45.

NOTE ON THE NUMBERS OF BERNOULLI AND EULER, AND A NEW THEOREM CONCERNING PRIME NUMBERS.

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FOLLOWING the accepted *Continental* notation, I denote by B_n* the positive value of the coefficient of t^{2n} in $\dfrac{t}{1-e^t}$ multiplied by the continual product $1 . 2 . 3 \ldots 2n$.

The law which governs the fractional part of B_n was first given in Schumacher's *Nachrichten*, by Thomas Clausen in 1840; and almost immediately afterwards a demonstration was furnished by Professor Staudt in *Crelle's Journal*, with a reclamation of priority, supported by a statement of his having many years previously communicated the theorem to Gauss.

The law is this, that the positive or negative fractional residue of B_n (according as n is odd or even) is made up of the simple sum of the reciprocals of all the prime numbers which, respectively diminished by unity, are contained in $2n$. The proof, which is of an inductive kind, is virtually as follows: Suppose the law holds good up to $(n-1)$ inclusive; if we expand† Σx^{2n} under the form $\dfrac{1}{e^{\frac{d}{dx}}-1}\, x^{2n}$, we shall evidently obtain $\dfrac{\Sigma x^{2n}}{x} \pm B_n$ under the form of a finite series, of which the terms are numerical multiples of the products of powers of x by the Bernoullian numbers of an order inferior to the nth. If, now, we make x equal to the product of all the primes which, diminished by unity, are contained in $2n$, it will at once be

* Were it not for the general usage being as stated in the text, I certainly think it would be far more convenient to use a notation agreeing with the Continental method as to sign, and nearly, but not quite, with Mr De Morgan's as to quantity, namely, to understand by B_n the coefficient of t^n in $\frac{1}{2}t\dfrac{e^t+1}{e^t-1}$ taken positively, so that B_n should be equal to zero for all the odd values of n, not excepting $n=1$.

[† Σx^{2n} denotes $1^{2n}+2^{2n}+\ldots+(x-1)^{2n}$. Cf. p. 227.]

seen (on inspection of the series) that all its terms become integer numbers, and consequently $\frac{\Sigma x^{2n}}{x} \pm B_n$ becomes an integer; and therefore the law will hold good up to n, since it may easily be shown, by an application of Fermat's theorem and elementary arithmetical considerations, that if N be the product of any prime numbers whatever, and if p is the general name of such of them as diminished by unity are factors of μ, then $\frac{\Sigma N^\mu}{N} + \Sigma \frac{1}{p}$ is an integer.

Hence, since the law holds good for $n = 1$, it is universally true. This theorem, then, of Staudt and Clausen, *inter alia*, gives a rule for determining what primes alone enter into the denominators of the Bernoullian numbers when expressed as fractions in their lowest terms; it enables us to affirm that only simple powers of primes enter into those denominators, and to know *à priori* what those prime factors are. This note is intended to supply a law concerning the *numerators* of the Bernoullian numbers, which I have not seen stated anywhere, and which admits of an instantaneous demonstration, *to wit*, that the whole of n will appear in the numerator of B_n, save and except such primes, or the powers of such primes, as we know by the Staudt-Clausen law must appear in the denominator.

I am inclined to believe that this law of mine was not known, at all events, in 1840, from the circumstance that in Rothe's Table, published by Ohm in *Crelle's Journal* in that year, which gives the values of B_n up to $n = 31$, the numerators are, with one exception (about to be named), all exhibited in such a form as to show such low factors as readily offer themselves, but for B_{23} the fact of the divisibility of the numerator by 23 is not indicated. This numerator is 596451111593912163277961, which in fact $= 23 \times 25932657025822267968607$. It is obvious, indeed, under my law, that whenever p is a prime number other than 2 and 3, the numerator of B_p must contain p, because in such case $p - 1$ cannot be a factor of $2p$. When $p = 3$ or $p = 2$, $2p$ always contains $(p - 1)$, so that 2 and 3 are necessarily constant factors of the Bernoullian denominators, and can therefore never appear in the numerators. In Schumacher the law of the denominator is given as "a passing" (or *chance*?) "specimen" of a promised memoir by Clausen on the Bernoullian numbers, as to which I shall feel obliged if any of the readers of this *Magazine* will inform me whether it has appeared anywhere, and if so, where. Now for my demonstration of the law of the numerators.

By definition, $B_n = \Pi(2n) \times$ coefficient of t^{2n-1} in $\frac{1}{e^t - 1}$. Let μ be any integer number; then $\pm(\mu^{2n} - 1) B_n = \Pi(2n) \times$ coefficient of t^{2n-1} in

$$\frac{\mu}{e^{\mu t} - 1} - \frac{1}{e^t - 1},$$

or in $$\frac{(\mu-1)-(e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^t)}{e^{\mu t}-1},$$

or in $$-\frac{e^{(\mu-2)t}+2e^{(\mu-3)t}+\ldots+(\mu-2)e^t+(\mu-1)}{e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^t+1}.$$

But obviously, by Maclaurin's theorem, the coefficient of t^{2n-1} in the expansion of this last generating function will be of the form $\pm\frac{1}{\Pi(2n-1)}\cdot\frac{I}{\mu^{2n-1}}$, where I is an integer, and therefore B_n will be of the form $\frac{2nI}{\mu^{2n-1}(\mu^{2n}-1)}$.

Suppose now, when $\frac{2nI}{\mu^{2n-1}(\mu^{2n}-1)}$ is reduced to its lowest terms, that p (a prime contained in $2n$) does not appear in the numerator, this can only happen by virtue of p being contained in $\mu^{2n-1}(\mu^{2n}-1)$; let now μ be taken successively 2, 3, 4, ... $(p-1)$, then $\mu^{2n}-1$ in all these cases is divisible by p; and therefore, by an obvious inverse of Fermat's theorem, $(p-1)$ must be contained in $2n$, that is, p must be a factor of the denominator of B_n under the Staudt-Clausen law, which proves my theorem.

As a corollary to the foregoing, using Herschel's transformation, we see that if μ be taken any integer whatever,

$$\pm B_n=\frac{2n}{\mu^{2n}-1}\cdot\frac{(1+\Delta)^{\mu-2}+2(1+\Delta)^{\mu-3}+\ldots+(\mu-1)}{\Delta^{\mu-1}+\mu\Delta^{\mu-2}+\mu\frac{\mu-1}{2}\Delta^{\mu-3}+\ldots+\mu}0^{2n}$$

$$=\frac{2n}{\mu^{2n}-1}\,\frac{\Delta^{\mu-2}+\mu\Delta^{\mu-3}+\mu\frac{\mu-1}{2}\Delta^{\mu-4}+\ldots+\mu\frac{\mu-1}{2}}{\Delta^{\mu-1}+\mu\Delta^{\mu-2}+\mu\frac{\mu-1}{2}\Delta^{\mu-3}+\ldots+\mu\frac{\mu-1}{2}\Delta+\mu}0^{2n};$$

and if we write 0^{2n+1} instead of 0^{2n}, the result vanishes. For the case of $\mu=2$, this theorem accords with one well known. As this subject is so intimately related to that of the Herschelian differences of zero, I may take this occasion of stating a proposition concerning the latter, which (simple as it is) appears to have escaped observation, namely, that $\frac{\Delta^r 0^{n+r}}{\Pi(r)}$ is in fact the expression for the sum of the homogeneous products of the natural numbers from 1 to r, taken n together. For

$$\frac{1}{(x-r)(x-r+1)\ldots(x-1)x}$$

$$=\frac{1}{\Pi(r)}\left\{\frac{1}{x-r}-\frac{r}{x-r+1}+\frac{r\cdot\frac{r-1}{2}}{x-r+2}\cdots\pm\frac{1}{x}\right\}.$$

Hence obviously

$$\frac{1}{\Pi(r)}\left\{r^n - r(r-1)^n + r \cdot \frac{r-1}{2}(r-2)^n \mp \&c.\right\},$$

that is

$$\frac{\Delta^r 0^n}{\Pi(r)} = \text{coefficient of } \frac{1}{x^n} \text{ in } \frac{1}{(x-r)(x-r+1)\dots(x-1)}$$

$$= \text{the sum of the } (n-r)\text{ary homogeneous products of } 1, 2, 3, \dots r.$$

Thus, then, we are able to affirm, from what is known concerning $\frac{\Delta^r 0^{r+n}}{\Pi r}$ (see Prof. De Morgan's *Calculus*), that the r-ary homogeneous product-sum of 1, 2, 3, ... n (which is of the degree $2r$ in n) always contains the algebraic factor $n(n+1)\dots(n+r)$.

Addendum.—Since sending the above to press, I have given some further and successful thought to the Staudt-Clausen theorem. Staudt's demonstration labours under the twofold defect of indirectness and of presupposing a knowledge of the law to be established. In it the Bernoullian numbers are not made the subject of a direct contemplation, but are regarded through the medium of an alien function, one out of an infinite number, in which they are as it were latently embodied; and the proof, like all other inductive ones, whilst it convinces the judgment, leaves the philosophic faculty unsatisfied, inasmuch as it fails to disclose the reason (the title, so to say, to existence) of the truth which it establishes. I present below an immediate and a direct proof of this beautiful and important proposition, founded upon the same principle as gives the law of the necessary factor in the numerators (namely, the arbitrary decomposition of the generating function of Bernoulli's numbers into partial fractions), and resting upon a simple but important conception, that of *relative* as distinguished from absolute integers.

I generalize this notion, and define a quantity to be an integer relative to r (or, for brevity's sake, to be an rth integer) when it may be represented by a fraction of which the denominator does not contain r.

The lemma* upon which my demonstration rests is the following, which

* This lemma is the converse of a self-evident fact, and it virtually embodies a principle respecting an arithmetical fraction strikingly analogous to a familiar one respecting an algebraical one; namely, in the same way as a rational algebraical function of x can be expressed *in one, and only one, way* as an integral function augmented by a sum of negative powers of linear functions of x, so a rational arithmetical quantity can be expressed in one, and only one, way as an integer augmented by the sum of negative powers of simple prime numbers multiplied respectively by numbers less than such primes. In drawing this parallel, the arithmetical quantity $\frac{c}{p^i}$, where $c<p$, is regarded as the analogue of the algebraical one $\frac{1}{(ax+b)^i}$, as is quite

is itself an immediate corollary from the arithmetical theorem that if $a, b, c, \ldots l$, with or without repetitions, are the distinct prime factors of the denominator of a fraction, the fraction itself may be resolved into the sum of simple fractions,

$$\frac{A}{a^\alpha} + \frac{B}{b^\beta} + \frac{C}{c^\gamma} + \&c. + \frac{L}{l^\lambda}$$

(itself a direct inference from the familiar theorem that if p, q be any two relative primes, the equation $px - qy = c$ is soluble in integers for all values of c). The lemma in question is as follows: If the quantity above described is representable under the several forms,

$$\frac{a'}{a^f} + \text{an } (a\text{th}) \text{ integer}, \quad \frac{b'}{b^g} + \text{a } (b\text{th}) \text{ integer}, \ldots \frac{l'}{l^k} + \text{a } (k\text{th}) \text{ integer},$$

then it is equal to

$$\frac{a'}{a^f} + \frac{b'}{b^g} + \ldots + \frac{l'}{l^k} + \text{an } \textit{absolute} \text{ integer}.$$

From what has been already shown, it is obvious that μ being any prime number, the highest power of μ which *can* enter into the denominator of $(\mu^{2n} - 1) B_n$ is μ^{2n}, and consequently $\mu^{2n} B_n$ is an integer relative to μ. Also it is clear that only those values of μ can appear in the denominator of B_n which, diminished by unity, are factors of $2n$. We have, moreover,

$$(-)^{n-1} (\mu^{2n} - 1) B_n = \Pi (2n) \times \text{coefficient of } t^{2n-1} \text{ in } \frac{\mu}{e^{\mu t} - 1} - \frac{1}{e^t - 1},$$

that is, coefficient of t^{2n-1} in $\dfrac{-N}{e^{\mu t} - 1}$, where

$$N = \Pi (2n) \{e^{(\mu-1)t} + e^{(\mu-2)t} + \ldots + e^t - (\mu - 1)\}$$

$$= \nu_1 t + \nu_2 t^2 + \ldots\ldots + \nu_{2n} t^{2n} + \&c.,$$

where obviously $\nu_1, \nu_2, \ldots \nu_{2n}$ are all integers, and the last of them

$$= (\mu - 1)^{2n} + (\mu - 2)^{2n} + \ldots + 2^{2n} + 1^{2n}.$$

proper, for both of them are fractions in their simplest forms, which would not be the case for the former were c equal to or greater than p, since in such case $\frac{c}{p^i}$ could be more simply expressed under the form $\frac{\gamma}{p^{i-1}} + \frac{\gamma'}{p^i}$.

This principle amounts to an affirmation that the equation in positive integers,

$$(b \ldots kl)\, x + (ab \ldots l)\, y + \ldots + (ab \ldots k)\, t - (ab \ldots kl)\, u = N,$$

where $a, b, \ldots k, l$ are relative primes, and $N < (ab \ldots kl)$, always admits of a solution, which may be termed the primitive one, and which will be unique, that namely in which $x, y, \ldots z, t$ are respectively less than $a, b, \ldots k, l$.

Suppose now that $2n$ contains $(\mu-1)$, then by Fermat's theorem

$$\nu_{2n} \equiv (\mu - 1) \quad [\text{mod } \mu].$$

Again, a very slight consideration* will serve to show that when μ is any prime other than 2, $e^{\mu t}-1$ is of the form

$$\mu\,(t + \mu\delta_1 t^2 + \mu\delta_2 t^3 + \ldots + \mu\delta_{2n-1}t^{2n} + \&\text{c.}),$$

where $\delta_1, \delta_2, \ldots \delta_{2n-1} \ldots$ are all integers *relative* to μ. Now suppose

$$\frac{\mu N}{e^{\mu t}-1} = q_0 + q_1 t + q_2 t^2 + \ldots + q_{2n-1}t^{2n-1} + \&\text{c.};$$

then by multiplication and comparison of coefficients we obtain the identities following:

$$q_0 = \nu_1, \quad q_1 + \mu q_0 \delta_1 = \nu_2, \quad q_2 + \mu q_1 \delta_1 + \mu q_0 \delta_2 = \nu_3, \ \ldots$$
$$q_{2n-1} + \mu q_{2n-1}\delta_1 + \ldots + \mu q_0 \delta_{2n-1} = \nu_{2n};$$

obviously therefore $q_{2n-1} = \mu \times$ (an integer relative to μ) $+ \nu_{2n}$. Hence

$$(-1)^n B_n = (\text{an integer relative to } \mu) - \frac{\nu_{2n}}{\mu}$$
$$= (\text{an integer relative to } \mu) + \frac{1}{\mu}.$$

And this relation obtains for any value of μ other than 2, which (or a power of which) *could* be contained in $2n$. When $\mu = 2$, the δ series will *not* all of them be the doubles of relative integers to 2; but the ν series, on account of the factor $\Pi(2n)$, will obviously, up to ν_{2n-1} inclusive, all contain 2 and $\nu_{2n} = 1$; consequently q_{2n} will be twice (an integer *quâ* 2) + 1, and B_n will

* For μ being a prime number greater than 2, if we put $\frac{\mu^r}{\Pi(r)}$ (the coefficient of t^r in $e^{\mu t}-1$) under the form of (an integer *quâ* μ) $\times \mu^i$, we have

$$i = r - E\left(\frac{r}{\mu}\right) - E\left(\frac{r}{\mu^2}\right) - E\left(\frac{r}{\mu^3}\right) - \&\text{c.}$$
$$= \text{ or } > r - \frac{r}{\mu-1} = \text{ or } > \frac{r}{2} > 1 \text{ when } r > 2\text{; also when } r = 2,\ i = 2 - E\left(\frac{2}{\mu}\right) = 2.$$

When $\mu = 2$, this would be no longer true; and in fact it is easily seen that in this case, whenever r is a power of 2, i will be only equal to 1.

For the benefit of my younger readers, I may notice that the *direct* proof of the theorem that the product of any r consecutive numbers must contain the product of the natural numbers up to r, or, in other words, that the trinomial coefficient $\frac{\Pi n}{\Pi\nu\Pi\nu'}$, where $\nu+\nu'=n$, is an integer, is drawn from the fact that this fraction may be represented as an integer *quâ* μ (any prime) multiplied by μ^i, where

$$i = \left[E\left(\frac{n}{\mu}\right) - E\left(\frac{\nu}{\mu}\right) - E\left(\frac{\nu'}{\mu}\right)\right] + \left[E\left(\frac{n}{\mu^2}\right) - E\left(\frac{\nu}{\mu^2}\right) - E\left(\frac{\nu'}{\mu^2}\right)\right] + \&\text{c.}$$

($E(x)$ meaning the integer part of x), so that i is necessarily either zero or positive, because the value of each triad of terms within the same parenthesis is essentially zero or positive. This is the natural and only direct procedure for establishing the proposition in question.

still be (an integer relative to μ) $+\frac{1}{\mu}$ as before. Hence it follows from the lemma that $(-1)^n B_n =$ an absolute integer $+ \Sigma \frac{1}{\mu}$, or

$$B_n = \text{an integer} + (-)^n \Sigma \frac{1}{\mu},$$

which is the equation expressed by the Staudt-Clausen theorem*.

My researches in the theory of partitions have naturally invested with a new and special interest (at least for myself) everything relating to the Bernoullian numbers. I am not aware whether the following expression for a Bernoullian of any order as a quadratic function of those of an inferior order happens to have been noticed or not. It may be obtained by a simple process of multiplication, and gives a means (not very expeditious, it is true) for calculating these numbers from one another without having recourse to the calculus of differences or Maclaurin's theorem, namely

$$-\frac{B_n}{\Pi(2n)} = (2^2-1)\frac{B_1}{\Pi(2)}\cdot\frac{B_{n-1}}{\Pi(2n-2)} + (2^4-1)\frac{B_2}{\Pi(4)}\cdot\frac{B_{n-2}}{\Pi(2n-4)}$$

$$+ \&\text{c.} \ldots + (2^{2n-4}-1)\frac{B_{n-2}}{\Pi(2n-4)}\cdot\frac{B_2}{\Pi(4)}$$

$$+ (2^{2n-2}-1)\frac{B_{n-1}}{\Pi(2n-2)}\cdot\frac{B_1}{\Pi(2)},$$

in which formula the terms admit of being coupled together from end to end, excepting (when n is even) one term in the middle.

To illustrate my law respecting the numerators of the numbers of Bernoulli, and its connexion with the known law for the denominators, suppose twice the index of any one of these numbers to contain the factor $(p-1)p^i$, where p is any prime; then this number will contain the first power of p in its denominator; but if the factor p^i is contained in double the index in question, but $(p-1)$ *not*, then p^i will appear bodily as a factor of the numerator.

* I ought to observe that in all that has preceded I have used the word *integer* in the sense of positive or negative integer, and the demonstration I have given holds good without assuming B_n to be positive. That this is the case, or, in other words, that the signs of the successive powers in $\frac{e^t-1}{e^t+1}$ are alternately positive and negative, may be seen at a glance by putting $t=2\sqrt{(-1)}\,\theta$, and remembering that all the coefficients in the series for $\tan\theta$ in terms of θ are necessarily positive, because $\left(\frac{d}{d\theta}\right)^i \tan\theta$ obviously only involves positive multiples of powers of $\tan\theta$ and $\sec\theta$.

It has occurred to me that it might be desirable to adhere to the common definition of "*Bernoulli's numbers,*" but at the same time to use the term Bernoulli's *coefficients* to denote the actual coefficients in $\frac{e^t+1}{2(e^t-1)}$; so that if the former be denoted in general by B_n and the latter by β_n, we shall have

$$\beta_{2n} = (-)^{n-1} B_n,$$

$$\beta_{2n+1} = 0.$$

In the absence of some such term as I propose, many theorems which are really single when affirmed of the *coefficients*, become duplex or even multifarious when we are restrained to the use of the *numbers* only.

Postscript.—The results obtained concerning Bernoulli's numbers in what precedes, admit of being deduced still more succinctly; and this simplification is by no means of small importance, as it leads the way to the discovery of analogous and unsuspected properties of Euler's numbers (namely the coefficients of $\frac{\theta^{2n}}{\Pi(2n)}$ in the expansion of $\sec\theta$), and to some very remarkable theorems concerning prime numbers in general.

In fact, to obtain the laws which govern the denominators and numerators of Bernoulli's numbers, we need only to use the following principles:—(1) That μ being a prime*, $\Sigma\mu^n \equiv 0$, or $\equiv -1$ to the modulus μ, according as $\mu-1$ is not, or is, a factor of n,—the second part of this statement being a direct consequence of Fermat's theorem, the first part a simple inference from its inverse. (2) That $e^{\mu t}-1$ is of the form $\mu t + \mu^2 t^2 T$, where T is a series of powers of t, all of whose coefficients are integers relative to μ, except for the case of $\mu = 2$, when $e^{\mu t}-1$ is of the form $2t + 2t^2 T$. We have then

$$(\mu^{2n}-1)(-1)^n B_n = \Pi(2n) \,.\, \text{coefficient of } t^{2n-1} \text{ in } \frac{e^{(\mu-1)t} + e^{(\mu-2)t} + \ldots + e^t - (\mu-1)}{e^{\mu t}-1};$$

this by actual division (in virtue of principle (2)) $= I + \frac{R}{\mu}$, where I is an integer relative to μ, containing n, and $R = 1^{2n} + 2^{2n} + \ldots + (\mu-1)^{2n}$. Hence $(-)^n B_n =$ an integer relative to μ, or to such integer $+\frac{1}{\mu}$, according as $2n$ does not or does contain $(\mu-1)$, which proves the law for the numerators; and so if μ^i is a factor of n, but $(\mu-1)$ not a factor of $2n$, $\frac{R}{\mu}$ will vanish, and $\mu^{2n}-1$ will not contain μ; hence $(\mu^{2n}-1)B_n$, and consequently B_n will be the product of μ^i by an integer relative to μ, which proves my numerator law.

[* $\Sigma\mu^n$ denotes $1^n + 2^n + \ldots + (\mu-1)^n$.]

So by extending the same method to the generating function $\frac{1}{e^t + \sqrt{(-1)}}$, it may very easily be proved that if we write

$$\sec\theta = E_0 + E_1 \frac{\theta^2}{1.2} + E_2 \frac{\theta^4}{1.2.3.4} + \dots + E_n \frac{\theta^{2n}}{1.2.3 \dots 2n} + \text{\&c.},$$

every prime number μ of the form $4n+1$, such that $(\mu - 1)$ is a factor of $2n$, will be contained in E_n; and every such prime, when of the form $4n-1$, will be contained in $E_n + 2(-)^{n-1}$.

I call the numbers $E_1, E_2, \dots E_n$ Euler's 1st, 2nd, ... nth numbers, as Euler was apparently the first to bring them into notice. In the *Institutiones Calculi Diff.* he has calculated their values up to E_9 inclusive: in this last there is an error, which is specified by Rothe in Ohm's paper above referred to; had Euler been possessed of my law this mistake could not have occurred, as we know that $E_9 + 2$ ought to contain the factors 19 and 7, neither of which will be found to be such factors if we adopt Euler's value of E_9, but both will be such if we accept Rothe's corrected value. But in still following out the same method, I have been led, through the study of Bernoulli's and the allied numbers, and with the express aid of the former, to a perfectly general theorem concerning prime numbers, in which Bernoulli's numbers no longer take any part. Fermat's theorem teaches us the residue of $r^{\mu-1}$ in respect to μ, namely, that it is unity; but I am not aware of any theorem being in existence which teaches anything concerning the relation of $\frac{r^{\mu-1}-1}{\mu}$ to μ (or, which is the same thing, of the relation of $r^{\mu-1}$ to the modulus μ^2). I have obtained remarkable results relative to the above quotient, which I will state for the simplest case only, namely, that where r as well as μ is a prime number. I find that when r is any odd prime,

$$\frac{r^{\mu-1}-1}{\mu} \equiv \frac{c_1}{\mu-1} + \frac{c_2}{\mu-2} + \frac{c_3}{\mu-3} + \dots + \frac{c_{\mu-1}}{1}, \text{ (to mod. } \mu),$$

where $c_1, c_2, c_3, \dots c_{\mu-1}$ are continually recurring cycles of the numbers $1, 2, 3, \dots r$, the cycle beginning with that number r' which satisfies the congruence $\mu r' \equiv 1$ (mod. r). Since we know that

$$\frac{1}{\mu-1} + \frac{1}{\mu-2} + \frac{1}{\mu-3} + \dots + \frac{1}{1} \equiv 0 \text{ (to mod. } \mu)$$

in place of the cycle $1, 2, 3, \dots r$, we may obviously substitute the reduced cycle

$$-\frac{r-1}{2}, \ -\frac{r-3}{2}, \dots -1, 0, 1, \dots \frac{r-3}{2}, \ \frac{r-1}{2}.$$

Thus*, for example, $\dfrac{3^{\mu-1}-1}{\mu}$, when μ is of the form $6n+1$,

$$\equiv \frac{1}{\mu-1} - \frac{1}{\mu-3} + \frac{1}{\mu-5} - \frac{1}{\mu-7} \dots + 1, \text{ (to mod. } \mu),$$

and when μ is of the form $6n-1$,

$$\equiv \frac{-1}{\mu-2} + \frac{1}{\mu-3} - \frac{1}{\mu-4} + \frac{1}{\mu-5} \dots - 1, \text{ (to mod. } \mu).$$

When r is 2, the theorem which replaces the preceding is as follows†:

$$\frac{2^{\mu-1}-1}{\mu},$$

when μ is of the form $4n+1$,

$$\equiv \frac{1}{\mu-1} + \frac{1}{\mu-2} - \frac{1}{\mu-3} - \frac{1}{\mu-4} + \frac{1}{\mu-5} + \frac{1}{\mu-6}$$

$$- \frac{1}{\mu-7} - \frac{1}{\mu-8} + \frac{1}{\mu-9} \pm \text{\&c., (to mod. } \mu),$$

and when μ is of the form $4n-1$,

$$\equiv - \frac{1}{\mu-1} + \frac{1}{\mu-2} + \frac{1}{\mu-3} - \frac{1}{\mu-4} - \frac{1}{\mu-5}$$

$$+ \frac{1}{\mu-6} + \frac{1}{\mu-7} \mp \text{\&c., (to mod. } \mu).$$

When r is not a prime, a similar theorem may be obtained by the very same method, but its expression will be less simple. The above theorems would, I think, be very noticeable were it only for the circumstance of their involving (as a condition) the primeness as well of the base as of the augmented index of the familiar Fermatian expression $r^{\mu-1}$,—a condition which here makes its appearance in the theory of numbers (as I believe) for the first time.

[* Cf. the formulae at the top of p. 230 above. The second of these had originally a wrong sign throughout, but has been corrected, after a sentence inserted by the author at the end of the paper 40 above (p. 241), not reproduced here.]

[† The sign of every term in the two following expressions should be changed.]