

NOTE ON CERTAIN DEFINITE INTEGRALS.

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IN the *Institutiones Calculi Integralis*, Euler has investigated the value of the definite integral $\int_1^0 \frac{\log x dx}{\sqrt{(1-x^2)}}$, and his mode of statement seems to imply that the result, as well as the demonstration, was his own. How this may be, in fact, I do not pretend to know: in the *Philosophical Magazine** of December, 1860, I have (under another notation) investigated the values of

$$\int_1^0 \frac{\log x dx}{\sqrt{(1-x^2)(1-c^2x^2)}} \text{ and } \int_1^0 \frac{\log \{1 + \sqrt{(1-c^2x^2)}\}}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}} dx,$$

and shown them to be equal, but of course with contrary signs, and the former to be expressed† by $\frac{1}{2} \log cF(c) + \frac{1}{4} \pi F(b)$. The relation of which (in regard to the form of the functions of which it is composed) to the integral of its differential without $\log x$ in the numerator is so strikingly analogous to the relation of Euler's more simple integral (namely, $\frac{1}{2} \pi \log 2$) to $\int_1^0 dx \frac{1}{\sqrt{(1-x^2)}}$ as to suggest the existence of some general theorem in which both these results are comprised.

In proving the equality of the two definite integrals in question, a third integral of different form from either came to light. In fact, it is shown in the paper referred to, that

$$\frac{\log \{1 + \sqrt{(1-t)}\}}{\sqrt{(1-t)}} = \frac{2}{\pi} \int_{\frac{1}{2}\pi}^0 d\phi \{ \log (\cos \phi) + (\cos \phi)^2 \log (\cos \phi) t + (\cos \phi)^4 \log (\cos \phi) t^2 + \text{etc.} \},$$

with a tacit supposition that t is contained within the limits (both *inclusive*) $+1$ and -1 , within which limits the series which expresses $\frac{\log \{1 + \sqrt{(1-t)}\}}{\sqrt{(1-t)}}$ in powers of t remains convergent.

[* p. 213 above.]

[† where $b^2 = 1 - c^2$.]

If now we make $1 - t = c^2$, and write θ in place of ϕ , we have

$$\int_{\frac{1}{2}\pi}^0 \frac{\log \cos \theta d\theta}{1 - t (\cos \theta)^2} = \int_{\frac{1}{2}\pi}^0 \frac{\log \cos \theta d\theta}{(\sin \theta)^2 + c^2 (\cos \theta)^2} = \frac{\pi}{2} \frac{\log (1 + c)}{c},$$

with the restriction that c must be positive.

The limits of convergency imply furthermore (as far as the demonstration given is concerned) that c^2 should not be greater than 2, but since neither side of the equation passes through a *critical* phase in any sense (that is, as regards either themselves or their successive differentials) for this, or indeed for any positive value of c , it seems to follow that the equation must continue good from $c=0$ to $c=\infty$, and that the limitation which the cessation of the convergency of the intermediary series might have required to be placed upon the subsistence of the equation may in effect be disregarded. Perhaps also it would be desirable to inquire whether the equality may not continue to subsist for imaginary values of c with a positive real part.

Knowing the value of $\frac{2}{\pi} \int_{\frac{1}{2}\pi}^0 \frac{d\theta \log \cos \theta}{\{(\sin \theta)^2 + c^2 (\cos \theta)^2\}^i}$, namely, $\frac{\log (1 + c)}{c}$ when $i=1$, we may obviously obtain an expression involving only logarithms and algebraical quantities for all integer values of i ; indeed, calling the above integral u_i , we easily obtain the formula of reduction,

$$u_{i+1} - u_i = -\frac{1 - c^2}{2ic} \frac{d}{dc} u_i,$$

from which it may readily be shown that u_i will be of the form

$$\left(\frac{A_1}{c} + \frac{A_2}{c^2} + \dots + \frac{A_i}{c^{2i-1}} \right) \log (1 + c) + \frac{B_1}{c} + \frac{B_2}{c^2} + \frac{B_3}{c^3} + \dots + \frac{B_{2i-2}}{c^{2i-2}},$$

where the two sets of numerators are *constant*, the law of which it may be desirable at some future time to investigate. It should be noticed that although $(1 + c)$ appears in the denominator of $\frac{du_i}{dc}$, it does not make its appearance in u_i by reason of the numerator $1 - c^2$ in the expression for Δu_i .

The series expressing $\frac{\log \{1 + \sqrt{1-t}\}}{\sqrt{1-t}}$ from which the value of u_1 has been derived, is the following:

$$\begin{aligned} \frac{\log \{1 + \sqrt{1-t}\}}{\sqrt{1-t}} &= \log 2 \left(1 + \frac{1}{2} t + \frac{1 \cdot 3}{2 \cdot 4} t^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} t^3 + \&c. \right) \\ &- \left\{ \frac{1}{1 \cdot 2} \frac{1}{2} t + \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} \right) \frac{1 \cdot 3}{2 \cdot 4} t^2 \right. \\ &\left. + \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} \right) \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} t^3 + \&c. \right\}, \end{aligned}$$

(see *Philosophical Magazine**, December, 1860, p. 530, where t^2 is used in place of t) but since

$$\frac{d}{dt} \log \{1 + \sqrt{(1-t)}\} = -\frac{1}{2\sqrt{(1-t)}} \frac{1 - \sqrt{(1-t)}}{t} = \frac{1}{2t} - \frac{1}{2t\sqrt{(1-t)}},$$

and consequently

$$\log \{1 + \sqrt{(1-t)}\} = \log 2 - \frac{1}{2^2} t - \frac{1 \cdot 3}{2 \cdot 4^2} t^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6^2} t^3 - \&c.,$$

the more natural (as the more obvious) mode of deducing the coefficients of the powers of t in the expansion of $\frac{\log \{1 + \sqrt{(1-t)}\}}{\sqrt{(1-t)}}$, would seem to be to multiply the series written above by the series

$$1 + \frac{1}{2} t + \frac{1 \cdot 3}{2 \cdot 4} t^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} t^3 + \&c.$$

This method of proceeding would however in fact have left obscure the true nature of those coefficients. Let us perform the multiplication indicated; we shall then obtain, by comparison with the expansion already obtained, the following very far from obvious, indeed very unlikely to be suspected identity, which it is desirable to put on record: namely,

$$\begin{aligned} & \frac{1}{2} \cdot \frac{1}{2} \frac{1 \cdot 3 \cdot 5 \dots (2i-3)}{2 \cdot 4 \cdot 6 \dots (2i-2)} + \frac{1}{4} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2i-5)}{2 \cdot 4 \cdot 6 \dots (2i-4)} + \dots \\ & + \frac{1}{2i-2} \frac{1 \cdot 3 \cdot 5 \dots (2i-3)}{2 \cdot 4 \cdot 6 \dots (2i-2)} + \frac{1}{2i} \frac{1 \cdot 3 \cdot 5 \dots (2i-1)}{2 \cdot 4 \cdot 6 \dots 2i} \\ & = \frac{1 \cdot 3 \cdot 5 \dots (2i-1)}{2 \cdot 4 \cdot 6 \dots 2i} \left\{ \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(2i-1) 2i} \right\}. \end{aligned}$$

Thus, for example, if $i = 4$,

$$\begin{aligned} & \frac{1}{2^2} \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3}{2 \cdot 4^2} \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6^2} \frac{1}{2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8^2} \\ & = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} \right). \end{aligned}$$

The expansion above referred to leads to the value of u_1 through the intervention of the equality [see p. 212 above]

$$\frac{2}{\pi} \int_{\frac{1}{2}\pi}^0 \log(\cos \theta) (\cos \theta)^{2n} d\theta = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \sum_{k=1}^{k=\infty} \frac{1}{(2n+2k-1)(2n+2k)},$$

established in the paper referred to: and it is not uninteresting to notice that the above equality enables us to determine the value of the integral on the left-hand side of the equation in a series of descending powers of n , from

[* p. 213 above.]

which doubtless many new conclusions may be deduced. The first term in this descending series being $\frac{1}{4n}$, we are enabled to fix the degree of the integral, when n becomes infinite, for in that case

$$\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} = \sqrt{\left(\frac{2}{\pi} \cdot \frac{1}{2n}\right)} = \frac{1}{\sqrt{(\pi n)}},$$

so that for $n = \infty$,

$$\int_{\frac{1}{2}\pi}^0 (\cos \theta)^{2n} \log \cos \theta d\theta = \frac{\pi}{2} \frac{1}{\sqrt{(\pi n)}} \cdot \frac{1}{4n} = \frac{\sqrt{(\pi)}}{(2n^2)^2}.$$

The consideration of the expansion above referred to, namely, of

$$\frac{1}{m+1} - \frac{1}{m+2} + \frac{1}{m+3} + \&c.$$

in a series of descending powers of $\frac{1}{m}$, or which is the same thing, of

$$\frac{\mu}{\mu+1} - \frac{\mu}{2\mu+1} + \frac{\mu}{3\mu+1},$$

in a series of ascending powers of μ , suggests an observation which may appear to amount to a mere futile distinction, but which, closely examined, will be found to have a real signification and importance.

The above series being obtained by means of the equivalence $\Sigma = e^{\frac{d}{dx}} - 1$ will readily be seen to import Bernoulli's numbers in such a manner into the development that the latter would commonly be said (like all the series of the same class) to be absolutely divergent, incapable, that is to say, of constituting an arithmetical equivalent to its generatrix for any value whatever of the variable μ . The distinction I would draw would be to say not that the circle of convergence of μ ceases to exist, but that it becomes indefinitely small, or which is the same thing, the *corona* of convergence for the series treated as a function of m , has its inner radius indefinitely large: so that for $\mu = 0$ or $m = \infty$, we may reason, and reason with perfect safety, upon the equality between the generating function and the series as subsisting in an arithmetical sense as regards not only μ or m , but all successive powers of the same. [I mean that supposing

$$\phi\mu = a_0 + a_1\mu + a_2\mu^2 + \dots,$$

we may affirm not only the equality

$$\phi\mu - a_0 = 0,$$

but also

$$\phi\mu - a_0 = 0, \quad (\phi\mu - a_0) \div \mu = a_1, \quad (\phi\mu - a_0 - a_1\mu) \div \mu^2 = a_2,$$

and so on when $\mu = 0$.]

This fact of the character of arithmetical equivalence within certain limits not absolutely departing even in the case of series considered irreclaimably divergent, may, I think, serve to account, *à priori*, for the phenomenon of many conclusions being capable of being truthfully drawn from reasonings upon them in which they are treated as though they were in an ordinary sense convergent, because, in fact, part of the attributes of ordinary convergency (all such indeed as are not nullified by the radius of convergency becoming infinitely small) must continue to adhere to such series.

The expansion for

$$\frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} \text{ \&c.},$$

which occurs in the expression for $\int_{\frac{1}{2}\pi}^0 \log \cos \theta (\cos \theta)^{2n} d\theta$ in a series proceeding according to powers of $\frac{1}{n}$, may be most readily obtained by means of the differences of zero, as follows: calling $2n = x$, we have

$$\begin{aligned} \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} + \text{\&c.} \\ = \{(1 + \Delta) - (1 + \Delta)^2 + (1 + \Delta)^3 \dots\} \frac{1}{x+0} \\ = \frac{1 + \Delta}{2 + \Delta} \left(\frac{1}{x} - \frac{1}{x^2} 0 + \frac{1}{x^3} 0^2 \pm \text{\&c.} \right) \\ = \frac{1}{x} - \frac{1}{2 + \Delta} \left(\frac{1}{x} - \frac{1}{x^2} 0 + \frac{1}{x^3} 0^2 \pm \text{\&c.} \right), \end{aligned}$$

so that the first term will be $\frac{1}{2x}$ or $\frac{1}{4n}$. This might also be shown in a strictly arithmetical method as follows: let

$$s = \frac{1}{(x+1)(x+2)} + \frac{1}{(x+3)(x+4)} + \dots \text{ ad inf.} = s_1 + s_2 + s_3 + \dots,$$

where

$$s_1 = \frac{1}{(x+1)(x+2)} + \frac{1}{(x+3)(x+4)} + \dots + \frac{1}{(\epsilon x - 1)\epsilon x},$$

$$s_2 = \frac{1}{(\epsilon x + 1)(\epsilon x + 2)} + \frac{1}{(\epsilon x + 3)(\epsilon x + 4)} + \dots + \frac{1}{(\epsilon^2 x - 1)\epsilon^2 x},$$

$$s_3 = \frac{1}{(\epsilon^2 x + 1)(\epsilon^2 x + 2)} + \frac{1}{(\epsilon^2 x + 3)(\epsilon^2 x + 4)} + \dots + \frac{1}{(\epsilon^3 x - 1)\epsilon^3 x},$$

\&c. = \&c.,

ϵ being any real positive value superior to unity, and x being infinite. Then, observing that each partial series is a mean between the products of the number of terms in it, by the first and last respectively, we have obviously s always intermediate to

$$\frac{\epsilon - 1}{2x} + \frac{(\epsilon - 1)}{2\epsilon x} + \frac{(\epsilon - 1)}{2\epsilon^2 x} + \&c.,$$

and

$$\frac{(\epsilon - 1)}{2\epsilon x} + \frac{(\epsilon - 1)}{2\epsilon^2 x} + \frac{(\epsilon - 1)}{2\epsilon^3 x} + \&c.;$$

that is, between $\frac{\epsilon}{2x}$ and $\frac{1}{2x}$, and consequently, is equal to $\frac{1}{2x}$, as before.