.3.10	21		1.4.4.4.4	1
1.1.2.4.9	21		2.2.2.2.9	1
1.1.2.5.8	21		2.2.2.3.8	6
i.1.2.6.7	19		2.2.2.4.7	6
1.1.3.3.9	9		2.2.2.5.6	6
1.1.3.4.8	21		2.2.3.3.7	9
i.1.3.5.7	19		2.2.3.4.6	21
1.1.3.6.6	9		2.2.3.5.5	9
1.i.4.4.7	8		2.2.4.4.5	9
1.1.4.5.6	21		2.3.3.3.6	6
1.1.5.5.5	2		2.3.3.4.5	18
1.2.2.2.10	6		2.3.4.4.4	6
i.2.2.3.9	20		3.3.3.3.5	1
1.2.2.4.8	21		3.3.3.4.4	2
1.2.2.5.7	21			
1.2.2.6.6	9		$\dfrac{\Pi 8\ \Pi 7}{\Pi 6\ \Pi 5\ (\Pi 2)^2} =$	588
1.2.3.3.8	21			

Here $s=5$, $r=12$,
$k=2$, $h=6$†

In the above Table, and those which follow, instead of using asterisks (danger-signals, so to say) as in the three Tables immediately preceding to indicate the existence of parallel equalities, I have employed the much more perfect method of connotation which actually exhibits such equalities to the eye.

† See second footnote, p. [669].

Ala	Δ
1 1 1 1 1 6	1
1 1 1 2 1 5	6
1 1 1 3 1 4	6
1 1 1 4 1 3	6
1 1 1 5 1 2	6
1 1 1 6 1 1	6
1 1 1 7 1 0	6
1 1 1 8 9	6
1 1 2 2 1 4	9
1 1 2 3 1 3	21
1 1 2 4 1 2	21
1 1 2 5 1 1	21
1 1 2 6 1 0	21
1 1 2 7 9	21
1 1 2 8 8	9
1 1 3 3 1 2	9
1 1 3 4 1 1	21
1 1 3 5 1 0	21
1 1 3 6 9	21
1 1 3 7 8	21
1 1 4 4 1 0	9
1 1 4 5 9	21
1 1 4 6 8	21
1 1 4 7 7	9
1 1 5 5 8	9
1 1 5 6 7	21
1 1 6 6 6	2
1 2 2 2 1 3	6
1 2 2 3 1 2	20

Ala	Δ
1 2 2 4 1 1	21
1 2 2 5 1 0	21
1 2 2 6 9	21
1 2 2 7 8	19
1 2 3 3 1 1	21
1 2 3 4 1 0	43
1 2 3 5 9	45
1 2 3 6 8	43
1 2 3 7 7	21
1 2 4 4 9	21
1 2 4 5 8	41
1 2 4 6 7	43
1 2 5 5 7	21
1 2 5 6 6	19
1 3 3 3 1 0	6
1 3 3 4 9	21
1 3 3 5 8	20
1 3 3 6 7	21
1 3 4 4 8	21
1 3 4 5 7	43
1 3 4 6 6	19
1 3 5 5 6	21
1 4 4 4 7	5
1 4 4 5 6	21
1 4 5 5 5	6
2 2 2 2 1 2	1
2 2 2 3 1 1	6
2 2 2 4 1 0	6
2 2 2 5 9	6

Ala	Δ
2 2 2 6 8	6
2 2 2 7 7	2
2 2 3 3 1 0	9
2 2 3 4 9	21
2 2 3 5 8	21
2 2 3 6 7	19
2 2 4 4 8	9
2 2 4 5 7	19
2 2 4 6 6	9
2 2 5 5 6	9
2 3 3 3 9	6
2 3 3 4 8	20
2 3 3 5 7	21
2 3 3 6 6	9
2 3 4 4 7	21
2 3 4 5 6	39
2 3 5 5 5	6
2 4 4 4 6	5
2 4 4 5 5	9
3 3 3 3 8	1
3 3 3 4 7	6
3 3 3 5 6	6
3 3 4 4 6	9
3 3 4 5 5	9
3 4 4 4 5	5
4 4 4 4 4	0

$$\left(\frac{\Pi 9}{\Pi 2\ \Pi 7}\right)^2 = 1296\dagger$$

Here $s=5$, $r=15$, $k=2$, $h=7$.

All three tables carry the heading: Class = 10, Order = 40.

† The *fall* in the arithmetical denumerant due to the existence of binomial parallelisms in the 4th or 5th diptychical class is easily recollected by aid of the following rule. Every such parallelism causes a fall of 2 units unless two elements in the parallel are identical, in which case the fall is of a single unit; but in applying this rule elements of the same name are not to be counted more than once over in the same parallel. Calling the two varieties of binomial

Anyone desirous of mastering the foregoing exposition (the cradle of a theory complementary to the existing theory of algebraical curves) will do well to possess himself of the ideas conveyed by the new or newly applied terms of art scattered up and down through the outline and resumed in the annexed stenograph : Continued Involutes ; Cyclodes, their Pole and Parabolic function ; Unicursal Relation ; Norwich Spiral ; Reducible Cyclodes and their Bifid functions ; Algebraico-Diophantine condition ; Symmetrical, Congeminate and Contrageminate forms ; Compound Logarithmic Waves and Wavelets ; Partitionments ; Representative Diptychs, their Alæ conformable and unconformable ; Monistic and Dualistic equation-systems ; Algebraical and Arithmetical Denumeration ; Zigzag Multiplication of continuous lines or columns ; Pneumatic Analysis ; Evaporation (partial and total) ; Consolidation ; Reduction ; Parallel Equalities ; Mononomial, Binomial, and Polynomial Parallels (Reducible and Irreducible) ; Connotation ; Order, Class, Genus, and Species of Cyclodes and Diptychs ; Special and Total Denumerants. The quotation at the head of the paper is from an article by M. Réville in the *Revue des deux Mondes* for the present July, " sur la Science des Religions." I have since found that a strikingly similar passage occurs in Dr Mansel's Bampton Lectures :—" The only test *of truth* is *the harmonious consent of all human faculties.*"

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conjugation A and B respectively, it will be found that we may have B or A, or A and B, or A and A, or A and A and A coexisting in the same diptych of the 5th class, causing respectively falls of 1, 2, 3, 4, 6 units in the corresponding values of Δ. It may not unlikely be possible to give rules for assigning the effect of any polynomial parallel on the denumerant in all cases; and then the theory will be greatly simplified, as we shall have only to deal with diptychs of the form $a^r b^s c^t \ldots | a^\rho \beta^\sigma \ldots$ freed from all latent parallelisms. I am virtually in possession of a complete theory of the special denumerants for diptychs of the form $[a^r . b^s . c^t \ldots]^2$ on the supposition of the absence of binomial or other polynomial parallel equalities. As regards the method of connotation for singling out the terms which enter into any parallel, a remark may be made which is of some importance, namely, that given such terms, and provided that the parallelism is *irreducible*, that is, not implied in parallelisms of a lower order, the resolution of the parallel into an equation can be effected in only one way. Hence, as we may always reject *reducible* parallels, the method of connotation is exempt from all ambiguity in its application to the single *alæ* of conformable diptychs. As to diptychs with the alæ both expressed, the method of connotation actually exhibits not only the parallels, but the equations which they respectively contain.

It is worth while to notice how gradually and slowly the final form of the theory is evolved. The first reducible cyclode is of the 2nd order. The plurality of such cyclodes only begins to show itself in the 6th order ; the existence of non-symmetrical ones, in the 8th ; congeminate pairs, in the 12th ; reduction in the value of Δ (the special denumerant) owing to the existence of parallel equalities (other than identities), in the 16th. In like manner, I discovered in the theory of partitions that it is not until we have entered upon the discussion of ternary equation-systems that we can be said to have cleared out of the narrows and to be sailing on the open sea; for only then do the most essential features of the theory and its geometrical basis begin to disclose themselves. A very similar remark applies to my unpublished generalization of Gauss's method of interpolation, which I have extended to multiple integrals of all degrees of multiplicity. So, too, it is well known that the algebraical or geometrical theory of forms does not exhibit itself in its true colours until we have passed beyond the case of the second degree, which is quite a world within itself *conditionally unlimited*, incongruent as this association of epithets may sound to the unpurged ears of the Hamiltonian school of metaphysicians.