87.

ASTRONOMICAL PROLUSIONS: COMMENCING WITH AN INSTANTANEOUS PROOF OF LAMBERT'S AND EULER'S THEOREMS, AND MODULATING THROUGH A CONSTRUCTION OF THE ORBIT OF A HEAVENLY BODY FROM TWO HELIOCENTRIC DISTANCES, THE SUBTENDED CHORD, AND THE PERIODIC TIME, AND THE FOCAL THEORY OF CARTESIAN OVALS, INTO A DISCUSSION OF MOTION IN A CIRCLE AND ITS RELATION TO PLANETARY MOTION*.

[*Philosophical Magazine*, XXXI. (1866), pp. 52—76.]

THE original demonstration by Lambert of the celebrated theorem which bears his name was a geometrical one. See *Monthly Notices* of the Astronomical Society, Vol. XXII. p. 238, where this demonstration is reproduced, or rather recapitulated, by Mr Cayley. See also Lambert's own *Insigniores Orbitæ cometarum proprietates*, Augusta Vindelicorum (Augsburg), 1761. It occupies seven or eight pages of this celebrated tract, and, elegant as may be considered the chain of geometrical enunciations from which it is deduced, is, as a specimen of geometrical style, little worthy of the inconsiderate commendations which have been heaped upon it, containing, as it does, a hybrid mixture of algebraical, geometrical, and trigonometrical ratiocination. The late Professor MacCullagh, as I am informed by my ingenious coadjutor, Mr Crofton, one of his hearers at Trinity College, Dublin, greatly improved upon Lambert's method, and succeeded in reducing it to a purely geometrical form. Lagrange has given no less than four distinct demonstrations of the same,—one a sort of verification by aid of trigonometrical formulæ in which

* Communicated by the author. A portion of this paper has appeared in the *Monthly Notices* of the Astronomical Society of London for December last [p. 496, above], namely, so much of it as relates to Lambert's theorem *proper*. The portion concerning circular motion formed the subject of a communication to the London Mathematical Society at the meeting of December 18, 1865. The part which presented itself last to the author's mind, and is consequently the least developed, is that which relates to the determination of the forces in any orbit to any two (or more) centres of force. The general expression for such forces will be found stated further on in a footnote, where the *equation of radial work* is defined and employed to obtain the solution in a form of unexpected simplicity.

the eccentric anomalies are introduced; a second of a similar nature, but dealing only with the true anomalies; a third founded on a property of integrals*; and a fourth, perhaps the most remarkable of any, derived from the general expressions for the time in an orbit described about two centres of force varying according to the law of nature, but one of them supposed to be situated in the orbit itself, and to become zero. Notwithstanding this plethora of demonstrations I venture to add a seventh, the simplest, briefest, and most natural of all, in which I employ a direct method to prove, from the ordinary formulæ for the time of a planet passing from one point to another, that, when the period is given, the time is a function only of the sum of the distances of these points from the centre of force, and of their distance from one another.

Let ρ, ρ' be the distances of the two positions from the sun, c their distance from one another, v, v' the true, u, u' the eccentric, m, m' the mean anomalies thereunto corresponding, e the eccentricity,

$$\omega = m - m', \quad s = \rho + \rho', \quad \Delta = \tfrac{1}{2}(s^2 - c^2);$$

then

$$\rho = 1 - e\cos u, \quad \rho' = 1 - e\cos u', \quad m = u - e\sin u, \quad m' = u' - e\sin u',$$

$$\rho\cos v = \cos u - e, \qquad \rho\sin v = \sqrt{(1-e^2)}\sin u,$$

$$\rho'\cos v' = \cos u' - e, \quad \rho'\sin v' = \sqrt{(1-e^2)}\sin u',$$

$$c^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(v'-v).$$

Writing for brevity p, p', q, q' for $\cos u$, $\cos u'$, $\sin u$, $\sin u'$, we have

$$s = 2 - ep - ep', \quad \omega = u - u' - eq + eq',$$

$$\Delta = \rho\rho' + \rho\rho'\cos(v'-v) = 1 + pp' + qq' - 2e(p+p') + e^2(1 - qq' + pp').$$

Let $$J = \frac{d(\Delta, s, \omega)}{d(e, u, u')};$$

then J is the determinant

$$\begin{vmatrix} \begin{Bmatrix} -2(p+p') \\ +2e(1+pp'-qq') \end{Bmatrix}; & \begin{Bmatrix} -p'q+pq'+2eq \\ -e^2(pq'+p'q) \end{Bmatrix}; & \begin{Bmatrix} -pq'+p'q+2eq' \\ -e^2(pq'+p'q) \end{Bmatrix} \\ -p-p' \quad ; & eq \quad ; & eq' \\ -q+q' \quad ; & 1-ep \quad ; & -1+ep' \end{vmatrix}.$$

* The property in question, discovered by Lagrange, is that the integral

$$\int \frac{rdr}{\sqrt{(L+Mr+Nr^2)}}$$

may be transformed into

$$\int \frac{(x^2+h)\,dx}{\sqrt{(a+bx+cx^2+dx^3+ex^4)}} - \int \frac{(y^2+h)\,dy}{\sqrt{(a+by+cy^2+dy^3+ey^4)}};$$

in applying it to Lambert's theorem a, b, c are made to vanish. This transformation and its consequences appear to us to deserve further study; as far as I know it has not been touched upon by the writers on elliptic functions.

Denoting this determinant by

$$\begin{vmatrix} A, & B, & C \\ D, & E, & F \\ G, & H, & K \end{vmatrix},$$

we find

$$(A, B, C) - 2H(D, E, F) + 2E(G, H, K) = (0, B, -B),$$
$$(A, B, C) - 2K(D, E, F) + 2F(G, H, K) = (0, -C, C),$$

so that

$$J = \begin{vmatrix} A, & B, & C \\ 0, & B, & -B \\ 0, & -C, & C \end{vmatrix} = 0.$$

Hence it appears that $d\omega$ is a linear function of ds and $d\Delta$; that is, ω is a function of s and Δ, or, what is the same thing, of s and c, and independent of e. If then, when $e=1$, the corresponding values of $\rho, \rho', v, v', u, u'$ are $r, r', \theta, \theta', \phi, \phi'$, we have

$$\cos\theta = -1, \quad \cos\theta' = -1, \quad \sin\theta = 0, \quad \sin\theta' = 0, \quad r - r' = c, \quad r + r' = s,$$

whence, writing

$$1 - \cos\phi = \frac{s+c}{2}, \quad 1 - \cos\phi' = \frac{s-c}{2},$$

we have finally

$$\omega = \phi - \phi' - \sin\phi + \sin\phi',$$

as was to be proved.

Essentially this demonstration is of the same nature as the first of Lagrange's four methods of proof above referred to, but with the difference that it leads up to and accounts beforehand for the success of the transformations therein employed.

Alluding to Lambert's cumbrous demonstration, Lagrange says of it, "His theorem merits the especial notice of mathematicians, both on its own account, and because it appears difficult to arrive at it by algebraical processes (*calcul*); so that it may be ranked among the small number of those in which geometrical seems to have the advantage over algebraical analysis." In the nature of things such advantage can never be otherwise than temporary. Geometry may sometimes appear to take the lead of analysis, but in fact precedes it only as a servant goes before his master to clear the path and light him on his way. The interval between the two is as wide as between empiricism and science, as between the understanding and the reason, or as between the finite and the infinite.

The result so simply obtained above is of course not restricted to the case of the ellipse, but applies to motion generally about a centre of force according to the law of nature.

Calling t the time, the syzygy shown to exist between δt, δs, δc, being independent of any supposition as to the value of e, or as to the reality of the functions employed, will of necessity continue to obtain where, e being greater than 1, the motion becomes hyperbolic. If μ be the absolute force, and 1, as before, the semi-major axis, writing

$$\epsilon = \sqrt{\frac{e-1}{e+1}}, \quad \epsilon \tan\frac{v}{2} = x, \quad \epsilon \tan\frac{v'}{2} = x',$$

the rest of the notation being preserved, we obtain, by direct integration and substitution,

$$\frac{s}{2} = \frac{1}{1-\epsilon^2}\left(\frac{\epsilon^2 + x^2}{1-x^2} + \frac{\epsilon^2 + x'^2}{1-x'^2}\right),$$

$$\frac{\Delta}{8} = \frac{1}{(1-\epsilon^2)^2}\frac{(\epsilon^2 + x^2)(\epsilon^2 + x'^2)}{(1-x^2)(1-x'^2)},$$

$$t = \mu^{\frac{1}{2}}\left\{\log\frac{1-x'}{1+x'} - \log\frac{1-x}{1+x} + 2(x-x')(1-xx')\right\}.$$

And we must needs find by actual computation the Jacobian

$$\frac{d(s, \Delta, t)}{d(\epsilon, x, x')} = 0.$$

Making $\epsilon = 0$, and giving x, x' their corresponding values in terms of s and Δ, there results

$$\frac{x^2}{1-x^2} = \frac{s+c}{2}; \quad \frac{x'^2}{1-x'^2} = \frac{s-c}{2};$$

and accordingly

$$t = \mu^{\frac{1}{2}}\left\{\begin{matrix}\log\{\sqrt{(s+c+2)} - \sqrt{(s-c)}\} - \log\{\sqrt{(s-c-2)} - \sqrt{(s-c)}\} \\ -\sqrt{\{(s+c+1)^2 - 1\}} \qquad + \qquad \sqrt{\{(s-c+1)^2-1\}}\end{matrix}\right\}.$$

It is worthy of notice that the effect of making $\epsilon = 0$ or $e = -1$ in this case, like that of making $e = 1$ in the case of elliptic motion, is to reduce the motion to that of a body in a straight line, but with the difference that for the elliptic the reduced motion is that of a body moving between the point of instantaneous rest and the centre of force or point of infinite velocity, whereas for the hyperbola it is that of a body moving on the same side of these two points.

The theorem for the case of the parabola was given by Euler (1744), but reproduced independently by Lambert in the *Insigniores Proprietates*, Sectiones 1, 2, in 1761.

I think the idea of the general theorem may not unlikely have originated in an observation of the accordance in form of the result for parabolic motion with that for motion in a straight line, an accordance easily verified to extend to motion in a circle. Such coincidence, to a mind open to the impressions of analogy, could hardly fail to suggest the necessity of the existence of a deeper-seated law, of which these extreme cases must represent the outcroppings. Euler's theorem is of course included as a particular case in Lambert's, and may be derived from it by making a infinite in the expression for t as a function of s, c, a; but it may also be obtained independently as follows. Calling $4m$ the latus rectum, retaining the rest of the notation, and writing $\tan\frac{v}{2}=q$, $\tan\frac{v'}{2}=q'$, we easily find

$$\tfrac{1}{2}\sqrt{\Delta}=m\,(1+qq'),$$

$$s=m\,(2+q^2+q'^2),$$

$$\frac{t}{\sqrt{2}}=m^{\frac{3}{2}}\,\{(q-q')+\tfrac{1}{3}(q^3-q'^3)\}.$$

Hence the Jacobian

$$\frac{d\,(\frac{1}{2}\sqrt{\Delta},\, s,\, \sqrt{2}\,.\,t)}{d\,(m,\, q,\, q')}$$

becomes a multiple of the determinant

$$\begin{vmatrix} 1+qq', & q', & q \\ 2+q^2+q'^2, & 2q, & 2q' \\ 3(q-q')+q^3-q'^3, & 2+2q^2, & -2-2q^2 \end{vmatrix}.$$

Calling this

$$\begin{vmatrix} A, & B, & C \\ D, & E, & F \\ G, & H, & K \end{vmatrix},$$

it will be found that

$$(A,B,C)-\frac{H}{2A-F}(D,E,F)+\frac{E}{2A-F}(G,H,K)=0,\ B,\ -B,$$

$$(A,B,C)+\frac{K}{2A+E}(D,E,F)-\frac{F}{2A+E}(G,H,K)=0,\ -C,\ C;$$

and consequently the Jacobian in question, as before, takes the form

$$\begin{vmatrix} A, & B, & C \\ 0, & B, & -B \\ 0, & -C, & C \end{vmatrix},$$

which is identically zero; so that t is a function only of s, c when a is given, and one solution is left free between m, q, q'.

Making $q = \infty$, we have

$$m(q-q') = \sqrt{(s-\sqrt{\Delta})} = \frac{\sqrt{(s+c)} - \sqrt{(s-c)}}{2},$$

$$m(q+q') = \sqrt{(s+\sqrt{\Delta})} = \frac{\sqrt{(s+c)} + \sqrt{(s-c)}}{2},$$

$$mq = \frac{\sqrt{(s+c)}}{2}, \quad mq' = \frac{\sqrt{(s-c)}}{2};$$

and thus $$t = \tfrac{1}{6}\{(s+c)^{\frac{3}{2}} - (s-c)^{\frac{3}{2}}\}.$$

There is a sort of pendant to Lambert's theorem which is worthy of notice. If we call $\rho - \rho' = v$ and $c^2 - \sigma^2 = D$, writing

$$ae(\sin u' - \sin u) = \Omega,$$

we have also

$$(1-e^2)\,a^2\{1 - \cos(u'-u)\} = D,$$

$$ea(\cos u - \cos u') = \sigma,$$

from which we easily obtain

$$\Omega = \sqrt{\frac{e^2c^2 - \sigma^2}{1-e^2}};$$

so that Ω is a function only of c, e, σ, as by Lambert's theorem it is a function only of c, a, s. Moreover, since

$$\sin\left(\frac{u'-u}{2}\right) = \sqrt{\frac{D}{2(1-e^2)}},$$

it is apparent that the change in the mean anomaly is a complete function of the two variables $\frac{\sqrt{D}}{b}$, $\frac{c}{a}$, as by Lambert's theorem it is of the two $\frac{\sqrt{\Delta}}{a}$, $\frac{c}{a}$. Comparing the value of Ω given immediately above with that which is contained in Lambert's theorem, the solution of a linear equation leads immediately, after certain simple reductions, to the equation

$$1 - e^2 = \frac{2(c^2-\sigma^2)}{ss' + c^2 \pm \sqrt{\{(s^2-c^2)(s'^2-c^2)\}}},$$

where $s' + s = 4a$. And as there is nothing to determine the signs of ρ or ρ', the above, by interchanging severally and independently ρ, ρ' with $-\rho, -\rho'$, represents eight values of e:—four corresponding to the change of ρ into $-\rho$, and ρ' into $-\rho'$, contained in the expression immediately above written, combined with the equation $s' + s = \pm 4a$; and four in the conjugate expression

$$1 - e^2 = \frac{2(c^2-s^2)}{\sigma\sigma' + c^2 \pm \sqrt{\{(c^2-\sigma^2)(c^2-\sigma'^2)\}}},$$

where $\sigma' + \sigma = \pm 4a$.

Since we have also (calling i the angle between c and a) $\cos i = \frac{\sigma}{ec}$ in the first case, and $\cos i = \frac{s}{ec}$ in the second case, the problem of determining the conic, of which one focus, the major axis, and two points are given, is thus completely solved. This of course comprehends the analytical solution of the problem of determining the magnitude and position of the orbit of a planet from the periodic time, two heliocentric distances, and the included angle, of which no mention is to be found in any of the ordinary books of astronomy, or even in the *Theoria Motus*, where one would naturally expect to find it.

There are thus eight values of e^2, and the solution is eightfold. The sign of $\cos i$ being left ambiguous does not raise the number to 16; for this ambiguity depends upon the fact of the direction of c being incapable of analytical representation; only one of these values of $\cos i$ will appertain to any stated case. If F be the given focus, P, Q the two given points, and G the second focus, by rotating the figure about the line FG, P, Q come into the positions P', Q'; c, s, σ remain unaltered; but the angles between $Q'P'$, FG, and between QP, FG, become supplementary. If we chose to effect a direct solution of the problem of determining the orbit without the aid of the eccentric anomalies, we should have to eliminate θ, θ' between the equations

$$\pm \rho = \frac{a(1-e^2)}{1 - e\cos\theta}, \quad \pm \rho' = \frac{a(1-e^2)}{1-e\cos\theta'}, \quad c^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\theta - \theta').$$

This elimination will be found to lead to a quadratic equation in e^2, the coefficients of e^6 and e^8 vanishing; and we thus obtain an eightfold solution as before, but in a more involved form. Or, again, we might write

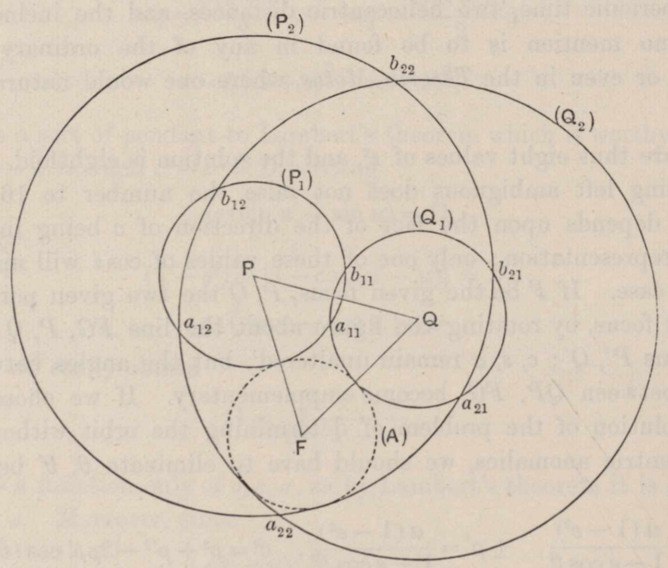
$$\{a(1-e^2) + ex\}^2 = \rho^2,$$
$$\{a(1-e^2) + ex'\}^2 = \rho'^2,$$
$$xx' + yy' = c^2,$$
$$x^2 + y^2 = \rho^2, \quad x'^2 + y'^2 = \rho'^2,$$

and between these five equations eliminate x, x', y, y'. By the general theory of elimination, e^2 should rise to the sixteenth degree in the resultant; but in fact it will rise only to the eighth. The following obvious geometrical construction will perfectly account *à priori* for the existence of the excluded infinite values of e^2.

Since $\quad FP \pm GP = \pm 2a$ and $FQ \pm GQ = \pm 2a$,

G will be any point in the intersection of either of two circles C_1, C_2 with either of the two K_1, K_2, where the squared radii of C_1, C_2 are $(2a+FP)^2$, and of K_1, K_2 $(2a+FQ)^2$, $(2a-FQ)^2$ respectively. Consequently there are eight real or imaginary positions of G at a finite, and eight at an infinite distance.

It is obvious that, if we restrict the orbit to being elliptical, there can never be more than two admissible solutions; but treating the question more generally, any even number of solutions whatever may exist from 0 to 8, both inclusive. I am indebted to my able friend, Dr Hirst, for the following figure, illustrating the interesting case where all eight solutions are real and hyperbolæ.



In this figure $\rho\,(=FP) > 2a, \quad \rho'\,(=FQ) > 2a,$

and likewise

$$\rho + \rho' - 4a > c,$$

$$4a - \rho + \rho' < c,$$

$$4a + \rho - \rho' < c.$$

Supposing FP, FQ to be each greater than $2a$, there will be no difficulty in verifying the following statement.

One pair of hyperbolæ, in which P, Q lie in the branch containing F, will be always real; a second pair, in which they lie in the opposite branch, will be real or imaginary according as s is greater or less than $c + 4a$, that is, according as $2a$ is less or greater than $\frac{s-c}{2}$. A third pair, in which the two given points are distributed between the two branches, will be real or imaginary respectively according as $2a$ is less or greater than $\frac{c+\sigma}{2}$; and a fourth pair, where the same distribution occurs, will be real or imaginary according as $2a$ is greater or less than $\frac{c-\sigma}{2}$.

It is of course only with the first kind of hyperbolæ, that in which the given points lie in the branch concave to the given focus, with which the problem, regarded as an astronomical one, is concerned. But in all cases the formulæ given for the determination of e and i admit of immediate adaptation to logarithmic computation. Thus, for example, if we take the one which meets the case of distribution between the two branches of an hyperbola, namely,

$$e^2 - 1 = \frac{2(s^2 - c^2)}{(\sigma\sigma' + c^2) \pm \sqrt{\{(c^2 - \sigma^2)(c^2 - \sigma'^2)\}}},$$

writing $e = \sec\phi, \quad s = c\sec\lambda, \quad \sigma = c\cos\mu, \quad \sigma' = c\cos\mu',$

we obtain

$$\tan\phi = \tan\lambda \sec\frac{\mu \pm \mu'}{2}$$

$$\pm \cos i = \cos\lambda \cos\phi.$$

Viewed as a question of analytical geometry, the investigation as to the reality of the curve would have to be treated in a more general manner; that is, without assuming, as I have done, the necessity of the inequalities $s < \rho_1 + \rho_2$, $\sigma > \rho_1 - \rho_2$, where ρ_1, ρ_2 represent the two given focal distances; for it is a very important, although hitherto strangely neglected geometrical principle, that every real conic is at one and the same time an ellipse and hyperbola; namely, either an actual ellipse accompanied by an ideal hyperbola, or an actual hyperbola accompanied by an ideal ellipse. This may immediately be made manifest by considering how the ordinary rectangular-coordinate equation to a conic, with its origin transferred to a focus, is connected with the property of the conic in respect to its two foci. Calling ρ, ρ' the two focal distances of any point, the equation to rectangular coordinates is obtainable by equating to zero the norm of the quantity $2a \pm \rho \pm \rho'$, where ρ represents $\sqrt{(x^2 + y^2)}$, and ρ' represents $\sqrt{\{(2ae + x)^2 + y^2\}}$, which norm will only be of the *second degree* in x, y, although a product of four factors each of the first degree in x, y, owing to the vanishing of the coefficients of the terms that ought to rise to the fourth degee in the variables. Calling, then, this norm N, we see that the quadratic equation $N = 0$ is satisfied alike by the equations $\rho + \rho' = 2a$ and $\rho - \rho' = 2a$, the difference being that one of these will belong to the apex of an actual, and the other to that of an ideal triangle, according to the sign of $e - 1$.

It may not be quite out of place here to show how immediately the knowledge of the existence of a third focus to the Cartesian ovals, that remarkable discovery of our illustrious Royal Society *Laureate* of the year, flows from a similar process to the one above. For taking the norm of the expression

$$a\sqrt{(x^2 + y^2)} + \sqrt{(b^2x^2 + b^2y^2 + 2bcx + c^2)} + d^2,$$

the equation to any such curve becomes

$$\{(b^2-a^2)(x^2+y^2)+2bcx+c^2\}^2$$
$$-2d^2\{(b^2+a^2)(x^2+y^2)+2bcx+c^2\}$$
$$+d^4=0,$$

that is,

$$(b^2-a^2)^2(x^2+y^2)^2+4bc(b^2-a^2)x(x^2+y^2)$$
$$+\{2(c^2-d^2)b^2-2(c^2+d^2)a^2\}(x^2+y^2)$$
$$+4b^2c^2x^2+4(c^2-d^2)bcx+(c^2-d^2)^2=0;$$

in which equation a^2, b^2, c^2, d^2 may obviously be replaced by α^2, β^2, γ^2, δ^2, provided

$$\left.\begin{aligned}\beta^2-\alpha^2&=b^2-a^2,\\ \beta\gamma&=bc,\\ \alpha^2\delta^2&=a^2d^2,\\ \gamma^2-\delta^2&=c^2-d^2\end{aligned}\right\}.$$

Writing for a^2, b^2, c^2, d^2, &c. a_1, b_1, c_1, d_1, and squaring the second equation, we obtain a symmetrical system of equations, namely,

$$\left.\begin{aligned}\beta_1-\alpha_1&=b_1-a_1, & \gamma_1-\delta_1&=c_1-d_1,\\ \beta_1\gamma_1&=b_1c_1, & \alpha_1\delta_1&=a_1d_1,\end{aligned}\right\}$$

for determining α_1, β_1, γ_1, δ_1. Throwing out the solution $\alpha_1=a_1$, $\beta_1=b_1$, $\gamma_1=c_1$, $\delta_1=d_1$, only one other solution will be found to exist, which, restoring the original variables, becomes

$$\alpha^2=\frac{b^2-a^2}{c^2-d^2}d^2,\quad \beta^2=\frac{b^2-a^2}{c^2-d^2}c^2,$$

$$\gamma^2=\frac{d^2-c^2}{a^2-b^2}b^2,\quad \delta^2=\frac{d^2-c^2}{a^2-b^2}a^2,$$

with the condition that $\beta\gamma=bc$.

The complete arithmetical determination of the signs to be given to the several quantities α, β, γ, δ requires a distinct and detailed examination, which it would be irrelevant to enter upon in this place; it is enough to see that a second focus G at the distance $\frac{c}{b}$ from a given one F may be moved along the line FG to a new focus H at the distance $\frac{\gamma}{\beta}$ from F, the *modulus* $\frac{b}{a}$ becoming simultaneously replaced by $\frac{\beta}{\alpha}$, and the constant $\frac{d}{a}$ by the constant $\frac{\delta}{\alpha}$. I am not aware that M. Chasles has ever disclosed the *aperçu* which led him to this unlooked-for discovery. It is to be hoped that he will not allow future ages to labour under the same doubt as to the source from which he drew it as we must, it is to be feared, ever remain under with regard to the origin of Newton's rule, recently demonstrated, or Lambert's theorem, the motive to this paper. In this age of the world *euristic* is even more important to the promotion of science, and its secrets less likely to be recovered than those of mere *apodictic*.

Since a focus may be regarded as the intersection of two tangents from the circular points of infinity, we may generalize the problem of constructing the orbit by considering it as a particular case of constructing the conic which passes through two given points, touches two given straight lines, and has a principal axis of given length.

Taking the two given lines supposed to be inclined to each other at the angle α as the axes of coordinates, the equation to the curve may be written under the form

$$(Ax + Cy + 1)^2 = 2B^2xy,$$

which, writing

$$x = \xi - \frac{C}{2AC - B^2}, \quad y = \eta - \frac{A}{2AC - B^2},$$

becomes

$$A^2\xi^2 + 2(AC - B^2)\xi\eta + C\eta^2 = \frac{2B^2}{2AC - B^2}.$$

Adding $\lambda(\xi^2 - 2\cos\alpha\xi\eta + \eta^2)$ to the left-hand side of the equation, the discriminant of that side so augmented becomes

$$(\sin\alpha)^2\lambda^2 + (A^2 + 2\cos\alpha AC + C^2 - 2\cos\alpha B^2)\lambda + B^2(B^2 - 2AC).$$

Hence, calling the squared reciprocal of the given principal semiaxis q, and writing $\lambda = \frac{2B^2}{2AC - B}q$, we obtain

$$4\sin\alpha^2 q^2 B^2 + 2(2AC - B)^2(A^2 + 2\cos\alpha \,.\, AC + C^2 - 2\cos\alpha \,.\, B^2)q + (B^2 - 2AC)^3 = 0;$$

combining which with any of the four couples of linear equations,

$$pA \pm \sqrt{(2pq)}B + qC + 1 = 0, \quad p'A \pm \sqrt{(2p'q')}B + q'C + 1 = 0,$$

obtained by substituting for x, y the coordinates of the two given points, we obtain six sets of quadruple solutions, making twenty-four *finite* solutions in all. This result is in perfect accordance with that which applies to the case of the tangents meeting at the focus; for when $\frac{1}{q}$ is the square of the principal semiaxis in which the focus lies, we have already found eight solutions; and when $\frac{1}{q}$ refers to the other semiaxis, we have

$$\frac{8}{q} = 8a(1 - e^2) = \frac{(s + s')^2(c^2 - \sigma^2)}{ss' + c^2 \pm \sqrt{\{(s^2 - c^2)(s'^2 - c^2)\}}};$$

which, considering c, s, σ given, leads to a biquadratic in s' which serves to fix the curve; and as there are four systems of values of s, σ arising from the permutations of the signs of ρ, ρ', we thus have four times four, or sixteen solutions over and above the previous eight, making twenty-four in all, as in the general case.

We might generalize the problem otherwise by supposing given, not the magnitude of a principal axis, but that of a diameter through the intersection of the two given tangents; or, again, in quite a different direction by supposing three points P, Q, R to be given in a *Cartesian oval* defined by the equation $k\rho - \rho' = 2a$, ρ referring to a given focus F, and ρ' to a second focus G to be determined, a also being given, but k being to be determined. It is easy to see that in this case also the position of G may be obtained by the intersections of circles; for the ratios $PG : QG : RG$ will be known; there will thus be eight pairs of solutions arising from the permutations of the signs of ρ_1, ρ_2, ρ_3 which measure FP, FQ, FR; and calling $\frac{FG}{2a} e$, it would be an interesting analytical question to express the eight systems of k and e in terms of ρ_1, ρ_2, ρ_3, and c_1, c_2, c_3 the three chords joining P, Q, R,—these six quantities, of course, being not independent but connected by the well-known equation between the mutual squared distances of any four points from one another on a plane.

Touching the Cartesian ovals, Mr Crofton has well remarked that a circle may be regarded as one of a very peculiar kind. For if we take any two points electrical images of one another, inverses, in Dr Hirst's language, or, as I prefer to call them, reciprocals or *harmonics* in respect to a given circle, the distances ρ, ρ' of any point in the circle from them will be connected by the equation $-k\rho + \rho' = 0$; so that any pair of harmonics whatever of a circle may be regarded as foci of such curves. The third focus correlated to each pair will evidently be the centre; for, calling its distance from any point in the circle ρ'', we have $0 . \rho + \rho'' = c$; in the first equation the modulus k is finite and the constant zero; in the second the modulus is zero and the constant finite. Consequently a circle is a Cartesian oval, not only as a particular case of a conic, but *proprio Marte* and porismatically in quite another sort of way*. Now it is well known that a conic may be described

* Thus it will be seen that, besides its derivation through the ellipse, the circle descends by a short cut immediately from the Cartesian oval; recalling to mind the singular condition of consanguinity of the ill-fated descendants of Laius, at once children and grand-children to their mother, sons and brothers to their father. Viewed as sprung from the ellipse, there should be but two coincident Cartesian foci to the circle; it is the fraternal aspect of the relationship which brings into view the existence of an infinite number of such foci in the circle; every point in fact being a focus. This is explained by considering the circle so descended, not (like a conic) as a Cartesian oval with a branch at an infinite distance, but without such branch, and as doubled upon itself; thus the circular points at infinity become each double points, and, as well remarked by Mr Cayley, every line through either such double point is analytically a tangent to the curve, and thus every point in the plane of the circle, being the intersection of two such tangents, ought to be, as it is, a *focus*.

by two forces (varying as the inverse square of the distance, and tending to its two foci). This led me to inquire whether some analogous theorem did not hold of a circle in respect to any of its pairs of foci, that is of harmonics; and I find such is the case, as the annexed simple investigation will make manifest.

Call the radius of the circle *unity*; c, $\frac{1}{c}$ the distances of two harmonics from its centre; $\frac{\mu}{\rho^n}$, $\frac{\mu'}{\rho'^n}$ two forces tending to these points respectively; then by duly assigning the initial velocity, we are at liberty to suppose the constant zero in the equation for *vis viva*, so as to be able to write

$$v^2 = \frac{2\mu}{(n-1)\rho^{n-1}} + \frac{2\mu'}{(n-1)\rho'^{n-1}};$$

we have also

$$\frac{\rho'}{\rho} = \frac{\frac{1}{c} - 1}{1 - c} = \frac{1}{c}.$$

In order that the circle may be described under the circumstances above supposed, it is necessary and sufficient that

$$v^2 = \frac{\mu}{\rho^n}\cos i + \frac{\mu'}{\rho'^n}\cos i',$$

i, i' being the angles which ρ, ρ' respectively make with the normal; that is

$$v^2 = \frac{\mu}{\rho^n}\left(\frac{1 - c^2 + \rho^2}{2\rho}\right) + \frac{\mu'}{\rho'^n}\left(\frac{1 - \left(\frac{1}{c}\right)^2 + \rho'^2}{2\rho'}\right)$$

$$= \tfrac{1}{2}\left(\frac{\mu(1-c^2)}{\rho^{n+1}} + \frac{\mu'\left(1 - \frac{1}{c^2}\right)}{\rho'^{n+1}}\right) + \tfrac{1}{2}\left(\frac{\mu}{\rho^{n-1}} + \frac{\mu'}{\rho'^{n-1}}\right).$$

Hence, in order to satisfy this identity, we must have

$$\mu\,\frac{1-c^2}{\rho^{n+1}} + \mu'\,\frac{1 - \frac{1}{c^2}}{\rho'^{n+1}} = 0,$$

or

$$\mu = \frac{1}{c^2}\left(\frac{\rho}{\rho'}\right)^{n+1}\mu' = c^{n-1}\,.\,\mu';$$

so that

$$\frac{\mu'}{\rho'^{n-1}} = \frac{\mu}{\rho^{n-1}}.$$

And accordingly the required identity will be completely satisfied if we further make

$$\frac{\mu}{2} = \frac{2\mu}{n-1}, \text{ or } n = 5,$$

which implies

$$\mu = c^4\mu', \text{ or } \frac{\mu}{c^2} = \frac{\mu'}{\gamma^2},$$

c, γ being the distances of the respective centres of force from the centre of the orbit.

The *vis viva* consists of two equal parts, $\frac{\mu}{2\rho^4}, \frac{\mu'}{2\rho'^4}$, each centre contributing, as it were, equally to its production. To find the time, calling u the angle which the orbit swept out subtends at the centre, we have

$$\left(\frac{du}{dt}\right)^2 = \frac{\mu}{\rho^4},$$

or

$$t = \int \frac{du\rho^2}{\mu^{\frac{1}{2}}} = \frac{1}{\mu^{\frac{1}{2}}} \int du\,(1 + c^2 - 2c\cos u);$$

and P, the periodic time, will be $\frac{2F}{\mu^{\frac{1}{2}}}(1 + c^2)$, or, restoring the value of a to the radius, the period becomes $\frac{2\pi}{\mu^{\frac{1}{2}}} a(a^2 + c^2)$, which of course is the same as

$$\frac{2\mu}{\mu'^{\frac{1}{2}}} a(a^2 + \gamma^2).$$

If we now suppose the two absolute forces μ, μ', and δ the distance between them, to be given, the problem of determining the magnitude and position of the orbit and the periodic time may be easily effected; for we have only to find c, γ and a from the equations

$$c\gamma = a^2,$$

$$\frac{c^2}{\gamma^2} = \frac{\mu}{\mu'},$$

$$\gamma - c = \delta,$$

from which results

$$a = \frac{(\mu\mu')^{\frac{1}{4}}}{\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}}}\delta,$$

$$c = \frac{\mu}{\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}}}, \quad \gamma = \frac{\mu'^{\frac{1}{2}}}{\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}}},$$

$$P = 2\pi \frac{\mu^{\frac{1}{4}} \cdot \mu'^{\frac{3}{4}} + \mu^{\frac{3}{4}} \cdot \mu'^{\frac{1}{4}}}{(\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}})^3}\delta^3.$$

Also the velocity at either apse is given by the formula $v = \frac{\mu}{(a \mp c)^2}$, which gives $\frac{(\mu'^{\frac{1}{2}} \pm \mu^{\frac{1}{2}})^2}{\delta}$ for the two limiting velocities.

Again, the general expression for the time is

$$t = \frac{(a^2+c^2)}{\mu^{\frac{1}{2}}}\left\{u - \frac{2ac}{a^2+c^2}\sin u\right\}.$$

Suppose, then, a planet to be describing an ellipse under the attraction of the sun, and a fictitious body moving in a circle described about its axis major to leave an apse simultaneously with the planet, and that its velocity parallel to the axis major always remains equal to that of the planet in the same direction. Then the arc swept out by such body subtends at the centre the angle which measures the eccentric anomaly of the planet, and may be termed its eccentric follower. The motion of this eccentric follower may be physically produced by supposing it to be attracted to two centres of force of proper absolute magnitudes and duly placed in the major axis, attracting according to the inverse fifth power of the distance; this is an immediate consequence from the preceding expression for t.

If we call M the absolute force of the sun, it will readily be seen that we must have

$$\mu = \frac{(a^2+c^2)^2}{a}M,$$

$$\mu' = \frac{(a^2+\gamma^2)}{a}M;$$

where c, γ are the distances of the two new centres of force from the centre of the planetary orbit, and satisfy the equation

$$\frac{2ac}{a^2+c^2} = e,$$

or

$$c^2 - \frac{2a}{e}c + a^2 = 0,$$

which gives

$$c = \frac{a-b}{e}, \quad \gamma = \frac{a+b}{e};$$

b representing the semi-minor axis. c being equal to

$$\frac{a\{1-\sqrt{(1-e^2)}\}}{e}, \quad c - ae = a\frac{\sqrt{(1-e^2)}}{e}\{\sqrt{(1-e^2)}-1\},$$

and is always negative, so that the interior centre of force always lies between the centre of the orbit and the sun; when e is small it lies about midway between these two points, but nearer to the latter than the former: for example, if we were to suppose $e=\frac{3}{5}$, we should have $ae=\frac{3a}{5}$, $\frac{a\{1-\sqrt{(1-e^2)}\}}{e}=\frac{a}{3}$, which differs not very much from $\frac{3a}{10}$.

It is perhaps remarkable—at all events I am not aware whether any one has remarked, that the motion of the *eccentric follower* of a planet may also be brought about by a single force placed at the sun itself, attracting according to the law which is consistent with the body describing a circle. This is immediately apparent; for if we call S the position of the centre of force, C the centre of the circle, c the distance of S from C, a the radius of the circle, P any point in it, calling i the angle SPC, u the angle PCS, we have

$$v = \frac{h}{p} = \frac{h}{\rho \cos i} = \frac{h}{a - c \cos u};$$

so that $\frac{dt}{du} = \frac{a^2}{h}\left\{1 - \frac{c}{a}\cos u\right\}$, which proves the point in question*.

The force f for this case has been given by Newton in the third section of the *Principia*; it can be obtained instantaneously from the equation

$$v^2 = af \cos i = -\frac{a}{2}\frac{dv^2}{d\rho}\cdot\frac{\rho^2 + a^2 - c^2}{2\rho};$$

so that

$$\frac{dv^2}{d\rho \, . \, v^2} = \frac{-4a\rho}{\rho^2 + a^2 - c^2},$$

or

$$v^2 = \frac{C}{(\rho^2 + a^2 - c^2)^2}; \; f = -\frac{C\rho}{(\rho^2 + a^2 - c^2)^3}.$$

Calling ρ' the remainder of the chord R of which ρ is a part,

$$\rho^2 + (a^2 - c^2) = \rho^2 + \rho\rho' = \rho R;$$

so that f varies as

$$\frac{1}{\rho^2 R^3},$$

as given in the *Principia*, and of course, if the force-centre is at the extremity of a diameter, f varies as $\frac{1}{\rho^5}$, which is the case in which our two reciprocal foci come together. When one of them is at the centre, the other goes off to infinity, and the actual amount of force exerted by it, $\frac{\mu'}{r'^5}$, or $\frac{\mu}{r^4}\cdot\frac{r'^4}{r'^5}$, becomes zero when $\frac{\mu}{r^5}$ is finite; so that this case returns to that of a single force at the centre of the circle. If we wished to find the general law of the respective forces f, f' at the two reciprocal foci suitable to produce motion in the circle we might proceed as follows:—Calling i, i' the angles

* Hence follows the *statical* proposition that the force which tending to any centre retains a point in a circular orbit may be resolved into two forces tending to two fixed centres, each varying as the inverse fifth power of the distance: this proposition will be generalized subsequently in the text.

between the radii vectores drawn to these points from any point in the circle and the radius at that point, and writing

$$V = 2\int dr\,.f,\quad V' = 2\int dr'.f',$$

we have to satisfy the equation

$$\frac{1}{2}\frac{dV}{dr}\cos i + \frac{1}{2}\frac{dV'}{dr'}\cos i' + V + V' = 0.$$

Writing $z = \frac{r}{\sqrt{(c)}} = \frac{r'}{\sqrt{(r)}}$, and taking $\psi(z)$, any arbitrary function of z, we may write

$$\frac{dV}{dr} + \frac{4r}{a^2 - c^2 + r^2}V = \frac{4r}{a^2 - c^2 + r^2}\psi(z),$$

or

$$\frac{dV}{dz} + \frac{4cz}{a^2 - c^2 + cz^2}V = \frac{4cz}{a^2 - c^2 + cz^2}\psi(z);$$

and then

$$\frac{dV'}{dz} + \frac{4\gamma z}{a^2 - \gamma^2 + \gamma z^2}V' = \frac{-4\gamma z}{a^2 - \gamma^2 + \gamma z^2}\psi(z).$$

Integrating these equations, we find

$$V = \frac{4c}{(a^2 - c^2 + cz^2)^2}\int dz\,[z\,(a^2 - c^2 + cz^2)]\,\psi z,$$

$$V' = \frac{-4\gamma}{(a^2 - \gamma^2 + \gamma z^2)^2}\int dz\,[z\,(a^2 - \gamma^2 + \gamma z^2)]\,\psi z;$$

also

$$f = \frac{\sqrt{(c)}\,z}{a^2 - c^2 + cz^2}(\psi z - V).$$

Hence, making

$$\psi z = \frac{c^{\frac{3}{2}}\phi' z}{2z\,(a^2 - c^2 + cz^2)},$$

we have

$$f = \frac{\phi'(z)}{(\gamma - c + z^2)^2} - \frac{4z\phi z}{(\gamma - c + z^2)^3},$$

$$f' = -\frac{\phi' z}{(c - \gamma + z^2)^2} + \frac{4z\phi z}{(c - \gamma + z^2)^3},$$

ϕ being any arbitrary function, and z representing $\frac{r}{\sqrt{(c)}}$ or $\frac{r'}{\sqrt{(r)}}$. A similar method will apply to the determination of the forces at the foci whereby any conic may be described*.

* Employing the equation

$$V + V' + \frac{1}{2}\left(\frac{dV}{dr} + \frac{dV'}{dr'}\right)\rho\cos i = 0,$$

replacing $\rho\cos i$ by its equivalent $\frac{rr'}{a}$, writing $r - a = a - r' = z$, and decomposing the equation above written into

$$V + \frac{1}{2}\frac{dV}{dz}(a^2 - z^2) = \phi z;\qquad V' + \frac{1}{2}\frac{dV'}{dz}(a^2 - z^2) = \phi z,$$

It may be worth while pointing out a somewhat singular consequence of the laws that have been above established for the motion of a body in a circle about two reciprocal points as centres of force. It is an immediate and now well known, although for a time singularly overlooked, consequence* of the linear form of the equation $(\Sigma f\cos i)\,\rho = C - 2\Sigma\int dr\,(f)$ [where f is any central force, and i the angle which it makes with ρ, the radius of curvature at any point], which equation† exhibits the sole necessary and sufficient condition for any determinate orbit being described, that, if several sets

integrating the two equations, and making suitable substitutions, thence results

$$\left.\begin{aligned} f &= \frac{r'}{r}\psi'(r-a) - 2\mu a\frac{\psi(r-a)}{r^2} \\ f' &= \frac{r}{r'}\psi'(a-r') - 2\mu' a\frac{\psi(a-r')}{r'^2} \end{aligned}\right\}.$$

* See an article by M. Ossian Bonnet among the valuable notes of M. Bertrand's edition of the *Mécanique Analytique*. The principle referred to must be taken with *analytical* latitude, or the range of its application will be unduly restricted. For instance, it is well known and easily demonstrable that a body starting from rest in a position where it is equally drawn by two forces converging to centres attracting according to the law of nature, will oscillate in the arc of an hyperbola. Here the principle seems inapplicable; for the hyperbola will be concave to one focus of attraction and convex to the other, but a curve actually described about either focus would be concave towards it. But in fact the principle does apply; for, analytically speaking, *any conic whatever* may be described about an attractive centre of force varying as the inverse square; only if it be convex to the centre of attraction, its *vis viva* will be a negative quantity, and the motion imaginary. In the case above supposed, the *vis viva* due to each centre of force acting singly will be equal, but with contrary signs, so that the body in such position must be supposed to be at rest; then, by virtue of the principle enunciated, it will for ever continue to move in the hyperbola, in which it would move *really* under the influence of one centre, *imaginarily* under that of the other,—the imaginary motion blended with the real continuous one changing the character of the latter into a reciprocating movement, which is in no way contradictory to M. Ossian Bonnet's theorem, which only determines the *locus*, but not the *direction* of the movement at any point.

† I am informed by the highest authority, the author of Reports on Mechanics, which have become classic, that he has never seen this equation anywhere before employed. It is of course an obvious generalization of Newton's rule, connecting the velocity with that due to a single central force acting through one-fourth of the chord of curvature. As it springs from a combination of the law of *vis viva* with that for centrifugal force, I propose to call it the *Equation of Radial Work*. By aid of it, it is easy to establish the following theorem, giving the most general binary system of forces acting to *two centres*, which will make a body describe *any* given orbit. Call V, V' the respective *force-functions* $\left(\text{so that } \frac{dV}{dr}, \frac{dV'}{dr'} \text{ are the two central forces}\right)$. Call P, P' the squared perpendiculars on the tangent from the two centres respectively, ϕ an *arbitrary* function of any *affection* of the position of the revolving body (for example, of the length of arc or radius of curvature at any point), then

$$V = \frac{\int\phi dP}{P}, \quad V' = \frac{-\int\phi dP'}{P'},$$

will be the general system in question. When the stored-up work for each point in the orbit is known, the *radial equation* gives the central forces without integration. Thus, for example, if a body move in an ellipse with uniform velocity acted on by forces towards the foci, the equation in question shows that they are equal, and vary as the inverse square of the conjugate diameter.

of forces taken separately can make a body describe a certain path, then all the sets acting collectively will make it describe the same path, provided the *vis viva* at the starting-point, on the latter supposition, is the sum of the *vires vivæ* on the former one.

Suppose, now, a zone of matter bounded by two arbitrary contours P, Q to lie anywhere within a circle C, and another zone bounded by two contours P', Q', the geometrical inverses (or reciprocals) of P, Q to lie outside the same. Then these two zones may be divided into corresponding rectangular elements by transversals drawn through the centre of the circle, points being taken all along every radius of one zone, and corresponding points along the radii of the other. If r, r' be the distances of the centres of any two corresponding elements thus obtained from the centre of the circle, $d\theta$ the angle between the two transversals which pass through both pairs of points in both figures, E, E', the areas of the respective elements, will be

$$d\theta \,.\, dr \,.\, r\,;\ d\theta\,(-dr')\,r',\ \text{that is, } -d\theta d\left(\frac{a^2}{r}\right)\frac{a^2}{r}\,;$$

so that

$$\frac{E}{E'} = \frac{r^4}{a^4} = \frac{r^2}{r'^2}.$$

Hence if the densities of E, E' be the same, and they *attract* with forces varying as the inverse fifth power of the distance, they will serve to make a body describe the circle in question, E, E' taking the place of μ, μ' and r, r' of c, γ in our previous formulæ; and as this is true of each pair of elements, it will be true of the two entire zones which they compose, the law of density being perfectly arbitrary, except that it must be the same for *corresponding points* in the interior and exterior zones. The contour Q may be made to coincide with Q' at the circumference of C if we please; and then, as a particular case of the proposition above, we may suppose the united zones to consist of homogeneous matter, or, if we please, of matter whose density at any point is only a function of the angular position of the line joining the point to the centre of the circle. Thus, if we suppose a plate of matter of uniform density and of indefinite extent, and attracting according to the law of the inverse fifth power, a point anywhere placed upon it may be made to move in any desired circle under the influence of the plate's attraction, if we cut away a portion of the plate surrounding the centre of such circle, and leave a proper margin exterior to the circle—the rule being that the intrados of the figure so obtained may be of any form whatever, provided the extrados be its electrical image or inverse. The initial velocity to be communicated to the moving point will of course be determined by the form of either of these bounding curves.

It is hardly necessary to add that instead of a zone we may take a patch of matter bounded by a contour of any form within the circle C, and then, finding the inverse of this contour so as to obtain a corresponding external patch, the two together, by the combined attractions of their particles according to the inverse fifth power of the distance, will serve to make a body describe the circle C; and conversely, since any two circles may be made reciprocals (inverses) to each other by duly determining the centre and radius of the circle of reference, it follows that any two circles of matter attracting according to the above law, will serve to keep a body moving in a certain third circle.

By calculating the attractions of these two circular images, and replacing them by forces tending to their centres, we shall be able to transform and generalize the results previously obtained. But first it will be expedient to recall attention to the form of the single central force which serves to make a body describe a circle. We have found that such force, when the centre lies within the orbit, is of the form $\frac{\mu\rho}{(\rho^2+k)^3}$; and it is easy to see that when external thereto, it takes the form $\frac{\mu\rho}{(\rho^2-k)^3}$, in either case k being the product of the two distances of the force-centre from the extremities of the diameter drawn through it; when the force is external, this product is the square of the tangent drawn to the circle from the centre. At the points of contact the force and velocity both become infinite, and the latter changes its sign.

In a physical sense, only the concave part of the circle will be described by virtue of attraction to the centre, the revolving body going off in a straight line towards the centre* when any point of contact is reached, and in like manner only the convex part by virtue of the repulsive force from the centre, the body going off in a straight line towards infinity on reaching such point; but inasmuch as in either case an infinitesimal deviation from the tangential direction will cause the remainder of the orbit to be described, we may consider, in an *analytical* sense, that the revolving body under the influences of such force describes the entire orbit. We may give the name of *clyclogenous* force to any central force of the form $\frac{-\mu\rho}{(\rho^2\pm k)^3}$, and, if we care to draw the distinction, call it internally cyclogenous or endocyclogenous when the k is positive, and externally cyclogenous or exocyclogenous when k is negative. If we call the cyclogenous-force-function V, so that $\frac{dV}{d\rho}$ is the cyclogenous force itself, we have, by integration, $V=\frac{1}{4}\cdot\frac{\mu}{(\rho^2\pm k)^2}$.

Let us now proceed to calculate the attraction of a circular plate (of radius r) of uniform density, whose particles attract according to the

[* Cf. p. 547, below.]

law of the inverse fifth power of the distance, upon an external particle at the distance ρ from the centre. If we call this $g\frac{dP}{d\rho}$, we have

$$P = \frac{g}{4}\int_0^r dr \int_0^{2\pi} \frac{rdr \,.\, d\theta}{(r^2+\rho^2-2r\rho\cos\theta)^2}.$$

By comparison of $\int_0^{2\pi} \frac{d\theta}{(r^2+\rho^2-2r\rho\cos\theta)^2}$ with the integral which represents twice the area of an ellipse of excentricity, $\frac{2r\rho}{r^2+\rho^2}$, we find instantaneously

$$P = \frac{\pi g}{4}\int_0^r \frac{2(\rho^2+r^2)\,rdr}{(\rho^2-r^2)^3} = \frac{\pi}{4}\frac{gr^2}{(\rho^2-r^2)^2}.$$

Thus P is of the form of the cyclogenous-force-function, so that the *force of attraction to the centre of a circular plate attracting according to the inverse fifth power of the distance, upon an exterior point, is an external cyclogenous force.* From this we may easily draw the conclusion that any circular orbit cutting orthogonally a circular plate whose particles attract according to the inverse fifth power of the distance may be described (or, at all events, the concave part of it be described) by virtue of such force of attraction.

Let us now consider the joint effect of two such circular plates, *images* of one another, lying one entirely within, the other entirely without a given circle. The centres of two such circles, it will be borne in mind, are *not* images of one another. Let r, r' be the radii of the two images, a of the image-making circle; call the distances of the centres of the images from that of the image-making circle c, c' respectively. The points of contact of the images with a common *exterior* tangent will be corresponding points, and this tangent will pass through the centre of the circle of reference; whence we easily derive $(c^2-r^2)(c'^2-r'^2)=a^4$, and by similar triangles

$$\frac{c}{c'}=\frac{r}{r'}.$$

Hence

$$(c^2-r^2)^2=\frac{c^2}{c'^2}a^4.$$

Whence, remembering that r must be less than c, we have

$$c^2-r^2=\frac{c}{c'}a^2, \quad c'^2-r'^2=\frac{c'}{c}a^2;$$

so that

$$r^2=c^2\left(1-\frac{a^2}{cc'}\right); \quad r'^2=c'^2\left(1-\frac{a^2}{cc'}\right)^*.$$

* Calling F, G the two centres, F', G' the images of F, G respectively, O the centre of the image-making circle, it is easily seen that $r^2=FO\,.\,FG'$, $r'^2=GO\,.\,GF'$.

Consequently, calling $1-\frac{a^2}{cc'}=\pm q$, if F, G, two points in the diameter of the image circle, be distant c, c' respectively from its centre, and two cyclogenous forces $\frac{\lambda c^2\rho}{(\rho^2\mp qc^2)^3}$, $\frac{\lambda c'^2\rho'}{(\rho'^2\mp qc^2)^3}$ tend to F and G, two such forces will serve to make a body describe a circle, and, as we shall see, will be statically equivalent to a single cyclogenous force tending to a fixed point, presently to be determined*.

It follows from what has been shown of any two corresponding elements in the two figures, that the total *vis viva* contributed by each at any moment of time, to the entire amount of stand-up work in the revolving body is the same; consequently, confining our attention to one of the image circles, we see that $v^2 \propto \frac{1}{(\rho^2\pm qc^2)^2}$. Hence using u to denote, as before, the angle at the centre, we have

$$\frac{du}{dt}\propto a^2+c^2\pm qc^2-2ac\cos u,$$

which is of the *form* which gives the motion of a planet in eccentric anomaly; consequently, by a proper adjustment of the constants, the motion due to the cyclogenous centres F, G may be made identical with the motion in a circle of radius a with centre at O, about the single cyclogenous-force centre at S. Call ae the distance of S from O, M the absolute force at S, then, comparing the *vis viva* on the two suppositions at the two apsidal points, and again availing ourselves of the law of equal production of *vis viva* from the two force-centres F, G, we obtain

$$\frac{2\lambda c^2}{[(a\mp c)^2+qc^2]^2}=\frac{M}{[(a\mp ac)^2+(a^2-a^2e^2)]^2}.$$

Hence

$$\frac{(a-c)^2+qc^2}{(a+c)^2+qc^2}=\frac{a-ae}{a+ae}$$

or

$$(1+q)\,c^2-\frac{2a}{e}\,c+a^2=0.$$

Calling c, c' the two roots of this equation, we have

$$\frac{1}{c}+\frac{1}{c'}=\frac{1}{2ae};$$

or, which is the same thing, the points O, F, S, G form a system of four points in harmonic relation.

Hence, if we take a system of points, $F, F', F''\ldots G, G', G''\ldots$ in *involution*, the double points of the system being at S and O, the cyclogenous force at S

* The proof of this through the medium of the two circular images requires $-q$ to be employed; but the laws of *analytical* continuity allow q to be made to change its sign.

will be statically equivalent to two cyclogenous forces directed to any two corresponding points F, G.

It is possible that this theorem may be modified so as to admit of further generalization, and be made to extend to an arbitrary system of points in involution, without regard to the condition of O being one of the double points; but I have not had time to consider this point.

In the particular case where F, G become images, in respect to the circular orbit annexed to the force at S, the cyclogenous centres F, G become centres of attraction, following the law of the simple inverse fifth power, as already found. Since in all cases the absolute forces at F, G are proportional to the squares of their distances from O, if we make $cc' = a'^2$, and draw the circle whose centre is at O and radius is a', and take two figures, images of one another in respect to this circle, by the same reasoning as applied to the case of $a' = a$ it may be proved that, provided the densities at corresponding points of such images be the same, and the particles attract according to a certain fixed cyclogenous law, their joint action will support a body in a circular orbit whose radius is a and centre at O. We might again assume two such images to be circular, calculate the law of attraction towards the centres according to the supposed law, and so return to a new system of conjugate points replacing F and G; but I have not had time to ascertain whether such transformation would or would not lead to a new theorem, or merely, as is possible, to a repetition, with a new set of constants, of the one already obtained.

It is hardly necessary to point out how strongly the analogies established in the preceding investigations point to the existence of some simple dynamical theory of the Cartesian ovals under the attraction of forces directed to their foci. The investigation of such theory cannot but tend materially to the elucidation of the essential properties of these most interesting and as yet little-understood curves, the natural parents of the conic sections viewed as focal curves.

In conclusion it may be observed that, in the foregoing paper, it has been seen how a single orbital force passing through a fixed centre may be resolved into others of a more simple form. This suggests a more general subject of investigation, where the force to be so resolved, instead of passing through a fixed point, is tangential, or, better, normal to a fixed curve or surface.

Such an inquiry by no means belongs to ideal mechanics; for it would correspond to the case of the motion under the earth's attraction of a body near the earth's surface, considered as a surface of fluid equilibrium.