65.

ON THE CENTRE OF GRAVITY OF A TRUNCATED TRIANGULAR PYRAMID, AND ON THE PRINCIPLES OF BARYCENTRIC PERSPECTIVE.

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THERE is a well-known geometrical construction for finding the centre of gravity of a plane quadrilateral, which may be described as follows.

Let the intersection of the two diagonals (say Q) be called the *cross-centre*; the intersection of the lines bisecting the middle points of pairs of opposite sides (say O) the *mid-centre* (which, it may be observed, is the centre of gravity of the four angles viewed as equal weights); then the centre of gravity is in the line joining these two centres produced past the latter (the mid-centre), and at a distance from it equal to one-third of the distance between the two centres; in a word, if G be the centre of gravity of the quadrilateral, QOG will be in a right line, and $OG = \frac{1}{3}QO$.

The frustum of a pyramid is the nearest analogue in space to a quadrilateral *in plano*, since the latter may be regarded as the frustum of a triangle. The analogy, however, is not perfect, inasmuch as a quadrilateral may be regarded as a frustum of either of two triangles, but the pyramid to which a given frustum belongs is determinate. Hence *à priori* reasonable doubts might have been entertained as to the possibility of extending to the pyramidal frustum the geometrical method of centering the plane quadrilateral. The investigation subjoined dispels this doubt, and will be found to lead to the perfect satisfaction, under a somewhat unexpected form, of the hoped-for analogy.

Let abc, $\alpha\beta\gamma$ be the two triangular faces, $a\alpha$, $b\beta$, $c\gamma$ the edges of the quadrilateral faces of a pyramidal frustum. Then this frustum may be resolved in six different ways into the sum total of three pyramids, as shown in the annexed double triad of schemes,

$$\begin{array}{lll} a,\ b,\ c,\ \alpha, & b,\ c,\ a,\ \beta, & c,\ a,\ b,\ \gamma, \\ b,\ c,\ \alpha,\ \beta, & c,\ a,\ \beta,\ \gamma, & a,\ b,\ \gamma,\ \alpha, \\ c,\ \alpha,\ \beta,\ \gamma, & a,\ \beta,\ \gamma,\ \alpha, & b,\ \gamma,\ \alpha,\ \beta, \end{array}$$

$$\begin{array}{lll} b,\ a,\ c,\ \beta, & a,\ c,\ b,\ \alpha, & c,\ b,\ a,\ \gamma, \\ a,\ c,\ \beta,\ \alpha, & c,\ b,\ \alpha,\ \gamma, & b,\ a,\ \gamma,\ \beta, \\ c,\ \beta,\ \alpha,\ \gamma, & b,\ \alpha,\ \gamma,\ \beta, & a,\ \gamma,\ \beta,\ \alpha. \end{array}$$

If, then, taking any one of the above schemes we draw a plane through the centres* of the three pyramids of which it is composed, the six planes thus drawn will meet in a point, which will be the centre of the frustum†.

Let the point in which αa, βb, γc meet when produced be the origin of coordinates, and $bc\beta\gamma$, $ca\gamma\alpha$, $ab\alpha\beta$ be taken as the planes of x, y, z; and let $4a, 0, 0$; $0, 4b, 0$; $0, 0, 4c$ be the coordinates of a, b, c, and $4\alpha, 0, 0$; $0, 4\beta, 0$; $0, 0, 4\gamma$ those of α, β, γ. Consider the first of the schemes above written.

$a+\alpha,$	$b,$	c	will be the coordinates of the centre of			$abc\alpha$,
$\alpha,$	$b+\beta,$	c	"	"	"	$bc\alpha\beta$,
$\alpha,$	$\beta,$	$c+\gamma$	"	"	"	$c\alpha\beta\gamma$;

because, as everyone knows, the centre of a pyramid is the same as that of its angles regarded as of equal weight. But again, if we define as the mid-centre the centre of the six angles of the frustum regarded as of equal weight, its coordinates will be

$$\frac{2a+2\alpha}{3},\quad \frac{2b+2\beta}{3},\quad \frac{2c+2\gamma}{3};$$

and if we substitute for each of the three centres last named points lying respectively in a right line with them and the mid-centre on the opposite side of the mid-centre and at distances from it double those of these centres themselves, these quasi-images of the centres in question will have for their coordinates

$$\begin{array}{ccc} 0\ , & 2\beta, & 2\gamma, \\ 2a, & 0\ , & 2\gamma, \\ 2a, & 2b, & 0\ . \end{array}$$

These points are accordingly the centres of the lines $\beta\gamma$, γa, ab respectively.

And a similar conclusion will apply to each of the six schemes. Hence using in general (p, q) to mean the middle of the line p, q, and by the

* I shall throughout in future for greater brevity hold myself at liberty to use the word *centre* to mean *centre of gravity*.

† I shall hereafter show that these six planes all touch the same cone, of which, as also of its polar reciprocal, I have succeeded in obtaining the equations.

collocation of the symbols for three points understanding the plane passing through them, it is clear

1. That the six planes,

$$(\beta, \gamma);\ (\gamma, a);\ (a, b);\quad (\gamma, \alpha);\ (\alpha, b);\ (b, c);\quad (\alpha, \beta);\ (\beta, c);\ (c, a);$$

$$(\gamma, \beta);\ (\beta, a);\ (a, c);\quad (\alpha, \gamma);\ (\gamma, b);\ (b, a);\quad (\beta, \alpha);\ (\alpha, c);\ (c, b),$$

will meet *in a single point* which may be called the *cross-centre*, being the true analogue of the intersection of the two diagonals of a quadrilateral figure in the plane.

2. That if we join this cross-centre (say Q) with O the mid-centre, and produce QO to G making $OG = \frac{1}{2}QO$, G will be the centre of the frustum $abc\alpha\beta\gamma$.

It may be satisfactory to some of my readers to have a direct verification of the above.

Let, then,

$$A = \frac{a^2bc - \alpha^2\beta\gamma}{abc - \alpha\beta\gamma}, \quad B = \frac{ab^2c - \alpha\beta^2\gamma}{abc - \alpha\beta\gamma}, \quad C = \frac{abc^2 - \alpha\beta\gamma^2}{abc - \alpha\beta\gamma}.$$

A moment's reflection will serve to show that A, B, C are the coordinates of the centre of the frustum.

Again, the first three of the six planes last referred to will be found to have for their equations respectively,

$$\beta\gamma x + \gamma a y + abz = 2a\gamma(b + \beta),$$

$$bcx + \gamma\alpha y + \alpha bz = 2b\alpha(c + \gamma),$$

$$\beta cx + cay + \alpha\beta z = 2c\beta(a + \alpha).$$

The determinant

$$\begin{vmatrix} \beta\gamma, & \gamma a, & ab \\ bc, & \gamma\alpha, & \alpha b \\ \beta c, & ca, & \alpha\beta \end{vmatrix} = (abc - \alpha\beta\gamma)^2.$$

The determinant

$$\begin{vmatrix} \gamma a, & ab, & 2a\gamma(b+\beta) \\ \gamma\alpha, & \alpha b, & 2b\alpha(c+\gamma) \\ ca, & \alpha\beta, & 2c\beta(a+\alpha) \end{vmatrix}$$

$$= 2a\alpha(bc - \beta\gamma)(abc - \alpha\beta\gamma),$$

$$= 2\{(\alpha^2\beta\gamma - a^2bc)(abc - \alpha\beta\gamma) + (a + \alpha)(abc - \alpha\beta\gamma)^2\}.$$

Hence if x, y, z be the coordinates of the intersection of the above-mentioned three planes,

$$x = -2A + 2(a + \alpha),$$

$$y = -2B + 2(b + \beta),$$

$$z = -2C + 2(c + \gamma);$$

and the same will evidently be true of the other ternary system of planes; so that all six planes intersect in a single point Q, of which x, y, z above written are the coordinates. And the coordinates of O being

$$\frac{2a+2\alpha}{3}, \quad \frac{2b+2\beta}{3}, \quad \frac{2c+2\gamma}{3},$$

and those of G being

$$A, \quad B, \quad C,$$

it is obvious QOG is a right line, and $OG = \frac{1}{2}QO$, as was to be shown.

The analogy with the quadrilateral does not end here. There is a construction* for the centre of a quadrilateral still easier than that above cited, which may be expressed in general terms by aid of a simple definition. Agree to understand by the *opposite* to a point L on *a limited line* AB a point M, such that L and M are at equal distances from the *centre* of AB but on opposite sides of it; then we may affirm that the centre of a quadrilateral is the centre of the triangle whose apices are the intersection of its two diagonals (that is, the cross-centre), and the *opposites* of that intersection on those two diagonals respectively. So now if we agree to understand by opposite points on a limited triangle two points in a line with the centre of the triangle and at equal distances from it on opposite sides, and bear in mind that the cross-centre of a pyramidal frustum is the intersection of *either* of two distinct ternary systems of triangles which may be called the two systems of cross-triangles†, we may affirm that the centre of a pyramidal frustum is the centre of a pyramid whose apices are its cross-centre, and the opposites of that centre on the three components of either of its systems of cross-planes. This is easily seen; for if we take the first of the two systems, their respective centres will evidently be

$$\begin{array}{ccc} \dfrac{4a}{3}, & \dfrac{2b+2\beta}{3}, & \dfrac{4\gamma}{3}, \\[2ex] \dfrac{4\alpha}{3}, & \dfrac{4b}{3}, & \dfrac{2c+2\gamma}{3}, \\[2ex] \dfrac{2a+2\alpha}{3}, & \dfrac{4\beta}{3}, & \dfrac{4c}{3}. \end{array}$$

* This is the mode of statement (except that the important notion of opposite points was not explicitly contained in it) which, accidentally meeting my eye in a proof sheet of some Geometrical Notes (by an anonymous author) intended for insertion in the forthcoming (if not forthcome) Number of the *Quarterly Journal of Mathematics*, led to the long train of reflections embodied in this paper, which but for that casual glance would never have seen the light. The same construction, under another and somewhat less eligible form, is given in the *Mathematician* (a periodical now extinct, edited by Dr Rutherford and Mr Fenwick, both of the Royal Military Academy), 1847, Vol. II. p. 292, and is therein stated by the latter gentleman to have, "as he believes, first appeared in the *Mechanics' Magazine*, and subsequently in the *Lady's Diary* for 1830."

† From the description given previously, it will be seen that *a cross-triangle* of the frustum is one which has its apices at the centres of either diagonal of any quadrilateral face and of the two edges coterminous but not in the same face with that diagonal.

Thus the three opposites to the cross-centre whose coordinates are

$$-2A+2(a+\alpha),\quad -2B+2(b+\beta),\quad -2C+2(c+\gamma),$$

will have for their x coordinates

$$\frac{2a}{3}-2\alpha+2A,$$

$$-2a+\frac{2\alpha}{3}+2A,$$

$$-\frac{2a}{3}-\frac{2\alpha}{3}+2A;$$

for their y coordinates

$$\frac{2b}{3}-2\beta+2B,$$

$$-2b+\frac{2\beta}{3}+2B,$$

$$-\frac{2b}{3}-\frac{2\beta}{3}+2B;$$

and for their z coordinates

$$\frac{2c}{3}-2\gamma+2C,$$

$$-2c+\frac{2\gamma}{3}+2C,$$

$$-\frac{2c}{3}-\frac{2\gamma}{3}+2C;$$

and consequently the centre of the pyramid whose apices are the cross-centre and its three opposites will be A, B, C, that is, will be the centre of gravity of the frustum, as was to be shown*.

* I at one time supposed that a, b, c; α, β, γ formed two systems of *diagonal* planes, and that there were thus two cross-centres; and dreamed a dream of the construction for the centre of gravity of the pyramidal frustum based upon this analogy, inserted (it is true as a conjecture only) in the *Quarterly Journal of Mathematics*; but the nature of things is ever more wonderful than the imagination of men's minds, and her secrets may be won, but cannot be snatched from her. Who could have imagined *à priori* that for the purposes of this theory a diagonal of a quadrilateral was to be viewed as a line drawn through two opposite angles of the figure regarded, not as themselves, but as their *own centres of gravity*! Some of my readers may remember a signal case of a similar autometamorphism which occurred to myself in an algebraical inquiry, in which I was enabled to construct the canonical form of a six-degreed binary quantic from an analogy based on the same for a four-degreed one, by considering the *square* of a certain function which occurs in the known form as consisting of two factors, one the function itself, the other a function morphologically derived from, but happening for that particular case to coincide with the function. This parallelism is rendered more striking from the fact of 4 and 6 being the *numbers* concerned in each system of analogies, those numbers referring to degrees in the one theory and to angular points in the other. It is far from improbable that they have their origin in some common principle, and that so in like manner the parallelism will be found

It is clear that these results may be extended to space of higher dimensions. Thus in the corresponding figure in space of four dimensions bounded by the hyperplanar quadrilaterals $abcd$, $\alpha\beta\gamma\delta$, which will admit of being divided into four hyperpyramids in twenty-four different ways, all corresponding to the type

$$\begin{array}{ccccc} a, & b, & c, & d, & \alpha, \\ b, & c, & d, & \alpha, & \beta, \\ c, & d, & \alpha, & \beta, & \gamma, \\ d, & \alpha, & \beta, & \gamma, & \delta, \end{array}$$

there will be a cross-centre given by the intersection of any four out of twenty-four hyperplanes resoluble into six sets of four each,—one such set of four being given in the scheme subjoined, where in general pqr means the point which is the centre of (p, q, r) and the collocation of four points means the hyperplane passing through them, namely,

$$\begin{array}{cccc} \beta\gamma\delta, & \gamma\delta a, & \delta ab, & abc, \\ \gamma\delta\alpha, & \delta\alpha b, & \alpha bc, & bc\alpha, \\ \delta\alpha\beta, & \alpha\beta c, & \beta cd, & cdb, \\ \alpha\beta\gamma, & \beta\gamma d, & \gamma da, & dac. \end{array}$$

The mid-centre will mean the centre of the eight angles $a, b, c, d, \alpha, \beta, \gamma, \delta$, regarded as of equal weight; and to find the centre of the hyperpyramidal frustum, we may either produce the line joining the cross-centre with the mid-centre through the latter and measure off three-fifths of the distance of the joining line on the part produced (as in the preceding cases we measured off two-fourths and one-third of the analogous distance), or we may take the four opposites of the cross-centre on the four components of any one of the six systems of hyperplanar tetrahedrons of which it is the intersection, and find the centre of the hyperpyramid so formed. The point determined by either construction will be the centre of gravity of the hyperpyramidal frustum in question. And so on for space of any number of dimensions. It will of course be seen that a general theorem of determinants* is contained

to extend in general to any quantic of the degree $2n$, and the corresponding barycentric theory of the figure with $2n$ apices (n of them in one hyperplane and n in another), which is the problem of a hyperpyramid in space of n dimensions. The probability of this being so is heightened by the fact of the barycentric theory admitting, as is hereafter shown, of a *descriptive* generalization, descriptive properties being (as is well known) in the closest connexion with the theory of invariants. Much remains to be done in fixing the canonic forms of the higher even-degreed quantics; and this part of their theory may hereafter be found to draw important suggestions from the hyper-geometry above referred to, if the supposed alliance have a foundation in fact.

* We learn indirectly from this how to represent under the form of determinants of the ith order, and that in a certain number of different ways, the general expressions

$$(l_1 l_2 \ldots l_i - \lambda_1 \lambda_2 \ldots \lambda_i)^{i-1}$$

and
$$l_1\lambda_1 (l_2 l_3 \ldots l_i - \lambda_2\lambda_3 \ldots \lambda_i)(l_1 l_2 \ldots l_i - \lambda_1\lambda_2 \ldots \lambda_i)^{i-2},$$

in the assertion that for space of n dimensions there will be $n!$ quasi-planes all intersecting in the same point, as also in the general relation connecting this point (the cross-centre) with the mid-centre and centre of gravity, of each of which it is easy to assign the value of the coordinates in the general case.

But returning to the case of the ordinary pyramidal frustum, the preceding results lead at once to an easy geometrical proof of the well-known analytical formula for finding the centre of gravity of a pyramidal frustum in the case where the base and its opposite plane are parallel.

As we know that the centre of gravity in this case is in the line joining the centres of the opposite faces, what is wanted here is merely the proportion of the segments into which this joining line is divided at the centre in question, or, in other words, the ratio to each other of the distances of the centre from the parallel faces.

Let $$ab : \alpha\beta = bc : \beta\gamma = ca : \gamma\alpha = l : \lambda.$$

Then obviously

$$\text{vol. } abc\alpha : \text{vol. } bc\alpha\beta = ab\alpha : b\alpha\beta = l : \lambda,$$

$$\text{vol. } bc\alpha\beta : \text{vol. } c\alpha\beta\gamma = bc\alpha : c\alpha\gamma = l : \lambda :$$

hence $$abc\alpha : bc\alpha\beta : c\alpha\beta\gamma = l^2 : l\lambda : \lambda^2;$$

also if h be the distance between abc, $\alpha\beta\gamma$, the distances of the centres of $abc\alpha$, $bc\alpha\beta$, $c\alpha\beta\gamma$ respectively from abc will be $\frac{h}{4}, \frac{h}{2}, \frac{3h}{4}$.

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a strange conclusion to be able to draw incidentally from a hyper-theory of centre of gravity! Thus, for example, on taking $i=4$, we shall find

$$\begin{vmatrix} bcd, & cda, & da\beta, & \alpha\beta\gamma \\ \beta\gamma\delta, & cda, & da\beta, & \alpha\beta\gamma \\ b\gamma\delta, & \gamma\delta a, & dab, & ab\gamma \\ bc\delta, & c\delta a, & \delta a\beta, & abc \end{vmatrix} = (abcd - \alpha\beta\gamma\delta)^3.$$

And again,

$$\begin{vmatrix} ad\,(bc + c\beta + \beta\gamma), & cda, & da\beta, & \alpha\beta\gamma \\ \beta a\,(cd + d\gamma + \gamma\delta), & cda, & da\beta, & \alpha\beta\gamma \\ \gamma b\,(da + a\delta + \delta a), & \gamma\delta a, & dab, & ab\gamma \\ \delta c\,(ab + ba + a\beta), & c\delta a, & \delta a\beta, & abc \end{vmatrix} = aa\,(bcd - \beta\gamma\delta)\,(abcd - \alpha\beta\gamma\delta)^2.$$

The number of these representations will not be twenty-four, that is, $4!$, but only twelve, the half of that number, because it will easily be seen that the cycles $abcd$, $\alpha\beta\gamma\delta$ will lead to the same determinants, only differently arranged, as the cycles $bcda$, $\beta\gamma\delta a$. I believe the law is, that the number of varieties of such representations is $(i)!$, or $\frac{1}{2}(i)!$, according as i is odd or even. The expression $ab - \alpha\beta$ at once conjures up the idea of a determinant. We now see that there is an equally natural determinantive representation, or system of representations, of $(abc - \alpha\beta\gamma)^2$, $(abcd - \alpha\beta\gamma\delta)^3$, &c.

Hence the distance of the centre of the frustum from abc will be $\frac{h}{4}\left(\frac{l^2+2l\lambda+3\lambda^2}{l^2+l\lambda+\lambda^2}\right)$, and so from $\alpha\beta\gamma$ it will be $\frac{h}{4}\left(\frac{\lambda^2+2l\lambda+3l^2}{l^2+l\lambda+\lambda^2}\right)$, agreeing with the well-known formula applicable to this case*.

But I pass on to a subject of much deeper interest.

The geometrical constructions included in the preceding inquiry (such for instance as depend on the properties of *centres* and *opposites*), like those which occur in the more ordinary theory of the triangle and pyramid, at once suggest the existence of descriptive propositions in which harmonic centres and harmonic opposites, and in general harmonic multiplications and divisions, take the place of the corresponding arithmetical operations.

To make my meaning perfectly clear, let us conceive a fixed plane; and by a harmonic succession of points $A, B, C, D \ldots$ in a line meeting the fixed plane† (which we may term the plane of relation) in O, let us understand that $ABCO$, $BCDO$, &c. form so many harmonic systems of points; B may be then called a harmonic centre of AC, A and C opposites to B; also we may call AB, BC harmonic steps of the succession, so that by multiplying a line AB n times, or making AX equal to n times AB, we are constructing the point X to which A will be transferred by n harmonic steps, of which AB is the first; and by n-secting a line AX, we mean finding a point B in it such that a succession of n harmonic steps, commencing with AB, will carry A to X.

In all this there is of course nothing new: these principles are familiar to all geometers, and have received their fullest development at the hands of Professor Cayley. We know *à priori* that the descriptive properties included in the preceding (or similar) constructions, such, for example, as that the six cross-triangles of a frustum all meet in a point, will remain true when, adopting a fixed plane of relation, we substitute harmonic centres in respect to that plane in lieu of arithmetical centres‡. Or, again, we may affirm that

* If we agree to denote by a, b, c; α, β, γ, the planes $\alpha\beta\gamma$, $b\gamma\alpha$, $c\alpha\beta$; abc, βca, γab respectively, it may easily be shown that each quaternary system of planes a, b, α, β; b, c, β, γ; c, a, γ, α passes through a single point; we have thus given three points which determine a plane; the intersection of this plane with the line a, b, c; α, β, γ is a sort of centre to the frustum, and must possess properties deserving closer investigation.

† It will of course be understood that in dealing with figures lying in the same plane, a line of relation (namely, the intersection of the plane of relation with the plane of the figures) may be substituted instead of the former plane, since the distances from the one and the other are in an invariable ratio; and so for different segments in a right line, we may substitute a point of relation on the line itself instead of the plane. I deal with a plane of relation as comprising implicitly all the subordinate cases; were it required to go out into space of four or a higher number of dimensions, it would of course become necessary to deal with hyper-planes of relation.

‡ Geometers have long been familiar with the idea of the *pole* or *harmonic centre* of a triangle in respect to a line in its plane; the principles now about to be developed will enable us to attach a precise signification to the *pole* or *harmonic centre* of every geometrical figure of any form whatever.

the lines joining the harmonic centres of the opposite edges of a tetrahedron will all *intersect* and *harmonically* bisect each other, and so on. But what is further wanted, and what I will proceed to supply, is a firm quantitative basis to this enlarged theory, so formed as that we shall be able in the general case to follow step by step the reasoning used in the common theory where the plane of relation goes off to infinity, and to assign to every point determined in the general constructions as distinctive a character as it possesses in the special ones. This may be done by the aid of very elementary considerations, which I proceed to unfold, and which will be seen at once to bring the general or perspective theory under the dominion of the so-called integral calculus or calculus of continuity.

The arithmetical centre of two *points* A, B is the centre of gravity of two equal atoms at A and B; let us then so assign the weights of the atoms A, B in the general case as to make their centre of gravity fall on the harmonic centre: this may evidently be done by considering their weights as proportional to their inverse distances from the plane of relation, and accordingly we shall understand by the weight of an atom at any point a quantity proportional to its inverse distance from the plane of relation. But, moreover, the centre of gravity of the homogeneous line AB ought to fall at this same point, which we may if we please consider as an inference at the limit from the same thing being true for equal atoms at distances dividing the line into any even number of equal parts. Hence in the general analogical theory we must take the infinitesimal intervals of our atoms at points in *harmonic* succession.

Let P, Q, R be any three such points, and let x, $x + dx$, $x + 2dx + d^2x$ be their respective distances from the plane of relation; and let q be the *frequency* at P, that is a quantity proportional to the number of atoms which occur in a given infinitesimal space about P; then evidently qdx is constant, and $qd^2x + dxdq = 0$; but by virtue of the harmonic relation between P, Q, R, we have

$$(x + 2dx + d^2x)(dx) = x(dx + d^2x),$$

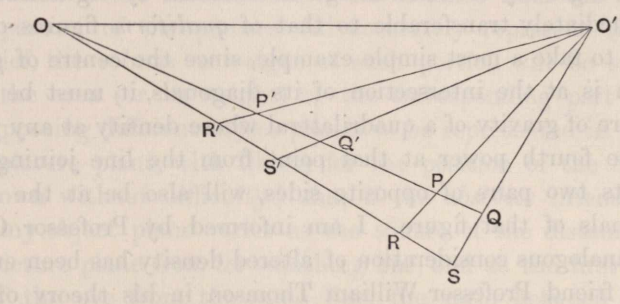
or

$$xd^2x = 2(dx)^2, \text{ or } -\frac{dq}{q} = 2\frac{dx}{x},$$

that is q varies as $\dfrac{1}{x^2}$.

Moreover the weight of each atom varies as $\dfrac{1}{x}$, hence the *density* of any element in a line must be taken to vary as the inverse cube of its distance from the plane of relation.

Let us now endeavour to obtain the law of density for any element of a *plane*. Let O, O' be any two points in the line in which the plane in question meets the plane of relation, and let the plane be divided into infinitesimal elements similar to $PQSR$ in the figure by pencils whose rays are in harmonic succession proceeding from O and O'; then one atom belongs to



every such element, which will be the analogue of a rectangular element in the common theory; but the area of this element, as compared with any similar element, say $P'Q'S'R'$ in the infinite sector QOS, varies as

$$OP \cdot RS + OR \cdot PQ,$$

where PQ, RS, by what has been last shown, vary as the square of the distance of the element from the plane of relation, and OP, OR vary directly as the distance; hence the *frequency* of the atoms at any element in *either sector* will vary as the inverse cube of its distance from the plane of relation, and hence this will be the law of frequency for elements all over the plane, and is irrespective of the particular positions of O, O'; and consequently, the density being proportional to the product of the frequency of the atoms by their atomic weights, the law of density is that it varies about any point as the inverse *fourth* power of its distance from the plane of relation. In like manner, by taking three points O, O', O'' in the plane of relation and dividing space into solid elements by plane bundles passing through OO', OO'', $O'O''$ respectively, it may be proved that the law of density for a solid figure will be that it varies as the inverse *fifth* power of the distance from the plane of relation*.

Atoms whose weights vary inversely as their distances from the plane of relation may be termed *like* atoms; lines, areas, and solids whose elements vary in *density* inversely as the cubes, fourth powers and fifth powers respectively, may be termed *qualiform* figures, or figures of *qualiform* density, the terms *like* and *qualiform* being adopted as the closest analogues to *equal* and *uniform*. It now becomes true, and may easily be verified, that

* The law of density for a solid is the inverse fifth power, for an area the inverse fourth power, and for a line the inverse third power. Here we must stop, for a *point* is that which has no parts: we can speak of the law of atomic *weights* at a point, but not of density, for the latter implies the existence of *elements* which are wanting to the point. In a hyper-ontological sense there would be no objection to saying that for an element of a point the law of density in this theory is as the inverse square, always remembering that no such element exists.

the centres of gravity of a *qualiform* finite line, triangle, and tetrahedron are respectively identical with the centres of gravity of *like* atoms placed at their apices*; and so every known or discoverable theorem whatever relating to the centre of gravity of uniform figures bounded by right lines or planes becomes immediately transferable to that of *qualiform* figures of the same kind. Thus, to take a most simple example, since the centre of gravity of a parallelogram is at the intersection of its diagonals, it must be and is true that the centre of gravity of a quadrilateral whose density at any point varies as the inverse fourth power at that point from the line joining the intersections of its two pairs of opposite sides, will also be at the intersection of the diagonals of that figure. I am informed by Professor Cayley that a somewhat analogous consideration of altered density has been employed by our eminent friend Professor William Thomson in his theory of images, in reference to the distribution of electricity, given in *Liouville's Journal*.

* As regards the finite line, these results may be very easily verified by the integral calculus. For the triangle, it may be made to depend on the preceding case by drawing from the point where the direction of any side intersects the plane of relation, rays dividing the triangle into infinitesimal portions; the centre of gravity of every one such portion will easily be seen to be in the right line joining the harmonic centre of the intersecting side with the opposite angle; and an analogous method applies to the tetrahedron.

The same results may also be obtained analytically. Thus, for example, for a qualiform triangle whose apices are distant h, k, l from the opposite sides, and $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ from the plane of relation, the distances of the centre of gravity from the respective sides will be

$$\frac{h\alpha}{\alpha+\beta+\gamma}, \quad \frac{k\beta}{\alpha+\beta+\gamma}, \quad \frac{l\gamma}{\alpha+\beta+\gamma}.$$

The masses, say M, of a qualiform line, triangle, or tetrahedron, using α, β; α, β, γ; $\alpha, \beta, \gamma, \delta$ for the inverse distances of the apices from the plane of relation, and V for the length, area, or volume, in the three cases respectively become expressible under the very noticeable forms

$$\frac{\alpha+\beta}{2}\alpha\beta V, \quad \frac{\alpha+\beta+\gamma}{3}\alpha\beta\gamma V, \quad \frac{\alpha+\beta+\gamma+\delta}{4}\alpha\beta\gamma\delta V,$$

their moments in respect to the plane of relation being respectively

$$\alpha\beta V, \quad \alpha\beta\gamma V, \quad \alpha\beta\gamma\delta V;$$

so that the *mean* density $\frac{M}{V}$ is in each case a simple symmetric function of the atomic weights of the apices (it being of course understood that the *absolute* atomic weight and frequency are each taken as unity). As the same figure may be variously partitioned, and the sum of the component areas and of their moments is unaffected by the mode of partition, the preceding formulæ obviously give rise to, or imply the existence of, a class of purely geometrical theorems relating to systems of points. It may be here observed that the moment of a qualiform figure in respect to its plane of relation represents the size, so to say, of (that is, the number of atoms contained in) the single molecule which, placed at the centre of gravity, will be the statical equivalent of such figure; for if n be this number, and d the distance of the centre from the plane of relation, and w the weight of the figure, since the atomic weight is $\frac{1}{d}$, we must have $\frac{n}{d}=w$, or

$$n = dw = \text{moment of } w \text{ in respect to the plane of relation.}$$

So in like manner, wherever the plane of relation is situated, two molecules A and B, placed at two *points*, will be equivalent to the molecule $A+B$ placed at their centre of gravity.

It is an easy inference* from what has been established concerning the law of *frequency*, that if in the perspective of any plane figure, by tinting or relief, we express the degree of crowding of any element, and proportion the tint or elevation to the inverse *cube* of its distance from the vanishing line, then any portion of the picture will accurately represent (and indeed if we use relief, the *volume* or weight of such portion will be strictly proportional to) the area (or its weight) of the corresponding part in the object plane. Supposing different object planes to be represented in perspective on the same picture plane, with liberty for the position of the eye to vary, it may be shown without difficulty† that if the *absolute* intensity of tint or relief for any object plane varies as the square of the distance of its trace upon the picture plane from its vanishing line, and as the first power of the distance of the eye from the same line, the ratio between corresponding portions of. object and picture will be alike for every plane.

In the corresponding problem for right lines, the relief or tint of any element in the perspective of a given right line must vary as the inverse square of the distance from the vanishing point, and the absolute intensity for different lines must vary as the product of the distance between the trace and the vanishing point into the distance of the eye from that point. In *barycentric* perspective we have seen the further consideration of atomic weight enters, so that the density follows the law of the inverse fourth and third powers for planes and lines respectively, instead of third and second powers as in geometrical perspective; in fact in the geometrical theory the quantities visibly represented correspond to the *moments*‡ in respect to

* It may here also incidentally be noticed that the area of the primitive of any perspective projection of a figure in a given plane is proportional to the *attraction* exercised upon it by the object plane indefinitely extended, the force of attraction between any two elements being supposed to vary inversely as the fifth power of the distance.

† For if we take T the trace of an object line, V its vanishing point, and through O (the eye) draw OPp meeting TV in P and the object line in p, Tp the quantity of $TP = \frac{\mu TP}{TV \, . \, PV}$, so that $\mu = TV \frac{Tp}{TP} PV = TV \, . \, OV$; and again, if tTt' be the trace of an object plane, V the foot of the perpendicular from O on the vanishing line VT perpendicular to tTt', P a point in VT, and p the point where OP meets the object plane, we have tpt' (the quantity of tPt') $= \mu \frac{tPt'}{TV \, . \, TV \, . \, PV}$, or

$$\mu = TV^2 \, . \, \frac{tpt'}{tPt'} \, . \, PV = TV^2 \, . \, \frac{Tp}{TP} \, . \, PV = TV^2 \, . \, OV.$$

The preceding calculations assume the expressions $\mu\alpha\beta$, $\mu\alpha\beta\gamma$ applicable to a linear and triangular space, given in a preceding footnote.

‡ And consequently if, in the pictorial representation of any plane surface, there is taken a triangular patch of given area, the quantity in the object corresponding thereto will vary inversely as the product of the distances of the three angles of the patch from the vanishing line,—a proposition in perspective which I imagine to be new, and at all events is certainly little known. This may be applied to determine instantaneously the area of an ellipse of which the perspective projection is a circle of radius r, and whose centre is at the distance h from the vanishing line. Writing μ equal to the distance of the vanishing line from the eye, multiplied by the square of its

the vanishing line of the quantities visibly represented in the barycentric theory *.

I have termed this a theory of barycentric perspective, because it includes a method whereby the centre of gravity of a plane figure is retained in perspective with the centre of gravity of its projection ; by what has pre-

distance from the trace of the ellipse upon the plane of the circle, the area of the ellipse (regarded as made up of infinitesimal sectors with the centre of the projection for their common vertex) becomes

$$\int_0^{2\pi} d\theta \frac{\frac{1}{2}\mu r^2}{h(h - r\sin\theta)^2} = \frac{\mu\pi r^2}{h^3\left[1 - \left(\frac{r}{h}\right)^2\right]^{\frac{3}{2}}} = \frac{\mu\pi r^2}{(h^2 - r^2)^{\frac{3}{2}}};$$

so that the area of any ellipse in a *given* plane, the perspective representation of which ellipse is a circle, will vary directly as the area of the circle, and inversely as the cube of the tangent drawn to meet it from the orthogonal projection of its centre on the vanishing line. More generally, if the figure in the plane of projection be an ellipse with semiaxes a, b, eccentricity e, inclination of minor axis to vanishing line α, and distance of one of its foci from that line h, then calling V the area of the primitive and μ the absolute ratio between a primitive element and its projection, we shall have

$$V = \frac{\mu}{2h}\int_0^{2\pi} d\theta \frac{r^2}{(h - r\sin\theta)^2}, \text{ where } r = \frac{a(1-e^2)}{1 + e\sin\theta}.$$

This integration may be performed with extreme facility, and gives

$$V = \mu\pi ab\,[h^2 + 2hea\cos\alpha - a^2(1 - e^2)]^{-\frac{3}{2}},$$

say

$$\frac{\mu}{D^3}\pi ab,$$

where to find D we may use the following construction:—*Draw a circle in the plane of, and concentric with, the projection, and such that a common tangent to the two shall be parallel to the vanishing line, and from the foot of the perpendicular upon that line from the centre draw a tangent to the circle, the length of the tangent so drawn will be D*; so that the area of any ellipse will be to the area of its perspective projection as the product of the square of the distance of the trace into that of the eye from the vanishing line is to the cube of the tangent just described,—a very remarkable proposition in perspective, if new. By varying the origin of our polar coordinates, as by taking it, for instance, at the centre of the projection or any other point, we may obtain a new class of definite integrals of known values, and which it might be exceedingly difficult to determine by any direct method. It may be added that all ellipses in the same plane will bear a constant ratio to their projections if these latter have a common tangent parallel to the vanishing line, and their centres be in another line also parallel to the same.

* The above statements, combined with the varying law of *frequency*, amount to the following propositions in perspective:—

1. If O be a *linear* element, P its perspective representation, H, h the distances of the eye and P from the line of O, and d of the eye from the line of P, then

$$O : P :: dH : (H - h)^2.$$

2. If O be a *plane* element, P its perspective, H, h the distances of the eye and P from the plane of O, and d the distance of the eye from the plane of P, then

$$O : P :: dH^2 : (H - h)^3.$$

These formulæ would become necessary in applying (as might be done perhaps advantageously) in some cases the integral calculus to the quantification of curved lines and surfaces by a perspective method more general than the one in ordinary use, which is essentially a method of orthogonal projection.

ceded, it appears that this may be effected by regarding its projection, not as of uniform density, but of a density following the law of the inverse cube of the distance. From this it follows that the distance of the perspective position in the picture of the centre of gravity of the primitive from the vanishing line becomes immediately known by a process of differentiation when the area of the primitive is expressed as a function of the distance of any arbitrarily fixed point in the plane of projection from the vanishing line. For if this area, which is the moment of the qualiform projection in respect to the vanishing line, be called M, and the mass of the same be termed Q, and if h, d be the distances of the origin and of the centre of gravity from the vanishing line, we have $d = \frac{M}{Q}$, where

$$M = \mu \int_0^{2\pi} \frac{r^2 d\theta}{(h - r\sin\theta)^2 h},$$

$$Q = \frac{1}{3}\mu \int_0^{2\pi} \frac{r^2 d\theta}{(h - r\sin\theta)^2 h}\left(\frac{1}{h - r\sin\theta} + \frac{1}{h - r\sin\theta} + \frac{1}{h}\right);$$

hence $\quad Q = -\frac{1}{3}\frac{dM}{dh},$

and $\quad d = \dfrac{\frac{1}{3}M}{\dfrac{dM}{dh}}.$

Thus, for example, if we wish to find the perspective position of the centre of gravity of the primitive of a given elliptic projection, we have found in a preceding footnote,

$$M = \mu\,(h^2 + 2hae\cos\alpha + a^2e^2 - a^2)^{-\frac{3}{2}};$$

hence
$$d = \frac{h^2 + 2hae\cos\theta + a^2e^2 - a^2}{h + ae\cos\alpha};$$

or, calling R the radius of the circle concentric with the given projection, and having with it a common tangent parallel to the vanishing line, and H the distance of the centre of this circle from that line, $d = \frac{H^2 - R^2}{H}$, an equation the geometrical interpretation whereof is readily obtained.

More generally, if we take $x\cos\alpha + y\sin\alpha - h = 0$ as the equation to the vanishing line, using, as before, M to denote the moment of the qualiform projection in respect to that line (well worthy in this theory of being termed the principal moment), or, which is the same thing, the area of the primitive, and take M_x for the moment of the same in respect to the axis of y, we shall have

$$M = \iint \frac{dx\,dy}{(x\cos\alpha + y\sin\alpha - h)^3},$$

$$M_x = \iint \frac{dx\,dy\,x}{(x\cos\alpha + y\sin\alpha - h)^4},$$

from which it is easy to deduce

$$M_x = \cos\alpha\left(M + \frac{1}{3}h\frac{d}{dh}M\right) + \frac{1}{3}\sin\alpha\frac{d}{d\alpha}M;$$

and consequently $\dfrac{M_x}{Q} - h\cos\alpha$, which is the distance of the perspective of the centre of gravity of the primitive in the direction of x from the foot of the perpendicular from the assumed origin upon the vanishing line, will be

$$\frac{3\cos\alpha \,.\, M + \sin\alpha\dfrac{dM}{d\alpha}}{\dfrac{dM}{dh}}.$$

And thus we are led to the remarkable proposition, that when we know the area of the primitive in terms of the parameters of its vanishing line, we can completely determine the perspective position of its centre of gravity by means of processes of differentiation only; so that a method closely akin to (if not identical with) that of potentials in the theory of attraction has a necessary place also in the theory of perspective.

If, as is most convenient, we fix the perspective of the centre of gravity of the object figure by its distance from the vanishing line and its distance from the line through the origin perpendicular to the vanishing line, we see, by making α successively zero and $\frac{1}{2}\pi$ in the above formula, that these distances are $\dfrac{3M}{\dfrac{d}{dh}M}$ and $\dfrac{\dfrac{d}{d\alpha}M}{\dfrac{d}{dh}M}$ respectively*. Analogous results may be obtained for

* In the case of the ellipse, we have found in a preceding footnote,

$$M = \mu\,(h^2 + 2aeh\cos\alpha + a^2e^2 - a^2)^{\frac{3}{2}},$$

so that

$$\frac{3M}{\dfrac{d}{dh}M} = \frac{h^2 + 2eah\cos\alpha + a^2e^2 - a^2}{h + ea\cos\alpha} = y,$$

$$\frac{\dfrac{dM}{d\alpha}}{\dfrac{dM}{dh}} = \frac{ea\sin ah}{h + ea\cos\alpha} = x,$$

where y and x are the coordinates of the point referred to in the text, if we take the vanishing line and a line perpendicular thereto from the focus for the axes of x and y. Consequently, if we remove the origin of coordinates to the centre of the ellipse, preserving the directions of the axes, and call x', y' the new coordinates, we shall have

$$x' = ae\sin\alpha - x = \frac{a^2e^2\sin\alpha\cos\alpha}{h + ae\cos\alpha},$$

$$y' = h + ae\cos\alpha - y = \frac{a^2[1 - e^2(\sin\alpha)^2]}{h + ae\cos\alpha},$$

$$\frac{y'}{x'} = \frac{1 - e^2(\sin\alpha)^2}{e^2\sin\alpha\cos\alpha},$$

solid figures, substituting the more general notion of homography for that of perspective, as will more fully appear in the sequel.

Remembering that M is the area of the primitive plane object, it seems to result as an indirect inference from the preceding theory, that whenever we can determine the area of an oval section (whether the bounding curve be the whole or a part of the curve of section) of an algebraical cone, then we can determine the position of the centre of gravity of that oval in its own plane by processes of differentiation only; and, *mutatis mutandis*, the same conclusion will admit of extension to solids bounded by algebraical surfaces; so that $\iint dx\,dy$ or $\iiint dx\,dy\,dz$ being given, subject to certain conditions of limit, $\iint (ax+by)\,dx\,dy$, $\iiint (ax+by+cz)\,dx\,dy\,dz$, subject to the same conditions, become known by algebraical and differentiation processes only, and so obviously for any number of variables*.

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which may easily be shown to be the equation to the diameter drawn to the point of the ellipse where the tangent is parallel to the vanishing line; and consequently the perspective of the centre of gravity of the original lies in this diameter, as evidently it ought to do, since every infinitesimal slice of the *qualiform* area contained between parallels to the vanishing line is of uniform density throughout, and is bisected by the diameter conjugate to the direction of that line.

* The inference made hesitatingly in the text, upon further reflection appears to me perfectly clear, and will become so, I think, to the reader with the aid of a few words of explanation.

Let Q be a closed curve of the kind supposed lying in a plane which will be treated as a constant plane of projection; and for greater simplicity, and in order to steady the ideas, imagine that the vanishing plane (meaning thereby the plane passing through the eye and the vanishing line), and the plane of the object to be put in perspective, are retained at a constant distance from each other and always perpendicular to the picture plane, and also that the height of the eye above the vanishing line is invariable. Take any fixed line and point in the picture Q, and determine the equation to the curve boundary of its primitive O corresponding to a given distance h between the fixed point and the variable vanishing line and to a given angle of inclination α between the fixed line and this variable line. Then by hypothesis the area of O, say M, is known in terms of its coefficients, which will be known functions of α and h; hence $\frac{dM}{d\alpha}$ and $\frac{dM}{dh}$ are known, and consequently the position of the perspective of the centre of gravity of O on the picture is known; and from this the position of that centre in its own plane can be constructed, and therefore will have been found by aid of algebraical and differentiation processes only, as was to be shown.

The above explanation may be made still more distinct if we suppose that we *begin* with an object Ω (the curve for which is expressed by an equation in its most general form), wherein we have, say, $\alpha=0$ and $h=1$; that from this we deduce the equation of P in the preceding investigation, and from P pass to O as before; then, having found the coordinates of the perspective of the centre of gravity of O as functions of h and α, make $\alpha=0$, $h=1$, and pass back to the coordinates of the centre of gravity in Ω, of which the centre of gravity last named then becomes the perspective.