

91.

NOTE ON A *MEMORIA TECHNICA* FOR DELAMBRE'S, COMMONLY CALLED GAUSS'S, THEOREMS.

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THE most subtle reagents employed in spherical analysis and transformation are the following four admirable formulæ, “commonly ascribed to Gauss, but in reality due to Delambre*”—

$$\cos \frac{c}{2} \cos \frac{A+B}{2} = \sin \frac{C}{2} \cos \frac{a+b}{2},$$

$$\cos \frac{c}{2} \sin \frac{A+B}{2} = \cos \frac{C}{2} \cos \frac{a-b}{2},$$

$$\sin \frac{c}{2} \cos \frac{A-B}{2} = \sin \frac{C}{2} \sin \frac{a+b}{2},$$

$$\sin \frac{c}{2} \sin \frac{A-B}{2} = \cos \frac{C}{2} \sin \frac{a-b}{2}.$$

Four out of the six binary combinations of these four equations give by simple division Napier's *Analogies*, a term which seems almost equally appropriate to designate Delambre's formulæ. It need hardly be remarked that whilst Napier's analogies may be immediately deduced from Delambre's formulæ, the converse is not true.

If we call the products on the left-hand side of the equations P, Q, R, S , and their polar reciprocals P', Q', R', S' , it is worthy of notice that the formulæ become

$$P = -P', \quad Q = R', \quad R = Q', \quad S = -S'.$$

* Todhunter's *Spherical Trigonometry*, p. 27. See also Davies's edition of *Hutton's Course*, Vol. ii. p. 37.

The formulæ may be expressed collectively by the easily remembered disjunctive elective equation

$$\frac{\cos c \cos \frac{A \pm B}{2}}{\sin \frac{c}{2} \sin \frac{A \pm B}{2}} = \frac{\cos C \cos \frac{a \pm b}{2}}{\sin \frac{C}{2} \sin \frac{a \pm b}{2}}.$$

The number of products on each side of the equation, if all the combinations of trigonometric affection and algebraical sign are exhausted, is 2^3 or eight. Out of each 8, 4 only are to be preserved and colligated each with each. Thus the number of systems capable of formation is

$$\left(\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \right)^2 (1 \cdot 2 \cdot 3 \cdot 4) = 24 \times 70^2 = 117600,$$

of which one only is valid. This accounts *à priori* for the difficulty of recollecting these formulæ, a difficulty often complained of and still oftener felt, and which is one reason of their being comparatively little used by junior students. Two observations easily retained in the memory will serve, I think, in a great degree to remove this difficulty.

Rule 1. On opposite sides of any one equation the trigonometric affections of the angles are contrary, and those of the sides similar.

Rule 2. The trigonometric affection of the uniliteral factor of each product governs the algebraic sign of the biliteral factor, in the following manner:—

Comparing products which lie on the *same* side of the equations, *like and unlike* affections go with like and unlike signs; comparing those which lie on *opposite* sides of the equations, *unlike and like* affections go with *like and unlike* signs.

These two rules are not quite sufficient in themselves; for they would be satisfied not only by the four true equations, but also by the four following false ones:—

$$\cos \frac{c}{2} \cdot \cos \frac{A - B}{2} = \sin \frac{C}{2} \cdot \cos \frac{a - b}{2},$$

$$\cos \frac{c}{2} \cdot \sin \frac{A - B}{2} = \cos \frac{C}{2} \cdot \cos \frac{a + b}{2},$$

$$\sin \frac{c}{2} \cdot \cos \frac{A + B}{2} = \sin \frac{C}{2} \cdot \sin \frac{a - b}{2},$$

$$\sin \frac{c}{2} \cdot \sin \frac{A + B}{2} = \cos \frac{C}{2} \cdot \sin \frac{a + b}{2}.$$

To make the system of rules complete so as to exclude *à priori* the construction of the four false deductions, it is necessary and sufficient to bear in mind

that, on the left-hand side of the equation, the *cosine*-affection of the uniliteral term is associated with the *plus* sign in the biliteral one*.

But even without this check the false equations may be put to the question and made severally to disclose their character as such by applying any one of them to the limiting case of a triangle on a sphere continuing of *finite radius*, but in which the angles become respectively 180° , 0 , 0 , and consequently the side opposite the first equal or capable of being equal to the sum of the other two. Thus writing in the first and third of the last written formulæ $C = 180^\circ$, $B = 0$, $A = 0$, we ought to be able to derive $c = a + b$, but find instead $a = b \pm c$ in the first, and $a = b + c$ in the third. And similarly in the second and fourth, writing $A = 180^\circ$, $B = 0$, $C = 0$, we ought to be able to derive $a = b + c$, but find instead $a = -b \pm c$ in the second, and $a = -b + c$, or $a + b + c = 360^\circ$ in the fourth. We might easily deduce other defective criteria from the reciprocal limiting case of a spherical triangle in which one side is zero and the two others each 180° , in which case the angle opposite the first augmented by 180° will equal the sum of the other two. Furthermore, using accents, as before, to denote polar reciprocation, the false system takes the form

$$P - P' = 0, \quad Q + R' = 0, \quad R + Q' = 0, \quad S - S' = 0,$$

in lieu of the true form,

$$P + P' = 0, \quad Q - R' = 0, \quad R - Q' = 0, \quad S + S' = 0.$$

A direct geometrical proof of these potent formulæ appears to be a *desideratum*.

* Rule 2, with the addition to it, may be easily retained in the memory by aid of the scheme below written,

	left	right
cos	+	-
sin	-	+

but, as subsequently shown in the text, the bordering of the square may be affixed at random, that is, the words *left* and *right* or *cos* and *sin* may be interchanged without leading to any error but of a kind susceptible of immediate detection and remedy.