51.

NOTE ON A DIRECT METHOD OF OBTAINING THE EXPANSION OF THE SINE OR COSINE OF MULTIPLE ARCS IN TERMS OF POWERS OF THE SINES OR COSINES OF THE SIMPLE ARC BY MEANS OF DE MOIVRE'S THEOREM.

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THE annexed appears to be the most direct and natural method for obtaining the known formulæ for the expansion of the sines and cosines of multiple arcs.

We know by De Moivre's theorem, that

$$\cos 2nx = (\cos x)^{2n} - 2n \cdot \frac{2n-1}{2} (\sin x)^2 (\cos x)^{2n-2} + \&\text{c.}$$

Let $(\sin x)^2 = \gamma$, then

$$\cos 2nx = (1-\gamma)^n - 2n \frac{2n-1}{2} \gamma (1-\gamma)^{n-1}$$

$$+ 2n \frac{2n-1}{2} \cdot \frac{2n-2}{3} \cdot \frac{2n-3}{4} \gamma^2 (1-\gamma)^{n-2}, \&\text{c.}$$

$$= A_0 - A_1\gamma + A_2\gamma^2 - A_3\gamma^3, \&\text{c.}$$

I use $\omega_r \phi x$ to indicate the coefficient of x^r in ϕx expanded in a series of powers of x. We have then

$$A_r = P_0 Q_0 + P_1 Q_1 + P_2 Q_2 + \&\text{c.},$$

where
$$P_0 = \omega_r (1+t)^n = \omega_r (1-t)^{-(n-r+1)} = \omega_{2r} (1-t^2)^{-(n-r+1)},$$

$$P_1 = \omega_{r-1} (1+t)^{n-1} = \omega_{r-1} (1-t)^{-(n-r+1)} = \omega_{2r-2} (1-t^2)^{-(n-r+1)},$$

$$P_2 = \omega_{r-2} (1+t)^{n-2} = \omega_{r-2} (1-t)^{-(n-r+1)} = \omega_{2r-4} (1-t^2)^{-(n-r+1)},$$

&c. &c.

$$Q_0 = 1 = \omega_0 (1+t)^{2n},$$

$$Q_1 = 2n \frac{2n-1}{2} = \omega_2 (1+t)^{2n},$$

$$Q_2 = 2n \frac{2n-1}{2} \cdot \frac{2n-2}{3} \cdot \frac{2n-3}{4} = \omega_4 (1+t)^{2n},$$

&c. &c.

Hence evidently,

$$A_r = \omega_{2r}\{(1-t^2)^{-(n-r+1)} \times (1+t)^{2n}\} = \omega_{2r}\{(1-t)^{-(n-r+1)} \times (1+t)^{n+r-1}\}\text{*}.$$

To fix the ideas, suppose $r = 2$, then

$$A_2 = \omega_4 \left\{ \begin{aligned} &\left\{1+(n-1)t+\frac{(n-1)n}{2}t^2 \right.\\ &\qquad \left. + \frac{(n-1)n(n+1)}{2.3}t^3 + \frac{(n-1)n(n+1)(n+2)}{2.3.4}t^4\right\} \\ &\times \left\{1+(n+1)t+\frac{(n+1)n}{2}t^2 \right.\\ &\qquad \left. + \frac{(n+1)n(n-1)}{2.3}t^3 + \frac{(n+1)n(n-1)(n-2)}{2.3.4}t^4\right\} \end{aligned} \right\}$$

$$= \frac{(n+1)n(n-1)(n-2)+(n+2)(n+1)n(n-1)}{1.2.3.4}$$

$$+ \frac{(n+1)n(n-1)(n-1)+(n+1)(n+1)n(n-1)}{1.2.3\times 1}$$

$$+ \frac{n(n-1)n(n+1)}{1.2\times 1.2}$$

$$= \frac{2n(n-1)n(n+1)}{1.2.3.4} + \frac{2n(n-1)n(n+1)}{1.2.3\times 1} + \frac{n(n-1)n(n+1)}{1.2\times 1.2}$$

$$= \omega_4\left(1+t+\frac{t^2}{1.2}+\frac{t^3}{1.2.3}+\frac{t^4}{1.2.3.4}\right)^2 (n^2-1)n^2;$$

and so in general, we shall have

$$A_r = n(n-r+1)(n-r+2)\dots(n+r-1)$$

$$\times\, \omega_{2r}\left(1+t+\frac{t^2}{1.2}+\dots+\frac{t^{2r}}{1.2\dots 2r}\right)^2$$

$$= \omega_{2r}\, e^{2t} \times n\{(n-r+1)\dots(n+r-1)\}$$

$$= \frac{2^{2r}}{1.2.3.4\dots 2r}\, n^2(n^2-1)(n^2-4)\dots\{n^2-(r-1)^2\},$$

and thus

$$\cos 2nx = 1 - \frac{n^2}{1.2}(2\sin x)^2 + \frac{n^2(n^2-1)}{1.2.3.4}(2\sin x)^4 \mp \text{\&c.}$$

In like manner we have

$$\cos(2n+1)x = \cos x\{(1-\gamma)^n - \tfrac{1}{2}(2n+1)2n\gamma(1-\gamma)^{n-1} + \text{\&c.}\}$$

$$= \cos x\{B_0 - B_1\gamma + B_2\gamma^2 + \text{etc.}\},$$

where

$$B_r = \omega_{2r}\{(1-t^2)^{-(n-r+1)}(1+t)^{2n+1}\}$$

$$= \omega_{2r}\{(1-t)^{-(n-r+1)} \times (1+t)^{n+r}\};$$

* Note well this simple change in the form of the generating function; in it the point and pith of the method resides.

and making, as before, $r = 2$, we see that

$$B_2 = \omega_4 \left\{ \begin{array}{l} \left\{1 + (n-1)t + \dfrac{(n-1)n}{2} t^2 \right. \\ \qquad \left. + \dfrac{(n-1)n(n+1)}{1.2.3} t^3 + \dfrac{(n-1)n(n+1)(n+2)}{1.2.3.4} t^4 \right\} \\ \times \left\{1 + (n+2)t + \dfrac{(n+2)(n+1)}{2} t^2 \right. \\ \qquad \left. + \dfrac{(n+2)(n+1)n}{1.2.3} t^3 + \dfrac{(n+2)(n+1)n(n-1)}{1.2.3.4} t^4 \right\} \end{array} \right\}$$

$$= \omega_4 (e^{2t}) \times (n-1)n(n+1)(n+2),$$

and so in general,

$$B_r = \omega_{2r} e^{2t} \{(n-r+1)(n-r+2) \dots n \dots (n+r-1)(n+r)\}$$

$$= \frac{(n-r+1)(n-r+2) \dots (n+r)}{1.2.3 \dots 2r} 2^{2r},$$

and thus

$$\cos(2n+1)x = \cos x \left\{1 - \frac{n(n+1)}{2}(2\sin x)^2 + \frac{(n-1)n(n+1)(n+2)}{1.2.3.4}(2\sin x)^4 \dots \&c.\right\}.$$

We might in like manner, and by precisely the same process, obtain the expressions for $\cos 2mx$, $\cos(2m+1)x$ in terms of $\cos x$, and of $\sin 2mx$, $\sin(2m+1)x$ in terms of $\sin x$ or of $\cos x$, but these results may, of course, be most readily found by means of obvious processes of differentiation in respect to the arc and by substitution of the complement for the arc itself in the results already obtained.

It may be worth while to show here how the same elementary theorem as we have employed above, furnishes, *uno ictu*, another important formula connected with multiple arcs:

$$\left(\frac{d}{dx}\right)^{n-1} (1-x^2)^{\frac{2n-1}{2}} = \left(\frac{d}{dx}\right)^{n-1} \left\{(1+x)^{\frac{2n-1}{2}} (1-x)^{\frac{2n-1}{2}}\right\},$$

by Leibnitz's Theorem,

$$= \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{3}{2} \sqrt{(1-x^2)} (1-x)^{n-1}$$

$$- (n-1) \times \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{5}{2} \times \frac{2n-1}{2} \sqrt{(1-x^2)} (1-x)^{n-2} (1+x)$$

$$+ (n-1) \frac{n-2}{2} \times \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{7}{2}$$

$$\times \frac{2n-1}{2} \cdot \frac{2n-3}{2} \sqrt{(1-x^2)} (1-x)^{n-3} (1+x)^2 \mp \&c.$$

$$= \frac{1.3.5\ldots(2n-1)}{2^{n-1}} \sqrt{(1-x^2)}$$

$$\times \{(1-x)^{n-1} - A_1(1-x)^{n-2}(1+x) + A_2(1-x)^{n-3}(1+x)^2 \mp \&c.\},$$

where
$$A_r = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \ldots \frac{2n-(2r-1)}{2}$$

$$\times \frac{(n-1)(n-2)\ldots(n-r)}{1.2\ldots r}$$

$$\times \frac{2}{3} \cdot \frac{2}{5} \ldots \frac{2}{2r+1}$$

$$= \frac{(2n-1)(2n-2)(2n-3)(2n-4)\ldots\{2n-(2r-1)\}}{2.3.4.5\ldots(2r+1)}$$

$$= \frac{1}{2n}\left[\frac{2n(2n-1)\ldots(2n-2r)}{1.2\ldots(2r+1)}\right].$$

Hence, making $x = \cos 2\phi$,

$$\left(\frac{d}{dx}\right)^{n-1} (1-x^2)^{\frac{2n-1}{2}} = \frac{1.3.5\ldots(2n-1)}{n}$$

$$\times \left\{2n(\sin\phi)^{2n-1}\cos\phi - \frac{2n(2n-1)(2n-2)}{1.2.3}(\sin\phi)^{2n-3}(\cos\phi)^3 \pm \&c.\right\}$$

$$= \frac{1.3.5\ldots(2n-1)}{n} \frac{\{\sin\phi + \sqrt{(-1)}\cos\phi\}^{2n} - \{\sin\phi - \sqrt{(-1)}\cos\phi\}^{2n}}{2\sqrt{(-1)}}$$

$$= (-)^{n-1}\frac{1.3.5\ldots(2n-1)}{n}\sin 2n\phi$$

$$= (-)^{n-1}\frac{1.3.5\ldots(2n-1)}{n}\sin\{n\sin^{-1}\sqrt{(1-x^2)}\},$$

or if we please to pass to the more general form by a linear transformation,

$$\left(\frac{d}{dx}\right)^{n-1} (A + 2Bx - Cx^2)^{\frac{2n-1}{2}}$$

$$= (-)^{n-1}\frac{1.3.5\ldots(2n-1)}{n} C^{\frac{2n-3}{2}} \sqrt{(AC+B^2)}\sin n \sin^{-1} \sqrt{\left(\frac{A+2Bx-Cx^2}{A+\frac{B^2}{C}}\right)}.$$