46.

NOTE ON THE HISTORICAL ORIGIN OF THE UNSYMMETRICAL SIX-VALUED FUNCTION OF SIX LETTERS.

[*Philosophical Magazine*, XXI. (1861), pp. 369—377.]

THE discovery and first announcement of the existence of the celebrated function of six letters having six values, and not symmetrical in respect to all the letters, is usually assigned to my illustrious friend M. Hermite, to whom M. Cauchy expressly ascribes it in a memoir inserted in the *Comptes Rendus* of the Institut for December 8, 1845, p. 1247, and again, January 5, 1846, p. 30.

M. Cauchy adds that the conversation he held with M. Hermite on this subject excited in himself a lively desire to sound to its depths the question of permutations, and to develope the consequences to be deduced from the application of the principles relative thereto, which he had himself long previously laid down.

I was not at that date in the habit of consulting the *Comptes Rendus*, or I should at once have made the reclamation of priority which I now do, not from any unworthy motive of self-love in so small a matter, but out of regard to historic truth. It is a year or two since I first learnt that the origin of this function was usually referred to M. Cauchy or M. Hermite; but although aware that its existence was known to myself long previous to the dates quoted, I did not recollect that I had ever communicated it to the world through the medium of the press, and I therefore kept silence on the subject.

Turning over, a few days ago, for another purpose, the pages of a back volume of this *Magazine*, my eye chanced to alight on a footnote to a paper of my own inserted therein*, under date of April 1844, "On the Principles of Combinatorial Aggregation," which I will take the liberty of quoting at length, as it proves incontestably the priority which I lay claim to.

[* p. 92 of Vol. I. of this Reprint.]

"When the modulus is four, there is only one synthematic arrangement possible, and there is no indeterminateness of any kind; from this we can infer, *à priori*, the reductibility of a biquadratic equation; for using ϕ, f, F to denote rational symmetrical forms of function, it follows that

$$F\left\{\begin{matrix} f\{\phi(a, b), \phi(c, d)\} \\ f\{\phi(a, c), \phi(b, d)\} \\ f\{\phi(a, d), \phi(b, c)\} \end{matrix}\right\} \text{ is itself a rational symmetric function of } a, b, c, d.$$

Whence it follows that if a, b, c, d be the roots of a biquadratic equation, $f\{\phi(a, b), \phi(c, d)\}$ can be found by the solution of a cubic: for instance, $(a+b)\times(c+d)$ can be thus determined, whence immediately the sum of any two of the roots comes out from a quadratic equation.

"To the modulus 6 there are fifteen different synthemes capable of being constructed. At first sight it might be supposed that these could be classed in natural families of three or of five each, on which supposition the equation of the sixth degree could be depressed; but on inquiry this hope will prove to be futile, not but what natural affinities do exist between the totals; but in order to separate them into families, each will have to be taken twice over; or in other words, the fifteen synthemes to modulus 6 being reduplicated, subdivide into six natural families of five each."

The six families above referred to (in which it is to be understood that $p \,.\, q$ and $q \,.\, p$ are identical in effect) are the following:—

$a.b$	$c.d$	$e.f$	$a.c$	$d.e$	$f.b$	$a.d$	$e.f$	$b.c$
$a.c$	$b.e$	$d.f$	$a.d$	$c.f$	$e.b$	$a.e$	$d.b$	$f.c$
$a.d$	$b.f$	$c.e$	$a.e$	$c.b$	$d.f$	$a.f$	$d.c$	$e.b$
$a.e$	$b.d$	$c.f$	$a.f$	$c.e$	$d.b$	$a.b$	$d.f$	$e.c$
$a.f$	$b.c$	$d.e$	$a.b$	$c.d$	$e.f$	$a.c$	$d.e$	$f.b$

$a.e$	$f.b$	$c.d$	$a.f$	$b.c$	$d.e$	$a.b$	$c.d$	$e.f$
$a.f$	$e.c$	$b.d$	$a.b$	$f.d$	$c.e$	$a.c$	$b.e$	$d.f$
$a.b$	$e.d$	$f.c$	$a.c$	$f.e$	$b.d$	$a.d$	$b.f$	$c.e$
$a.c$	$e.b$	$f.d$	$a.d$	$f.c$	$b.e$	$a.e$	$b.d$	$c.f$
$a.d$	$e.f$	$b.c$	$a.e$	$f.b$	$c.d$	$a.f$	$b.c$	$d.e$

And it will be observed that every two families have one, and only one, syntheme in common between them; and precisely in the same way as in the note above quoted it is especially shown that the one single natural family

$a.b$	$c.d$
$a.c$	$b.d$
$a.d$	$b.c$

gives rise to a function of four letters with only one value, so the six functions analogously formed with these six families obviously give rise to six functions, which change into one another when any interchange is effected between the letters which enter into them; so that any one of these is a function of six letters having only six values. I conceive that, after this reference, no writer on the subject wishing to specify the function in question would hesitate to call it after my name.

I may also take occasion to observe that, in connexion with my researches in combinatorial aggregation, long before the publication of my unfinished paper in the *Magazine*, I had fallen upon the question of forming a heptatic aggregate of triadic synthemes comprising all the duads to the base 15, which has since become so well known, and fluttered so many a gentle bosom, under the title of the fifteen school-girls' problem; and it is not improbable that the question, under its existing form, may have originated through channels which can no longer be traced in the oral communications made by myself to my fellow-undergraduates at the University of Cambridge long years before its first appearance, which I believe was in the *Ladies' Diary* for some year which my memory is unable to furnish.

In order to relieve this notice from the mere personal character which it may thus far appear to bear, I will state another question concerning the combinatorial aggregation of fifteen things which may serve as a pendant to the famous school-girl problem.

The number of triads to the base 15 is $\frac{15 \times 14 \times 13}{3 \cdot 2 \cdot 1} = 5 \times 91$. Let it be required to arrange these into 91 synthemes, in other words, to set out the walks of 15 girls for 91 days (say a quarter of the year) in such a manner that the same three shall never *all* come together more than once in the quarter. Of the various ways in which it is probable this problem may be solved, the following deserves notice. Let 15 letters be arbitrarily divided into 5 sets, namely,

$$a_1 b_1 c_1 ; \quad a_2 b_2 c_2 ; \quad a_3 b_3 c_3 ; \quad a_4 b_4 c_4 ; \quad a_5 b_5 c_5 .$$

The sets as they stand will represent one of the 91 arrangements sought for, which I call the basic syntheme. The remaining 90 may be obtained as follows in 10 batches of 9 each. Write down the 10 index distributions following:—

1 2 3; 4 5	1 4 5; 2 3
1 2 4; 3 5	2 3 4; 1 5
1 2 5; 3 4	2 3 5; 1 4
1 3 4; 2 5	2 4 5; 1 3
1 3 5; 2 4	3 4 5; 1 2.

Take any one of these distributions, as for instance 2 3 5; 1 4, and proceed as follows:—In respect of 2, 3, 5, conjugate the three sets $\begin{matrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_5 & b_5 & c_5 \end{matrix}$; and in respect of 1, 4, conjugate the two remaining sets $\begin{matrix} a_1 & b_1 & c_1 \\ a_4 & b_4 & c_4 \end{matrix}$.

From the ternary conjugation form the nine arrangements,

	$a_2\ a_3\ a_5$	$b_2\ b_3\ b_5$	$c_2\ c_3\ c_5$
	$a_2\ a_3\ b_5$	$b_2\ b_3\ c_5$	$c_2\ c_3\ a_5$
	$a_2\ a_3\ c_5$	$b_2\ b_3\ a_5$	$c_2\ c_3\ b_5$
	$a_2\ b_3\ a_5$	$b_2\ c_3\ b_5$	$c_2\ a_3\ c_5$
	$a_2\ b_3\ b_5$	$b_2\ c_3\ c_5$	$c_2\ a_3\ a_5$
	$a_2\ b_3\ c_5$	$b_2\ c_3\ a_5$	$c_2\ a_3\ b_5$
	$a_2\ c_3\ a_5$	$b_2\ a_3\ b_5$	$c_2\ b_3\ c_5$
	$a_2\ c_3\ b_5$	$b_2\ a_3\ c_5$	$c_2\ b_3\ a_5$
	$a_2\ c_3\ c_5$	$b_2\ a_3\ a_5$	$c_2\ b_3\ b_5$,
which call	$L_1\ L_2\ L_3$	$L_4\ L_5\ L_6$	$L_7\ L_8\ L_9$.

Again, from the binary conjugation, form the nine arrangements,

	$a_1\ b_1\ c_4$	$a_4\ b_4\ c_1$
	$a_1\ b_1\ b_4$	$a_4\ c_1\ c_4$
	$a_1\ b_1\ a_4$	$c_1\ b_4\ c_4$
	$a_1\ c_1\ c_4$	$a_4\ b_4\ b_1$
	$a_1\ c_1\ b_4$	$a_4\ b_1\ c_4$
	$a_1\ c_1\ a_4$	$b_1\ b_4\ c_4$
	$b_1\ c_1\ c_4$	$a_4\ b_4\ a_1$
	$b_1\ c_1\ b_4$	$a_4\ a_1\ c_4$
	$b_1\ c_1\ a_4$	$a_1\ b_4\ c_4$,

which call $M_1\ M_2\ M_3 \quad M_4\ M_5\ M_6 \quad M_7\ M_8\ M_9$.

Now combine the L with the M system, each L with some M in any order whatever; the 9 combinations or appositions thus obtained will give a batch of 9 synthemes; and proceeding in like manner with each of the 10 distributions of the indices 1, 2, 3, 4, 5, we shall obtain 90 synthemes, which together with the basic syntheme complete the system required. The M system corresponding to any distribution of the indices is the system which contains the synthematic arrangement of the bipartite* triads which can be constituted out of six things, separated in two sets or parts, and is unique. The L system is *one* of those which represents the synthematic arrangement

* See note at end of paper.

of the tripartite* triads of nine things separated into three sets or parts. I have set out above one in particular of these for the sake of greater clearness; but any other system having the same property will serve the same purpose, and a careful study will serve to show that the total number of *L*'s corresponding to a given distribution of indices will be ( )*. Consequently the total number of *LM*'s that we can form for a given distribution will be ( ) × 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9; and the number of *distinct* synthematic arrangements satisfying the given conditions corresponding to any assumed basic syntheme will be this number raised to the tenth power; and as this vastly exceeds the total number of permutations of fifteen things, we see, without even taking into consideration the diversity that may be produced by a change of the base, that this method must give rise to many distinct types of solution (arrangements being defined to belong to the same or different types, according as they admit or not of being deduced from each other by a permutation effected among their monadic elements). The common character of all these allotypical aggregations, and which serves to constitute them into a natural order or family, consists in their being derived from a base formed out of five sets, such that the monopartite triads corresponding to the base form one syntheme, and the other 90 synthemes each contain a conjugation of the tripartite triads belonging to three out of the five sets of the base with the bipartite triads belonging to the other two sets thereof. There is, moreover, no reason to suppose, or at all events no safe ground for affirming, that this family exhausts the whole possible number of types to which the arrangements satisfying the proposed condition admit of being reduced. A further question which I have somewhere raised, and which brings the two problems of the school-girls into *rapport*, is the following:—"To divide the system of 91 synthemes satisfying the conditions above stated into thirteen minor systems, each of which satisfies the conditions of the old problem, that is, of containing all the duads that can be made out of the fifteen elements once and once only"; or to put the question in a more exact form, to exhibit thirteen systems, each satisfying this last condition, which shall together include between them all the triads that can be made out of the fifteen elements.

The reader would have reason to be dissatisfied with the author's reticence, were he to leave altogether unmentioned the synthematic aggregation of the *binomial* triads appertaining to the same three triliteral sets or *nomes*; but space forbids my doing more at present than giving one of these aggregates, and indicating the number and mode of generation of all from this one. It will readily be seen that any such aggregate will be made up of two sub-aggregates, which I shall call A and B respectively, of which one bears

* Some day or another a new combinatorial calculus must come into being to furnish general solutions to the infinite variety of questions of *multifariousness* to which the theory of combinatorial aggregation, *alias* compound permutations, gives rise.

the same relation to the disposition of the nomes in the order 123 456 789, as the other to their disposition in the order 123 789 456. Thus we may take for our A and B the following, which will each contain 9 synthemes, the total number of synthemes in the two together being 18*:—

(A)			(B)		
124	567	893	127	894	763
125	468	739	128	795	436
126	459	783	129	786	453
134	568	279	137	895	246
135	469	278	138	796	245
136	457	289	139	784	256
234	569	187	237	896	154
235	467	189	238	794	156
236	458	179	239	785	146

The system of triads contained in A may be arranged in twelve different aggregates similar to the one given, and the same will be true for the triads in the B; so that the total number of the combined systems will be 144. All the permutations which leave A or B (separately considered) unaltered will form a natural group,—the theory of groups in this, as in every other case, standing in the closest relation to the doctrine of combinatorial aggregation, or what for shortness may be termed syntax. I have elsewhere given the general name of Tactic to the third pure mathematical science, of which order is the proper sphere, as is number and space of the other two. Syntax and Groups are each of them only special branches of Tactic. I shall on another occasion give reasons to show that the doctrine of groups may be treated as the arithmetic of ordinal numbers. With respect to the twelve varieties of the A or B aggregates, they may be obtained from the one given by combining the substitutions corresponding to the six permutations of the three constituents of one nome, as 7, 8, 9, with the permutation of any two constituents of another, as 5, 6. But I have said enough for my present purpose, which is to point out the boundless untrodden regions of thought in the sphere of order, and especially in the department of *syntax*, which remain to be expressed, mapped out, and brought under cultivation. The difficulty indeed is not to find material, of which there is a superabundance, but to discover the proper and principal centres of speculation that may serve to reduce the theory into a manageable compass.

* Thus, since there is evidently one mononomial syntheme, the total number of synthemes of all three kinds will be $1+18+9=28=\frac{8\times 7}{2}$, as it should be, the total number of triads being $\frac{9.8.7}{3.2}$, and $\frac{9}{3}$ of them going to a syntheme.

I put on record (as a Christmas offering on the altar of science) for the benefit of those studying the theory of groups, or compound permutations (to which the prize shortly to be adjudicated by the Institute of France for the most important addition to the subject may tend to give a new impulse), and with an eye to the geometrical and algebraical verities with which, as a constant of reason, we may confidently anticipate it is pregnant, an exhaustive table of the monosynthematic aggregates of the trinomial triads that are contained in a system of three triliteral nomes. Let these latter be called respectively 123; 456; 789; then we have the annexed:—

Table of Synthemes of Trinomial Triads to base 3 . 3.

(1)	(2)	(3)	(4)
147 258 369	147 258 369	147 258 369	147 258 369
148 259 367	148 259 367	148 259 367	148 259 367
149 257 368	149 257 368	149 257 368	149 267 358
157 268 349	157 268 349	157 269 348	157 268 349
158 269 347	158 269 347	158 267 349	158 269 347
159 267 348	159 267 348	159 268 347	159 247 368
167 248 359	167 249 358	167 248 359	167 248 359
168 249 357	168 247 359	168 249 357	168 249 357
169 247 358	169 248 357	169 247 358	169 257 348
(5)	**(6)**	**(7)**	**(8)**
147 258 369	147 258 369	147 258 369	147 258 369
148 267 359	148 267 359	148 269 357	148 269 357
149 268 357	149 257 368	149 257 368	149 267 358
157 249 368	157 268 349	157 268 349	157 268 349
158 269 347	158 269 347	158 249 367	158 249 367
159 248 367	159 248 367	159 267 348	159 247 368
167 259 348	167 259 348	167 248 359	167 248 359
168 257 349	168 249 357	168 259 347	168 259 347
169 247 358	169 247 358	169 247 358	169 257 348

The discussion of the properties of this Table, and the classification of the eight aggregates into natural families, must be reserved for a future occasion.

Note.—A triad is called tripartite if its three elements are culled out of three different parts or sets between which the total number of elements is supposed to be divided; bipartite if the elements are taken out of two distinct sets; unipartite if they all lie in the same set. The more ordinary method for the reduction of synthematic arrangements from a given base to a linear one which I employ, consists in the separate synthematization *inter se* of all the combinations of the *same* kind as regards the number of parts from which they are respectively drawn. Thus, for example, if the distribution of the $\frac{30 \times 29 \times 28}{6}$ triads to the base 30 into $\frac{29 \times 28}{2}$ synthemes be required, this may be effected by dividing the 30 elements in an arbitrary manner into 15 parts, each part containing 2 elements. These 15 parts being now themselves treated as elements, are first to be conjugated as in the old 15-school-girl problem, and each of these 7 conjugations can be made to furnish 6 synthemes containing exclusively bipartite triads. The same 15 parts are then to be conjugated as in the new school-girl problem, and the 91 conjugations thus obtained will each furnish 4 synthemes, containing exclusively the tripartite triads. These bipartite and tripartite synthemes will exhaust the entire number of triads of both kinds, and accordingly we shall find

$$7 \times 6 + 91 \times 4 = 406$$

$$= \frac{29 \times 28}{2}.$$

A syntheme, I need scarcely add, is an aggregate of combinations containing between them all the monadic elements of a given system, each appearing once only. In the more general theory of aggregation, such an aggregate would be distinguished by the name of a monosyntheme. A disyntheme would then signify an aggregate of combinations containing between them the duadic elements, each appearing once only, and so forth. Thus the old 15-school-girl question in my nomenclature would be enunciated under the form of a problem "to construct a triadic disyntheme, separable into monosynthemes to the base 15"; the new school question, as a problem "to divide the whole of the triads to base 15 into monosynthemes"; the question which connects the two, as a problem "to exhibit the whole of the triads to base 15 under the form of 13 disynthemes, each separated into 7 monosynthemes."

A question of a more general kind, and embracing this last, would be the problem of dividing the whole of the same system of triads into 13 disynthemes, without annexing the further condition of monosynthematic divisibility. So there is the simpler question of constructing a single disyntheme to the base 15 without any condition annexed as to its decomposability into 7 synthemes.