35.

ON THE EQUATION

$$P(m) + E\left(\frac{m}{m-1}\right) P(m-1) + E\left(\frac{m}{m-2}\right) P(m-2) + \ldots + E(m) = m\frac{m+1}{2}.$$

[*Quarterly Journal of Mathematics*, III. (1860), pp. 186—190.]

$P(m)$ I use to denote the number of integers less than m and prime to it except when $m=1$, in which case $P(m)=1$. $E\left(\frac{m}{r}\right)$ I use to denote the integer part of $\frac{m}{r}$, or the whole of $\frac{m}{r}$ if $\frac{m}{r}$ is an integer.

Then evidently if we use $\frac{m}{r;}$ to denote *unity* when m contains r and *zero* in all other cases

$$E\left(\frac{m}{r}\right) - E\left(\frac{m-1}{r}\right) = \frac{m}{r;}.$$

Again, it is well known that the factors of any binomial function, as for instance $x^{12}-1$, are made up of the prime factors of all the binomial factors of $x^{12}-1$ as x^2-1, x^3-1, x^4-1, x^6-1, $x^{12}-1$, and consequently that

$$m = \frac{m}{1;} P(1) + \frac{m}{2;} P(2) + \frac{m}{3;} P(3) + \ldots + \frac{m}{m;} P(m),$$

which equation may also be easily proved independently (*vide* note at end).

Let now

$$E\left(\frac{m}{m}\right) P(m) + E\left(\frac{m}{m-1}\right) P(m-1) + E\left(\frac{m}{m-2}\right) P(m-2)$$
$$+ \ldots + E\left(\frac{m}{1}\right) P(1) = u_m.$$

Then
$$E\left(\frac{m-1}{m-1}\right) P(m-1) + E\left(\frac{m-1}{m-2}\right) P(m-2)$$
$$+ \ldots + E\left(\frac{m-1}{1}\right) P(1) = u_{m-1}.$$

Hence $u_m - u_{m-1} = \frac{m}{m;} P(m) + \frac{m}{m-1;} P(m-1) + \ldots + \frac{m}{1;} P(1) = m.$

Hence $$u_m = m\frac{m+1}{2} + C,$$

and since $u_1 = 1$ we must make $C = 0$, and

$$u_m = m\frac{m+1}{2},$$

as was to be shown.

NOTE.—Proof of the equation

$$P(m) + \frac{m}{(m-1);} P(m-1) + \frac{m}{(m-2);} P(m-2) + \ldots + 1 = m.$$

Let $a, b, c \ldots$ be the prime factors of m, so that

$$m = a^\alpha . b^\beta . c^\gamma \ldots,$$

and, for example, suppose

$$m = a^\alpha . b^\beta . c^\gamma.$$

Then the numbers contained in m may be divided into groups as follows: one group in which a, b, c all appear, another in which only two of the letters a, b, c appear, a third in which only one of them appears, and finally *unity* in which none of them appears.

The sum of the numbers of integers prime to m and less than it for the factors in the first group

$$= (a^{\alpha-1} + a^{\alpha-2} + \ldots + 1)(b^{\beta-1} + b^{\beta-2} + \ldots + 1)(c^{\gamma-1} + c^{\gamma-2} + \ldots + 1) \times (a-1)(b-1)(c-1)$$

$$= (a^\alpha - 1)(b^\beta - 1)(c^\gamma - 1).$$

In like manner the sum of the numbers of such integers for the factors in the second group

$$= (a^\alpha - 1)(b^\beta - 1) + (a^\alpha - 1)(c^\gamma - 1) + (b^\beta - 1)(c^\gamma - 1),$$

for the third group

$$= (a^\alpha - 1) + (b^\beta - 1) + (c^\gamma - 1),$$

and for *unity*

$$= 1.$$

Hence the total sum of such factors

$$= a^\alpha . b^\beta . c^\gamma$$

$$= m,$$

as was to be shown, and so in the like manner whatever may be the number of prime constituents $a, b, c \ldots$ in m.

Q. E. D.

P.S. 1. By successive integration the theorem first established may be generalized, and preserving the same notations as before, it emerges into the following proposition: [cf. the form below]

$$\Sigma_\infty^0{}_i\, P(i^r)\times\left[\frac{\left(E\frac{m}{i}\right)\left(E\frac{m}{i}+1\right)\ldots\left\{E\frac{m}{i}+(r-1)\right\}}{1\,.\,2\ldots r}\right]$$

$$=\Sigma_m^1{}_m\,(m^r).$$

Thus let $$r=2.$$

Then $$\Sigma_\infty^0{}_i\, P(i^2)\left\{\frac{\left(E\frac{m}{i}\right)\left(E\frac{m}{i}+1\right)}{2}\right\}$$

$$=\Sigma m^2=\frac{m(m+1)(2m+1)}{2\,.\,3},$$

or observing that

$$P(i^q)=i^{q-1}\,.\,P(i),$$

$$\Sigma_\infty^0{}_i\, iP(i)\left\{E\left(\frac{m}{i}\right)E\left(\frac{m}{i}+1\right)\right\}$$

$$=\frac{m(m+1)(2m+1)}{3}.$$

Example, let $$m=5,$$

$$5P(5)=20,$$

$$4P(4)=8,\quad E(\tfrac{5}{4})=1,$$

$$3P(3)=6,\quad E(\tfrac{5}{3})=1,$$

$$2P(2)=2,\quad E(\tfrac{5}{2})=2,$$

$$E(\tfrac{5}{1})=5,$$

$$20\times2+8\times2+6\times2+2\times6+5\times6$$

$$=110,$$

$$\frac{5\times6\times11}{3}=110.$$

Or we may use the theorem under the form following:

$$\Sigma_n^1{}_i\left[P(i^r)\times S\left\{E\left(\frac{n}{i}\right)\right\}^{r-1}\right]=S(n^r),$$

where it is to be observed that

$$Sq^r \text{ means } 1^r+2^r+\ldots+q^r.$$

Example, let $r=3,$

then $$S\left(E\frac{n}{i}\right)^2 = \frac{\left(E\frac{n}{i}\right)\left(E\frac{n}{i}+1\right)\left(2E\frac{n}{i}+1\right)}{2\,.\,3},$$

$$Sn^3 = \left\{n\left(\frac{n+1}{2}\right)\right\}^2,$$

accordingly $$\Sigma_n^1 i\left\{P(i^3)\times\frac{\left(E\frac{n}{i}\right)\left(E\frac{n}{i}+1\right)\left(2E\frac{n}{i}+1\right)}{6}\right\}$$

$$=\left(n\frac{n+1}{2}\right)^2.$$

Thus let $n=4,$

then $$E(\tfrac{4}{4})=1,\ \frac{1\,.\,2\,.\,3}{2\,.\,3}=1,\ \ P(4^3)=16\times 2=32,$$

$$E(\tfrac{4}{3})=1,\ \frac{1\,.\,2\,.\,3}{2\,.\,3}=1,\ \ P(3^3)=\ \ 9\times 2=18,$$

$$E(\tfrac{4}{2})=2,\ \frac{2\,.\,3\,.\,5}{2\,.\,3}=5,\ \ P(2^3)=\ \ 4\times 1=\ \ 4,$$

$$E(\tfrac{4}{1})=4,\ \frac{4\,.\,5\,.\,9}{2\,.\,3}=30,\ P(1^3)=\qquad\qquad 1,$$

$$32+18+20+30=100,$$

$$\left(\frac{4\,.\,5}{2}\right)^2=100.$$

P.S. 2. The fundamental theorem in its simplest terms is as follows: If $i_1, i_2 \dots i_r$ be any arbitrary positive integers

$$n^r = (\Sigma)^r\left[P\{(i_1)^{r-1}\}\,P\{(i_2)^{r-2}\}\dots P(i_{r-1})\times\frac{n}{i_1 i_2\dots i_r}\right];$$

the $(\Sigma)^r$ meaning merely the sign of summation r times repeated.

Example, let $r=2,\quad n=4,$

4 is divisible by $1\times 1,\ 2\times 1,\ 4\times 1,$

$$1\times 2,\ 2\times 2,$$

$$1\times 4,$$

$$P(1)=1,\ P(2)=1,\ P(4)=2,$$

$$1\times 1\times 1+1\times 1\times 1+1\times 1\times 2$$

$$+2\times 1\times 1+2\times 1\times 1$$

$$+4\times 2\times 1$$

$$=4+4+8=16=4^2.$$

It is obvious that this theorem must be capable of being reduced to an algebraical identity by writing $n=a^\alpha.b^\beta.c^\gamma\dots$ as I have shown in the *note* above for the case $r=1$.

The proof is left to the ingenuity of the reader.