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NOTE ON SUCCESSIVE INVOLUTES TO A CIRCLE*.

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It is surprising that the families and groups of families of forms capable of being educed by successive involution from a circle should not have attracted the attention of geometers. I find, if any, not more than a passing allusion to their existence in Dr Salmon's *Higher Plane Curves*, in the *Integral Calculus* of Mr Todhunter, or in the memoirs of the late Dr Whewell in the *Cambridge Philosophical Transactions* (vols. VIII. and IX.), although these latter are exclusively devoted to the study of various curves of mechanical and kinematical origin by aid of the so-called intrinsic equation, which is in fact the natural expression of, and key to, the properties of such like curves. And yet this form of equation almost instantaneously furnishes the general polar equation to the entire system of circular involutes, and exhibits at once their leading properties†.

Foremost amongst these stands the algebraical form of equation (and that a quantic), which connects not only the arc, but also the squared radius vector with the angle of contingence, and consequently the two former with one another. In a marvellous and, so to say, transcendental fashion, these curves participate in the nature of algebraical curves—their apses, cusps, and points of retrocession being counted by the order of the involution, and becoming imaginary in pairs.

I need hardly say that by a second involute I mean an involute of the first, by a third an involute of a second, and so on in continual progression. To any given curve all its first involutes form a system of parallel curves, so

* The germ of this Note was communicated to the Mathematical Section of the British Association at the Norwich Meeting.

† Professor Rankine and Mr Merrifield have made a useful application of the *second* involute of the circle to the calculation of the stability of the finite rotation of vessels. In the Turkey carpet under my eyes whilst this is being written, I perceive graceful and complicated figures of winding and intersecting scrolls and convolutions, which render it, I think, not at all improbable that the successive involutes of the circle would furnish or suggest many patterns available for decorative purposes: the enormous variety of each kind of involute, which of course increases with the order of derivation, adds to the probability of this conjecture.

that in general the number of *form*-parameters to a curve will be augmented by i when we pass to its general involute of the ith order. In the case of the circle, however, owing to its homogeneity, the first involute, like the curve itself, contains only one *form*-parameter (it being, in other words, a property of the first involute, that when rotated round a certain point, the curves so generated continue always parallel to each other); and so the number of *form*-parameters in the general ith involute will contain i, and not $i+1$ parameters, as the general formula would require.

I shall use ϕ, s, r, θ to denote the angle of contingence, arc, radius vector, and vectorial angle of the curves under consideration.

Starting from the circle $s = a\phi$, a set of corresponding successive involutes will, as is well known, be represented by

$$s_1 = \frac{a\phi^2}{2} + b\phi\,; \qquad s_2 = \frac{a\phi^3}{6} + \frac{b\phi^2}{2} + c\phi,$$

and so on, according to the obvious law

$$s_i = \int d\phi \,.\, s_{i-1}.$$

Now in general for any *curve* whatever, if we call p the perpendicular on the tangent from an arbitrary pole, q the projection of the radius on the tangent, we have

$$q = -\frac{dp}{d\phi}\,; \tag{1}$$

also

$$p + \frac{d^2p}{d\phi^2} = \frac{ds}{d\phi}, \tag{2}$$

$$p^2 + \left(\frac{dp}{d\phi}\right)^2 = r^2. \tag{3}$$

These equations are of course not new*; they are given by Mr Todhunter

* We have only to take P, P', two consecutive points, and on PP', $P'T$, the tangents at P, P', draw perpendiculars from an arbitrary point O, and we obtain at once, by inspection,

$$\delta p = -q\delta\phi, \quad \delta q + \delta s = p\delta\phi,$$

whence

$$q = -\frac{dp}{d\phi}, \quad p + \frac{d^2p}{d\phi^2} = \frac{ds}{d\phi}.$$

Or, again, proceeding analytically, we have

$$x - A = \int ds \cos\phi, \quad y - B = \int ds \sin\phi\,;$$

whence, integrating by parts,

$$x - A = G\cos\phi - G'\sin\phi,$$

$$y - B = G\sin\phi + G\ \cos\phi.$$

where

$$G = s - s'' + \ldots; \quad G' = s' - s''' + \ldots;$$

whence

$$r = G^2 + G'^2 \text{ and } G + G'' = \frac{ds}{d\phi}.$$

in the later editions of his *Integral Calculus*, accompanied with a reference to another English treatise, from which he has taken them; but in themselves easily as they can be obtained, they contain the whole theory of the remarkable curves to which this note refers. In the case before, $\frac{ds}{d\phi} = F\phi$, where F is (a quantic in, that is) a rational integral function of ϕ. Hence we have for the solution of (1), (2),

$$p = F - F'' + F'''' + \ldots + A \cos\phi + B \sin\phi,$$
$$-q = F' - F''' + \ldots - A \sin\phi + B \cos\phi;$$

wherefore r^2 is known in terms of F and the arbitrary constants A and B, whose values depend on the position of the origin from which r is reckoned, by a due choice of which they may be made to vanish. One will readily suppose that this eligible position of the pole must be the centre of the generating circle; and the proof is as follows:

If $r_1, r_2, \ldots r_i$ be any radii vectores corresponding to $s_1, s_2, \ldots s_i$, it is well known, and follows from the definition of the involute, that

$$r^2_{i+1} = r_i^2 - 2r_i s_i \frac{dr_i}{ds_i} + s_i^2. \tag{4}$$

From which also we may deduce

$$q = r\frac{dr}{ds} = G', \quad p^2 = r^2 - q^2 = G^2.$$

This last demonstration would at first sight seem to be only valid for the case of the G series coming to an end, that is, of $\frac{ds}{d\phi}$ being a rational integral function in ϕ; but it would be quite legitimate to infer at once from it the *universality* of the equations above written connecting r^2 with s and ϕ; for we may write down the general differential equation of the second order

$$d\cos^{-1}\frac{dr}{ds} - d\theta = d\phi,$$

that is,
$$\frac{d \cdot \frac{dr}{ds}}{\sqrt{\{(ds)^2 - (dr)^2\}}} - \frac{\sqrt{(ds^2 - dr^2)}}{r} = d\phi,$$

in which $\frac{ds}{d\phi}$ may be considered as given, and r or r^2 to be determined. The equations in question, having been shown to be true for a form $\frac{ds}{d\phi}$ containing an indefinite number of arbitrary constants, evidently can only amount to a transformation of, and may be used in supersession of the equation last written. It may be worth while to set out this latter under a more familiar form of notation. If, then, we use y for r and x for ϕ, and call $\frac{ds}{d\phi} = X$ (any function of x), it becomes

$$\frac{-Xy'' + X'y'}{X\sqrt{(X^2 - y'^2)}} + \frac{\sqrt{(X^2 - y'^2)}}{y} = 1,$$

an apparently very complicated form of equation, but admitting of the simple solution $y^2 = u^2 + u'^2$, where u satisfies the linear differential equation $u + u'' = X$.

Now, suppose that for any number i the origin has been so chosen that

$$r_i^2 = (s_{i-1} - s_{i-3} + \dots)^2 + (s_{i-2} - s_{i-4} + \dots)^2,$$

then $$r_i \frac{dr_i}{ds_i} = \frac{1}{s_{i-1}} \left\{ \begin{array}{l} (s_{i-1} - s_{i-3} \dots)(s_{i-2} - s_{i-4} \dots) \\ + (s_{i-2} - s_{i-4} \dots)(s_{i-3} - s_{i-5} \dots) \end{array} \right\}$$

$$= s_{i-2} - s_{i-4} + \dots .$$

Hence $$r^2_{i+1} = (s_i - s_{i-2} + s_{i-4} \dots)^2 + (s_{i-1} - s_{i-3} + \dots)^2 ;$$

and the supposed relation, if true for any value i, is true for all superior values; but when the origin is at the centre, $s_1 = a\phi$, $r_1^2 = a^2$; and consequently the equation (4) is true universally*.

Thus, then, when the origin is at the centre, whilst for the ith involute to a circle the arc is any quantic $\int d\phi F$ of degree $(i+1)$ in ϕ, r^2 is a quantic in ϕ of degree $2i$ of the particular form $(G\phi)^2 + (G'\phi)^2$, where $G\phi$ may be supposed, if we please, to be any quantic of the order i in ϕ†; and then $\frac{ds}{d\phi}$ the radius of curvature at s, ϕ, is expressed by $G\phi + G'\phi$‡.

* The above result might have been deduced more directly from the equation $\frac{dp_i}{d\phi} = p_{i-1}$, which is true for any curve and its evolute. In fact this paper need never have been written (for all that it contains is a straightforward inference from four equations which may be found scattered up and down in elementary treatises), had it been the custom to regard those equations as forming collectively a connected apparatus. I mean the four following, where the unaccented and accented s and p refer to any curve and its evolute respectively, ϕ being the angle of contingence common in magnitude to the two :

(1) $\frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2}$, (2) $r^2 = p^2 + \left(\frac{dp}{d\phi}\right)^2$,

(3) $s' = \frac{ds}{d\phi}$, (4) $p' = \frac{dp}{d\phi}$,

the last of them more familiarly known under the form

$$p'^2 = r^2 - p^2.$$

The third and fourth equations show respectively that s and p are each quantics in ϕ ; the first gives the connexion between the constants which enter into these quantics, and the second, combined with the first, the relation between s and r (in other words, the rectification of the curve), that between r and θ (where θ is the vectorial angle) being contained in a fifth equation,

$$\theta = \phi + \sin^{-1}\frac{p}{r}.$$

† Accordingly we see that the spiral of Archimedes, as is well known, is the locus of the feet of the perpendiculars upon the tangents to the first involute from the centre of the circle; and, much more generally, if we substitute for each radius vector in this spiral any given quantic thereof, we obtain the corresponding first pedal to an involute whose order of derivation is the degree of the quantic. For example, by squaring or cubing the radius vector of the spiral of Archimedes (of course leaving the vectorial angle unchanged), we may form the pedal to particular species of the second and third involutes respectively.

‡ Since the radius of curvature, radius vector, and perpendicular on the tangent arc are all known rational integral functions of the same quantity, it becomes a simple problem of elimination to determine the central force competent to make a body describe an involute of any order

It may be here noticed that substituting for ϕ, $\phi+\lambda$, where λ is arbitrary, amounts only to a rotation of the curve through the angle λ, so that, as regards the intrinsic form of the curve, no generality is sacrificed by imposing one condition upon the coefficients in $G\phi$, or, if we please, in making any of the coefficients in it except the first to vanish.

From what precedes, and from the general theory of elimination, it follows that in general the relation between i^2 and s is expressed by a rational integral equation of the degree $(i+1)$ in the former and $2i$ in the latter. But this is subject to an obvious exception in the case of $i=1$; for then, calling $G\phi = a\phi$, we have

$$\frac{ds}{d\phi} = a\phi, \quad s = \frac{a\phi^2}{2} + b, \text{ and } r^2 = a^2\phi^2 + a^2 = 2as + (a^2 - 2ab);$$

so that the degrees in r^2 and s are here 1 and 1 in lieu of 2 and 2, as given by the general rule.

As regards the polar equation to the general involute,

$$r^2 = (G\phi)^2 + (G'\phi)^2,$$

it is obvious that, agreeably to the well-known case for the first involute,

$$\theta = \sin^{-1}\frac{dr}{ds} + \phi = \sin^{-1}\frac{G'}{r} + \phi^*,$$

where G' and ϕ are given by the solution of an algebraical equation of the $2i$th degree, and which will therefore *usually* be incapable of expression in

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to a circle. Thus it will be found that the half-pitch second involute may be described under the action of a central force varying as the inverse cube of the shortest distance from the generating circle. So, again, the first involute may be described under the action of a central force, the component of which in the direction of the tangent to the generating circle (or say the centrifugal force) varies as the inverse cube of this tangent, the centre of force in each case being of course situated at the centre of the circle.

* ϕ and G' will form $2i$ systems of values. Will they be all applicable to the true involute, and how about the sign to be given to r? It must, I think, be a matter of some delicacy and difficulty to answer these questions. For take even the first involute, where

$$\theta = \sin^{-1}\frac{a}{r} + \sqrt{\left(\frac{r^2-a^2}{a^2}\right)},$$

we know, as a matter of fact, that if the first term is made to decrease as r increases, the positive sign of the square roots only must be employed, and of course the negative sign if the first term increases with r. Were we to reverse this rule, instead of the involute we should obtain what may be termed the counter-involute; that is, a figure formed by points, each the *opposite* of every point in the involute in respect to its centre of curvature. Or, again, if a pair of parallel rulers were made always to touch a circle at opposite points, and the under parallel to *roll* round the circle, whilst each point in this line describes an involute, each point in the one above would describe a counter-involute. Or, again, if a string, by aid of a pin, were unwrapped *back* upon itself from a circle, the extremity would describe the extraneous curve. From this last observation it would seem as if the forced intrusion of a foreign curve into the polar equation of the involute resulted from the impossibility of affixing an absolute sign to the length of an arc—the condition of drawing a tangent always equal in length to the varying arc of a curve admitting of satisfaction without breach of continuity in two distinct modes.

finite terms beyond the second involute. In the case of this involute the reducing equation is not a general biquadratic, but a form involving only square and no cube roots—being in fact reducible to a quadratic in ϕ^2, as will at once be seen from the fact that we may write

$$(\alpha\phi^2 + \gamma)^2 + 4\alpha^2\phi^2 = r^2.$$

Since $\frac{ds}{d\phi}$, that is $G + G''$, is a quantic of the degree i or ϕ, we learn that there may be i cusps to the ith involute, or any less number differing from i by an even integer. Also, since $\frac{dr^2}{ds} = G'$, the number of apses (in regard to the centre) may be any number inferior to and differing from i by an odd integer*. Also, since G represents the perpendicular on the tangent, the number of points where the tangent passes through the centre will follow the same law (although, of course, the two numbers need not be equal) as the number of the cusps. The cusps, of course, can only exist at points where the involute meets the parent curve.

Between any two cusps of an involute evidently must be comprised an odd number of the cusps of its parent curve; but, of course, not *vice versâ*; thus, for example, in the second involute, if there are no cusps, it will easily be seen that the curve possesses a simple loop enclosing the cusp of the first involute (its evolute), and consequently cutting the two branches of the latter, and so in general the disappearance of consecutive cusps in any involute will give rise to loops enclosing those cusps of the parent curve on the branches adjoining to which (on each side), cusps of the derived curve are wanting; (by a branch, I mean, of course, the portion of curve included between any two cusps, or between either of the two terminal cusps and infinity;) whether the absence of cusps of the involute on $2i$ consecutive branches of the parent curve implies the necessary existence of i distinct loops, one round every alternate one of the $2i-1$ cusps in which those branches meet, requires further consideration. It is clear that in an analytical sense the length of the arc of the parent curve included between any two cusps of the second curve must be taken as zero; the correct view (at least, for the purposes of this theory) being that the angle of convergence *continually* increases or decreases up to positive or negative infinity as we

* Thus we see that the apsidal distances from the centre are the arithmetical magnitudes of the roots of the equation formed by equating to zero the *discriminant* of $G+r=0$, which is of course of the degree $(i-1)$ in r.

If we consider the apses and cusps of any involute to form a combined group, an odd number of the points of this group will always be included between any two intersections of the curve with a circle concentric with the parent circle; for the limiting equation to $G^2+G'^2-r^2=0$ is $G'(G+G'')=0$. Every point in this combined group is a point of maximum or minimum elongation from the centre.

pass in one direction from point to point in a curve. Accordingly we ought not to say, as is usually done, that at a cusp the tangent is suddenly reversed in direction, but, rather, that the increment of the arc on passing through a cusp changes sign, as it ought to do according to first principles; for the *flow* of the incremental arc, from being concurrent with, becomes opposite to that of the rotating tangent line which carries it, or *vice versâ*. Thus in the common cycloid (a curve of infinite length to the eye and with an infinite number of cusps) we have $s = c \cos \phi$, which, subject to this interpretation, is perfectly true and self-consistent for the whole extent of the curve from infinity to infinity. In that case we have a visible representation of *quantity* undergoing an infinite number of periodic changes, although the *subject matter* of the quantity is continually changing and never recurs. In the case of the involutes of the circle, the number of those periodic changes is of course finite and equal to the number of the cusps. If $A, B, C, D, \ldots L$ be the cusps in natural order on the curve whose involute is to be found, and if we call x the radius of curvature of the point of the involute corresponding to A (x being taken positive when this radius is in the position into which it would be brought by unwinding a string from the infinite branch adjacent to A), and if we form the series $x - AB + BC - CD \ldots \pm KL$, where AB, $BC, \ldots$ are the arithmetical lengths of the branches, it is clear that at each term of this series in which a change of sign in the sum takes place the involute will have a cusp; if the number of branches is odd and x is negative, the sum may remain negative at whatever term we stop, and then there will be no cusp in the involute so engendered; but when the number of points $A, B, C, \ldots L$ is even, then it is easy to see that one of the infinite branches must contain a cusp of the involute and the other be vacant. The second involute, whether cusped or not, manifestly consists of two parts symmetrically arranged about its apse. If we form a third involute by unwinding from this apse as origin, the figures so formed will again be symmetrical, and the cusps will lie at the vertices of an isosceles triangle; and now *every* involute of this symmetrical third involute will again be symmetrical, and so on continually, the number of conditions imposed on the parameters in order to ensure symmetry in the involute of the ith order being thus the integer part of $\frac{i-1}{2}$. When this symmetry obtains, the algebraical equation requisite for determining the polar equation depends on the solution of an equation of only the ith instead of the $2i$th degree; for it is clear that in this case the functions G^2 and G'^2 may be made to contain only powers of ϕ^2. Thus we may very easily write down the general polar equation to the absolutely general second involute, and might, if it were worth while, do as much for the symmetrical class of third and fourth involutes, of which the former will contain two and the latter three arbitrary parameters, by solving a cubic and biquadratic equation in these two cases respectively.

It is easy to see also that the *arco-radial* equation to the symmetrical involute of an odd order is of only half the degrees in s and r^2 that it is of in the general case, and for the symmetrical involute of an even order, although of the same degrees in r^2 and s as in the general case, involves only the even powers of s.

A few words upon the second involute, and I have done; for it is difficult to deal with theory in any detail so as to be intelligible, or even safe, without the suggestive and regulative aid of drawn figures, which I have not yet been able to obtain in a form fit for use.

The two principal classes to distinguish in the second involute are the cusped and uncusped species. The cusped second involute winds round the parent curve upon which the extremities of its finite branch rest. The uncusped species crosses itself, and intersects each branch of the first involute of which it encloses the cusp, its node being on one side of it and its apse on the other. The transition case is when the unwinding begins from the cusp of the first involute; the second involute so obtained has a very singular point at that cusp, which may be regarded as a coincident pair of cusps*.

The general connecting equations for this involute may be put under the form

$$r^2 = \left(\frac{a}{2}\phi^2 + b\right)^2 + a^2\phi^2,$$

$$\frac{ds}{d\phi} = \frac{a}{2}\phi^2 + (a+b),$$

$$\theta = \sin^{-1}\frac{a\phi}{r} + \phi,$$

where a is the radius of the circle; and there will be a loop or cusps according as $a+b$ is positive or negative; when $b=-a$, we have the transition case, for which

$$r^2 = \frac{a^2}{4}\phi^4 + a^2, \quad s = \frac{a}{6}\phi^3,$$

and the arco-radial equation becomes

$$(r^2 - a^2)^3 = \frac{81}{4}a^2s^4.$$

* Mr Crofton has noticed, in an ingenious paper published in the *Mathematical Messenger*, that this involute is the locus of the centres of all the circles cutting orthogonally the originating circle and the parent first involute. This is seen very easily as follows:

$$p = a\left(\frac{\phi^2}{2} - 1\right), \quad s' = p + p'' = a\frac{\phi^2}{2},$$

$$r^2 = p^2 + p'^2 = a^2\frac{\phi^2}{4} + a, \text{ or } r^2 - a^2 = s'^2,$$

showing that the tangents to the circle and first involute from any point in the second are equal to one another.

Another and more remarkable case occurs when $G^2 + G'^2$ becomes a perfect square, for then the degrees in s and r will sink to half their usual values: this occurs when $b = -\frac{a}{2}$, which is the case of a looped curve bisecting at its apse the radius drawn from the centre of the circle to the generating first involute.

We have in that case

$$G = \frac{a}{2}(\phi^2 - 1),$$

$$r = \frac{a}{2}(\phi^2 + 1) \text{ or } \phi = \sqrt{\left(\frac{2r - a}{a}\right)},$$

$$s = \frac{a}{2}\left(\frac{\phi^3 + 3\phi}{3}\right),$$

whence $9as^2 = (2r - a)(r + a)^2$,

and $\theta = \sin^{-1}\sqrt{\left(\frac{2ar - a^2}{r^2}\right)} + \sqrt{\left(\frac{2r - a}{a}\right)} = \text{vers}^{-1}\frac{a}{r} + \sqrt{\left(\frac{2r - a}{a}\right)}$,

a form even simpler than that of the first involute.

We may write this equation under the form

$$\theta = -2\cos^{-1}\sqrt{\left(\frac{\frac{1}{2}a}{r}\right)} + \sqrt{\left(\frac{r - \frac{1}{2}a}{\frac{1}{2}a}\right)};$$

or, turning round the line from which θ is reckoned through a quarter of a revolution,

$$\theta = 2\sin^{-1}\sqrt{\left(\frac{\frac{1}{2}a}{r}\right)} + \sqrt{\left(\frac{r - \frac{1}{2}a}{\frac{1}{2}a}\right)}.$$

Let now $\theta = 2\vartheta, \quad \frac{a}{2}\cdot r = \rho^2$

(which is the same thing as if for x and y we substituted $x^2 - y^2$ and $2xy$), then

$$\vartheta = \sin^{-1}\frac{\frac{1}{2}a}{\rho} + \frac{\sqrt{\left\{\rho^2 - \left(\frac{a}{2}\right)^2\right\}}}{a}.$$

This is the polar equation to a known curve (of the kind used by Captain Moncrieff in his barbette gun-carriage). It is of the class of curves generated by a fixed point on a wheel rolling on a plane. Such a curve may be termed the *convolute* of a circle of a *pitch* denoted by the ratio of the distance of the fixed point *below* the centre to the radius of the revolving circle; thus a convolute of zero-pitch is the spiral of Archimedes, a convolute of unit pitch the first involute to the circle: the general equation to a convolute, when the distance below the centre is d and the radius a, is given by the

Rev. James White in the last September Number of the *Educational Times* and is easily shown to be

$$\vartheta = \sin^{-1}\frac{d}{\rho} + \frac{\sqrt{(\rho^2 - d^2)}}{a}.$$

Similarly, we may define the pitch of the second involute to be the ratio of the distance of its apse from the centre to the radius; and then we are conducted to the observation that whilst the convolute of full pitch is the first involute, the convolute of half pitch, on applying to it one of the simplest forms of M. Chasles's or Mr Roberts's method of transformation (given in Dr Salmon's *Higher Plane Curves*, p. 236), namely, doubling the vectorial angle and squaring the radius vector, becomes converted into the second involute of half pitch. Since for this curve

$$r = \frac{a}{2}(\phi^2 + 1) = \frac{ds}{d\phi},$$

we see that it may be completely defined, without reference to any theory of involutes, as the curve whose radius of curvature at any point is equal to its radius vector reckoned from a given origin. It is the curve which completely satisfies the equation $rd\cos^{-1}\frac{dr}{ds} = s$, the two arbitrary parameters which the complete integral of this equation should contain being furnished by the linear magnitude and angle of swing of the curve round the given origin*.

I conclude with the remark that if we regard the s and r^2 of the successive involutes as *rectilinear* coordinates to a variable point, the arco-radial

* This evolute possesses the property, which serves to characterize it completely, of cutting the originating *circle* (its second evolute) orthogonally. For when $r^2 = a^2$, $G^2 = 0$, that is, the tangent to the curve passes through the centre. Moreover, since $G = 0$ gives $\phi = 1$, it follows that the curve cuts out of the circle an arc equal in length to the diameter. Summarizing such of its principal properties as have fallen in our way, we see that it bisects the line joining the centre of the originating circle and the cusp of the first involute; that it cuts the said circle orthogonally; that its radius of curvature is everywhere equal to its elongation from the centre; that it is a trajectory to a central force varying as the inverse cube of the shortest distance from the periphery of the originating circle; that its arco-radial equation is of only half the number of dimensions of the general involute of the same order; and that by the simplest form of quadratic transformation (namely, that which leaves unaltered the inclination of the tangent to the radius vector) it changes into the half-pitch circular convolute; not to add that its polar equation is even simpler than that of the first involute. Certainly, then, as it seems to me, it ought to take permanent rank among the spirals which have a specific name on the geometrical register; and for want of a better, with reference to the place where its properties first came into relief, it might be termed the *Norwich spiral*. Where it meets the first involute we have

$$\left(\frac{\phi^2+1}{2}\right)^2 = \frac{r^2}{a^2} = \phi^2 + 1,$$

or

$$(\phi^2+1)(\phi^2-3) = 0;$$

so that at the real intersections the radius vector is $2a$, and the perpendicular on the tangent, namely, $\left(\frac{\phi^2-1}{a}a\right)$, is a, showing that the tangent and radius vector at those points are inclined to each other at an angle of 30°.

equation will represent a peculiar class of unicursal algebraical curves. Thus the first involute will represent a pair (one for each branch of the curve) of coincident right lines, and the general second involute (taking r^2 and s^2 as the coordinates) a pair of coincident semicubical parabolas.

In making s vary continuously on passing a cusp, the corresponding abscissa from increasing must begin to decrease, or *vice versâ*, according to the principles previously noticed.

Thinking of the recovery of the cusps and apses from the arco-radial equation, I have been led to consider a morphological property of a more general class of unicursal equations, which I think is likely to bear valuable fruit, and may possibly form the subject of another communication.