81.

ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS OF EQUATIONS.

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In the first part of my "Trilogy of Algebraical Researches," printed in the *Philosophical Transactions*, will be found a proof of Newton's Rule for the discovery of imaginary roots carried as far as equations of the 5th degree inclusive. The method, however, therein employed offered no prospect of success as applied to equations of the higher degrees. I take this opportunity, therefore, of announcing that I have recently hit upon a more refined and subtle method and idea, by means of which the demonstration has been already extended to the 6th degree, and which lends itself with equal readiness to equations of all degrees. Ere long I trust to be able to lay before the Society a complete and universal proof of this rule—so long the wonder and opprobrium of algebraists. For the present I content myself with stating that the new method consists essentially, first, in the discerption of the question as applied to an equation of any specified degree into distinct cases, corresponding to the various combinations of signs that can be attached to the coefficients; secondly, in the application of the fecund principle of variation of constants, laid down in the third part of my "Trilogy," and, in particular, of the theorem that if a rational function of a variable undergoes a continuous variation flowing in one direction through any prescribed channel, then at the moment when it is on the point of losing real roots, not only must it possess two equal roots (a fact familiar to mathematicians as the light of day), but also its second differential, and the variation, when for the variable is substituted the value of such equal roots, must assume the same algebraical sign*. By aid of the processes afforded by this principle, which admits of an infinite variety of modes of application, according to the form imparted to the channel of variation, and constitutes in effect for the examination of algebraical forms an instrument of analysis as powerful as the

* The above is on the supposition that there is no ternary or higher group of equal roots.

microscope for objects of natural history, or the blowpipe for those of chemical research, the problem in view is resolved with a surprising degree of simplicity; so much so that, as far as I have hitherto proceeded with the inquiry, the computations, algebraical and arithmetical, which I have had occasion to employ may be contained within the compass of a single line. The new method, moreover, enjoys the prerogative of yielding a proof of the theorem in the complete form in which it came from the hands of its author (but which has been totally lost sight of by all writers, without exception, who have subsequently handled the question), namely, in combination with, and as supplemental to, the Rule of Descartes. On my mind the internal evidence is now forcible that Newton was in possession of a proof of this theorem (a point which he has left in doubt and which has often been called into question), and that, by singular good fortune, whilst I have been enabled to unriddle the secret which has baffled the efforts of mathematicians to discover during the last two centuries, I have struck into the very path which Newton himself followed to arrive at his conclusions.

Since the above note was sent in to the Society, I have completed the demonstration for the 7th degree, and in the course of the inquiry have had occasion to consider the conditions to be satisfied in order that a rational function of x, with r equal roots a, may undergo no loss of real roots for any assigned variation imparted to the function: for the theory of the 7th degree the case of three equal roots has to be considered, and the conditions in question are that the variation itself may contain the equal root a, and that its first differential coefficient may have the contrary sign to that of the third differential coefficient of the function which it varies when a is substituted for x—a theorem which is, of course, capable of extension to the case of an equation passing through a phase of any number of equal roots*.

* The above is on the supposition that one of the three equal roots remains unaffected in magnitude by the variation, whilst the other two change. If all three are to change simultaneously, infinitesimals beyond the first order and with fractional indices have to be brought into consideration; in that case, on making $x=a$, the variation need not become absolutely zero, but must contain no infinitesimal of the first order. And a further limitation becomes necessary in addition to the conditions stated in the text, in order that no loss of real roots may be incurred in consequence of the variation.