31.

ON PONCELET'S APPROXIMATE LINEAR VALUATION OF SURD FORMS.

[*Philosophical Magazine*, xx. (1860), pp. 203—222.]

M. PONCELET'S method of approximately representing surd forms, and more particularly the square roots of homogeneous quadratic functions, by linear functions of the variables, is given in *Crelle's Journal*, Vol. XIII. 1834, pp. 277—291, under the title "Sur la Valeur approchée des radicaux." By this method, as applied to two variables, the resultant of two forces in a plane may be approximately expressed as a linear function of its two components, a case fully considered by M. Poncelet; and tables have been worked out applicable to this case, which appear to have been found of great utility in some important problems of mechanical and practical engineering. But the illustrious author of this beautiful method has left his theory imperfect in respect of its application to three variables.

To supply this slight but not unimportant omission, and to indicate how this more general case admits of being treated, more especially with reference to the approximate representation of the resultant of three forces in space as a linear function of its three components, is the object of this communication. At the close of the memoir referred to, M. Poncelet uses these words:—"Il serait inutile de pousser plus loin cet examen (referring to a discussion of the form $\sqrt{(a^2-b^2)}$), attendu que dans les applications de la mécanique aux machines les radicaux de la forme $\sqrt{(a^2-b^2)}$ sont rarement à considérer. Nous en dirons autant de ceux de la forme $\sqrt{(a^2+b^2+c^2)}$, qui représentent la résultante de trois forces rectangulaires entre elles et situées dans l'espace. D'ailleurs, si l'on connait les limites entre lesquelles demeurent compris les rapports des composantes a, b, c, ou de leurs résultantes partielles $\sqrt{(a^2+b^2)}$, &c., on pourra toujours ramener ce cas au premier de ceux que nous avons examinés," meaning to the case of $\sqrt{(a^2+b^2)}$. Now, in the first place, it is not clear how this reduction can be effected in general, or indeed in the vast majority of cases that might be proposed. For instance, if we have given

$a < \sqrt{(b^2+c^2)}$, $a > b$, $a > c$, I do not see how after, according to M. Poncelet's process, $\sqrt{(a^2+b^2+c^2)}$ is put under the form $\alpha a + \beta\sqrt{(b^2+c^2)}$ by aid of the limit $a < \sqrt{(b^2+c^2)}$, any use can be made of the other limits $a > b$, $a > c$ in further reducing this to the ultimate form $\alpha a + \alpha'\beta b + \beta'\beta c$. Or if we take the still simpler case, where a, b, c are left unlimited, in whatever way we attempt to proceed we shall obtain different approximations, according to the order in which we effect the successive reductions.

Furthermore, in those few exceptional cases where the process indicated by M. Poncelet leads to the use of all the limits given, the form arrived at is not and never can be the true *best* form, defined as such, according to M. Poncelet's own principles, as that which within the given limits has its *maximum* proportional error the least possible. Thus M. Poncelet indicates as the linear form for $\sqrt{(a^2+b^2+c^2)}$, when the given limits are $a^2 > b^2+c^2$, $b^2 > c^2$, $\cdot 96046a + \cdot 38201b + \cdot 15827c$, with a maximum error textually quoted from his memoir, ·0507. It will be seen hereafter that the true best linear form gives a maximum error about one-tenth less than this. But it would be quite easy to give examples in which the maximum error by Poncelet's process should exceed in an indefinite proportion the necessary maximum error. This, for instance, would be the case if we imposed the limitations

$$x^2+y^2 > \lambda z^2, \quad y^2+z^2 > \lambda x^2, \quad z^2+x^2 > \lambda y^2,$$

on taking λ inferior but indefinitely near to 2.

The geometrical method of demonstration given by M. Poncelet for the case of two variables, labours under the inconvenience of *beginning* with a figure of three dimensions, and consequently does not admit of being carried beyond that case, although the results for three variables geometrically stated, when the conditions of the question are set under an appropriate form, are precisely analogous to that obtained by M. Poncelet for two variables; for whilst his construction is begun in space, his result subsides to a representation *in plano*. But between these two cases there is a very marked distinction; which is, that whilst for a surd radical with two variables every change in the limits proposed gives rise to a change in the corresponding linear form, such is never the case with a surd form with three or more variables, unless the limits be expressed by a *single linear* inequality between the variables which enter into the surd form, and the surd form itself. Thus, for instance, if $\sqrt{(x^2+y^2+z^2)}$ is to be represented linearly within the limits $z > x$, $z > y$ (for greater conciseness I throughout suppose the variables to be positive), the linear representation will be precisely the same as for the single limit $z > \sqrt{(x^2+y^2)}$, or, which is the same thing, $z - \sqrt{\tfrac{1}{2}}\,\sqrt{(x^2+y^2+z^2)} > 0$; and accordingly for the problem with three variables there is usually a preliminary question to be solved, namely, to find the single inequality of the kind proposed which involves the satisfaction of the given limits, and is capable of being substituted for them without increasing the maximum proportional error. This preliminary question may be reduced, as will be seen, to an elementary geometrical form, and is strictly tantamount to the problem following:—Imagine a pincushion with a number of pins stuck into it, to find the least ring which can be made to take them all in,—a problem proposed by myself some four or five years ago with reference to points in a plane, in the *Quarterly Mathematical Journal*, and of which Professor Peirce of Cambridge University, U.S., has favoured me with a complete solution, which is equally applicable to the sphere, the case with which we shall be principally concerned in what follows.

I shall begin, then, with supposing R to be an integer homogeneous quadratic function of x, y, z, where x, y, z, R are subject to the linear inequality $Ax+By+Cz-\sqrt{R}>0$. The geometrical solution, as such, will be seen to be equally applicable to the case of two, and the analytical representation to which it leads to any number of variables.

The problem to be solved is to find a linear form $Lx+My+Nz$ such that the greatest value of $\dfrac{Lx+My+Nz}{\sqrt{R}}-1$ shall have the least possible arithmetical magnitude, without regard to *sign* as positive or negative, for all values of x, y, z satisfying the proposed inequality.

It is clear that, as the entire question is one of ratios, we may subject x, y, z to the condition expressed by $R=1$ without affecting the result; in other words, we may consider x, y, z as the coordinates of a point limited to lie on the segment of the surface $R=1$ cut off by the plane $Ax+By+Cz=1$. Suppose, then, that $Lx+My+Nz$ is the linear form sought. The proportional error is $Lx+My+Nz-1$; so that if we draw the plane

$$Lx+My+Nz-1=0,$$

the error is expressible geometrically (paying no attention to sign) as the quotient of the perpendicular upon this plane from any point x, y, z in the segment, namely, $\dfrac{Lx+My+Nz-1}{\sqrt{(L^2+M^2+N^2)}}$, divided by the perpendicular from the origin to the same plane, namely, $\dfrac{1}{\sqrt{(L^2+M^2+N^2)}}$. Hence, then, the geometrical question to be resolved is simply to draw a plane for which the greatest value of this quotient, restricted to points within the segment, shall be the least possible. From this it is immediately seen to follow, that the portion of the surface cut off by the plane $Lx+My+Nz-1=0$ must be a portion of the segment cut off by the given plane $Ax+By+Cz-1=0$. And its actual position may be determined by means of a principle generally known, but which, as it will occupy but a few words, it may be well to deduce from first principles.

Suppose there are $(r+1)$ quantities, each containing the same system of r parameters; for greater brevity, say three quantities, p, q, r, each functions of the same two parameters λ, μ: let us call the greatest of the quantities p, q, r, corresponding to assigned values of λ, μ, the *dominant*; so that, according as we change λ, μ, the name of the dominant is liable to change; and that we wish to find M the minimum value of the dominant upon the supposition that the variations of p, q, r in respect to λ or μ are never simultaneously zero, and may be made positive or negative at will; then M will be found from the equations $M = p = q = r$. For if we had $M = p$ and $p > q$, $p > r$, by varying at will λ or μ we could make δp negative; and consequently since by hypothesis p differs sensibly from q and r, the dominant of $p + \delta p$, $q + \delta q$, $r + \delta r$ would necessarily be less than that of p, q, r, and thus M would not be the minimum dominant.

In like manner, if $M = p = q$, $p > r$, we could by means of the equations

$$\frac{dp}{d\lambda}\delta\lambda + \frac{dp}{d\mu}\delta\mu = -\epsilon,$$

$$\frac{dq}{d\lambda}\delta\lambda + \frac{dq}{d\mu}\delta\mu = -\eta,$$

so determine $\delta\lambda$, $\delta\mu$ as to diminish simultaneously p and q; and thus the dominant of $p - \epsilon$, $q - \eta$, $r + \delta r$ would, as before, be less than that of p, q, r. The same reasoning applies to any number $(r+1)$ functions of r variables. And if the number of functions should exceed $r+1$, it would still serve to show that when the dominant is a minimum, $(r+1)$ out of the whole number of the functions must all alike represent that dominant. Thus leaving for a moment in our original problem the case of three variables, and going down to that of only two variables, in which case we have to deal with a curve of the second order in lieu of a surface, and are to suppose that a segment of such curve is cut off by a right line A, and are required to draw another right line B such that the maximum square of the quotient of a perpendicular upon B from any point in the segment by the perpendicular from the centre upon B is to be a minimum, we evidently have to solve the same problem as if we had to find the least value of the dominant of three quantities involving two parameters, two being the number of constants required to fix the line B; those three quantities being the squares of the fractions whose numerators are the three perpendiculars from the extremities of A, and from the vertex of the arc cut off by B upon B, and their denominators the perpendicular upon B from the origin; accordingly the line B must be so chosen as to make the three perpendiculars in the numerators, without reference to sign, all equal, so that B is parallel to A, and bisects the sagitta of the segment cut off by A, that is, the longest perpendicular from any point in the segment upon A.

In the case of R being, as originally supposed, a function of x, y, z, we may take an indefinite number of points in the section of the surface $R=1$ made by the plane $Ax+By+Cz-1=0$, and the summit of the segment made by the plane to be determined $Lx+My+Nz=1$, and may show by the same reasoning as above (there being now three parameters) that four of these perpendiculars must be equal *inter se*, which proves, to begin with, that at all events the two planes must be parallel; and then the reasoning applied to two functions of one parameter will further show that this plane must bisect the sagitta of the segment cut off by the *given plane* $Ax+By+Cz-1=0$*. And we have now a geometrical solution of the question, which it is important to observe is in general, but, as will be presently seen, not universally applicable to the case when the limiting relations of x, y, z are defined by means of the position of a variable point limited to lie within a triangular area upon the surface $R=1$, whose sides are determined by the traces upon that surface of three planes drawn through the origin; the plane drawn through the angular points of this triangle will then take the place of the plane $Ax+By+Cz-1=0$ in the preceding investigation.

The next thing to be done is to obtain the quantities L, M, N in terms of A, B, C, and the coefficients of R, which is an easy matter to accomplish. Let

$$R = ax^2+by^2+cz^2+2fyz+2gzx+2hxy=\phi(x, y, z),$$

and call ξ, η, ζ the coordinates at the summit of the segment; the equation to the tangent plane at that point, which is of the form $Ax+By+Cz=0$, will be identical with

$$(a\xi+h\eta+g\zeta)X+(h\xi+b\eta+f\zeta)Y+(g\xi+f\eta+c\zeta)Z=1.$$

Hence
$$a\xi+h\eta+g\zeta=\frac{A}{\sigma},$$

$$h\xi+b\eta+f\zeta=\frac{B}{\sigma},$$

$$g\xi+f\eta+c\zeta=\frac{C}{\sigma},$$

and
$$\frac{A}{\sigma}\xi+\frac{B}{\sigma}\eta+\frac{C}{\sigma}\zeta=1;$$

* The absolute liberty of the plane sought for ($Lx+My+Nz=1$) to take up all positions in space, and the absence of singular points in the segment cut off by the plane $Ax+By+Cz=1$, suffice to show that the conditions of variation necessary for the legitimate application of the theorem employed above are satisfied. If the minimum dominant is not at one of the *points of equality* given by the theorem, it must lie either at some minimum, or at all events at some singular point of one of the functions of the system to which the dominant belongs, or else at some point corresponding to the contour, so to say, if there be one, of the space within which the parameters are contained. In the case before us, the parameters, however chosen, to fix the position of the plane are perfectly independent, so that there is no limiting contour; and it is obvious that the functions representing the distances concerned from this variable plane have no

and therefore $$\frac{1}{\sigma^2}\frac{P\phi(A, B, C)}{\Delta\phi(A, B, C)} = 1,$$

where $\Delta\phi$ is the *discriminant*, and $P\phi$ the *polar reciprocal* of $\phi(A, B, C)$. Hence

$$\sigma = \sqrt{\frac{P}{\Delta}}\,*,$$

and the perpendicular upon the tangent plane is

$$\frac{1}{\sqrt{(A^2+B^2+C^2)}}\sqrt{\frac{P}{\Delta}}.$$

Consequently the mean between this and the perpendicular upon the given plane is

$$\frac{1}{\sqrt{(A^2+B^2+C^2)}}\frac{\sqrt{P}+\sqrt{\Delta}}{2\sqrt{\Delta}};$$

and therefore the equation to the plane required is

$$Ax + By + Cz = \frac{\sqrt{P}+\sqrt{\Delta}}{2\sqrt{\Delta}},$$

so that $$L = \frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}A,\quad M = \frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}B,\quad N = \frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}C,$$

$Lx + My + Nz$ being the approximate representation of $\sqrt{\{\phi(x, y, z)\}}$, and the maximum error being evidently

$$\frac{\sqrt{P}-\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}.$$

These results are perfectly general, and apply to a quadratic radical of an integer homogeneous quadratic function of any number of variables; thus for $\sqrt{\{\phi(x, y, z, t)\}}$ the linear representative form is

$$\frac{2\sqrt{\Delta}\,.\,A}{\sqrt{P}+\sqrt{\Delta}}x + \frac{2\sqrt{\Delta}\,.\,B}{\sqrt{P}+\sqrt{\Delta}}y + \frac{2\sqrt{\Delta}\,.\,C}{\sqrt{P}+\sqrt{\Delta}}z + \frac{2\sqrt{\Delta}\,.\,D}{\sqrt{P}+\sqrt{\Delta}}t,$$

maxima or minima values. I do not (nor ought I to) pretend to have presented the theoretical principles involved in the limitation of the general *law of equality* with all the logical rigour and precision of which the subject might admit, as this would be beside my present object, which is not to call in question the grounds of admitted truth applicable to the question in hand, but to advance it one step further in the direction of practical application.

* We see from the above, that if $Ax+By=1$, or $Ax+By+Cz=1$ be the equation to the chordal line or plane of a segment of a line or surface of the second degree, the ratio of the perpendiculars to such line or plane from the centre of the line or surface and the vertex of the segment respectively, or, which is the same thing, of a ray to any point in the segment to the portion of this ray produced, intercepted between the line or surface and the tangent at the vertex, is expressed by $\sqrt{\Delta} : \sqrt{P}$. It may at first sight appear strange that P should be of the *form* of a *contravariant* (in lieu of a covariant); but it must be remembered that the axes to which the line or surface and its chord are referred are supposed to be orthogonal, and for orthogonal substitutions, contravariants and covariants are indistinguishable.

and the greatest proportional error is still

$$\frac{\sqrt{P}-\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}};$$

D signifying the discriminant, and P the polar reciprocal of $\phi(A, B, C, D)$.

For the sphere, the perpendicular upon any tangent plane being 1, the linear form ought to be that obtained from the equation $Ax + By + Cz = K$, where

$$\frac{K}{\sqrt{(A^2+B^2+C^2)}} = \frac{1}{2}\left(1+\frac{1}{\sqrt{(A^2+B^2+C^2)}}\right),$$

or
$$K = \tfrac{1}{2}\{\sqrt{(A^2+B^2+C^2)}+1\},$$

that is to say, the approximation is

$$\frac{2A}{1+\sqrt{(A^2+B^2+C^2)}}\,x + \&\text{c.},$$

the maximum error being

$$\frac{\sqrt{(A^2+B^2+C^2)}-1}{\sqrt{(A^2+B^2+C^2)}+1},$$

which is easily seen to agree with the general formulæ above given.

When, as is usually the case in applying these results, the plane $Ax + By + Cz - 1 = 0$ is not directly given, but is to be found as the plane passing through three given points whose coordinates are a, b, c; a', b', c'; a'', b'', c'' respectively, we may use the equations

$$A = \frac{F}{Q}, \quad B = \frac{G}{Q}, \quad C = \frac{H}{Q},$$

where
$$F = (b'c'' - b''c') + (b''c - bc'') + (bc' - b'c),$$
$$G = (c'a'' - c''a') + (c''a - ca'') + (ca' - c'a),$$
$$H = (a'b'' - a''b') + (a''b - ab'') + (ab' - a'b),$$
$$Q = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}.$$

But it may also sometimes be needful in practice, as will presently appear, to determine the plane with immediate reference to only two points upon the surface.

Application to the surd form which represents the resultant of three forces at right angles to each other.

Here $R = \sqrt{(x^2+y^2+z^2)}$, and $R=1$ represents a sphere. Two cases will be shown to arise. The first, the more frequent one, is that already alluded to, where a limiting plane has to be drawn through three given points. For this case, using F, G, H in the sense in which they have immediately above been employed, the linear representation of $\sqrt{(x^2+y^2+z^2)}$ becomes

$$\frac{2F}{Q+N}x + \frac{2G}{Q+N}y + \frac{2H}{Q+N}z,$$

with a maximum proportional error

$$\frac{N-Q}{N+Q},$$

N representing

$$\sqrt{(F^2+G^2+H^2)}.$$

The second case is where the limiting plane has to be drawn through two points upon the sphere so as to cut it in a circle, of which the line joining the two points is a diameter.

In this case, calling the coordinates of the two points respectively α, β, γ; α', β', γ', and writing $\alpha\alpha' + \beta\beta' + \gamma\gamma' = m$, it is easily seen that the perpendicular upon the limiting plane is $\sqrt{\frac{1+m}{2}}$, and consequently the perpendicular upon the plane

$$Lx + My + Nz = 1 \text{ is } \frac{1}{2}\left\{1 + \frac{\sqrt{(1+m)}}{2}\right\}.$$

Also this plane being parallel to the limiting plane, is perpendicular to the line joining the origin to the point

$$x : y : z :: \frac{\alpha+\alpha'}{2} : \frac{\beta+\beta'}{2} : \frac{\gamma+\gamma'}{2},$$

and therefore

$$L = \frac{\alpha+\alpha'}{\rho}, \quad M = \frac{\beta+\beta'}{\rho}, \quad N = \frac{\gamma+\gamma'}{\rho},$$

and
$$\frac{\rho}{\sqrt{\{(\alpha+\alpha')^2+(\beta+\beta')^2+(\gamma+\gamma')^2\}}} = \frac{1}{2}\left\{1+\sqrt{\frac{1+m}{2}}\right\};$$

that is to say,

$$\rho = \sqrt{\{2(1+m)\}} \cdot \frac{1}{2}\left(1+\sqrt{\frac{1+m}{2}}\right)$$
$$= \tfrac{1}{2}\left[\sqrt{\{2(1+m)\}} + (1+m)\right];$$

so that the linear form required is

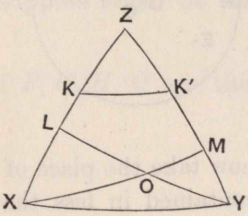
$$[\sqrt{\{2(1+m)\}}+1+m]\left\{\frac{\alpha+\alpha'}{2}x+\frac{\beta+\beta'}{2}y+\frac{\gamma+\gamma'}{2}z\right\},$$

with a maximum proportional error

$$\frac{\sqrt{2}-\sqrt{(1+m)}}{\sqrt{2}+\sqrt{(1+m)}}.$$

(m is of course identical with the cosine of the angle between the radii joining the two given points.)

The conditions of inequality which obtain between x, y, z may be, and usually will be, such as correspond to the limitation of the point (x, y, z) to an area contained within a triangle or polygon upon the surface of the sphere. Thus take X, Y, Z each a quadrant apart from the other, the points where the surface of the sphere $x^2+y^2+z^2=1$ is pierced by the axes. If no limitation is placed upon the values of x, y, z further than the one throughout supposed of their remaining always positive, the limiting area will be XYZ. If we suppose

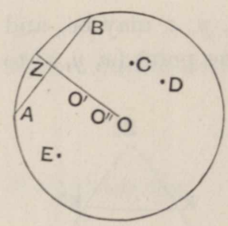


$$z > k\sqrt{(x^2+y^2)},$$

we may take $\tan XK = k$, and drawing the small circle KK', ZKK' will be the limiting area; if, again, $z < k\sqrt{(x^2+y^2)}$, $KK'YX$ will be the limiting area; if, again, $z < k\sqrt{(x^2+y^2)}$, $z > lx$, $z > my$ be the limiting conditions, taking $\tan LX = l$, $\tan MY = m$, and drawing LY, XM to intersect in O, $KK'MOL$ will be the corresponding area, and so in general. Even so simple a set of conditions as $z > x$, $z > y$ it is seen will give rise to a quadrilateral area, limited in the figure by $ZLOM$, when $ZL = ZM = 45°$. Thus, then, we approach the preliminary question to which allusion has been already made, which is to determine the *least circle* that will cut off from a given sphere a segment containing a given system of points lying upon it. The solution is precisely the same, substituting arcs of great circles for right lines, as the problem of drawing upon a plane the least circle containing a set of points given in the plane.

We may, in the first place, obviously reject all those points that are contained within the contour formed by arcs joining the remaining points, so that the case of points lying at the angles of a convex polygon alone remains to be studied. Now if we confine our attention even to the simplest case of a system of three points, we shall see at once that two cases arise. If a circle be drawn through them, and these three points do not lie in the same semicircle, no smaller circle than this can be drawn to contain the three; but if they do lie in the same semicircle, it is obvious that a circle described upon the line joining the outer two as a diameter will be smaller than the circle passing through all three, and will contain them all. It was this simple but striking fact in the geometry of situation which led me to propose the question for any number of points in the *Quarterly Mathematical Journal*; and as Prof. Peirce's exhaustive method of solution has not appeared in print, I may take this occasion of presenting it.

Let A, Z, B, C, D, E be the given points. Let AZB be a circle whose centre is drawn through A, Z, B, chosen so as to include all the others; then if A, Z, B are not contained in the same semicircle, AZB is the circle required. But if AZB be less than a semicircle, as in the figure, we may first reject the consideration of all the points contained between the arc AB and its chord. We must then find O', O'', &c., the centres of the circles passing through A, B, C; A, B, D, &c.: these will all lie in the same straight line $O'O''O$. Selecting the one nearest to O, say O'', we describe the corresponding circle, in which AC will now take the place of AB in the former circle. If the points A, B, C are not contained in less than a semicircle, that is, if ABC is an acute-angled or right-angled triangle, ABC is the circle required; but if they do lie within the same semicircle so that ABC forms an obtuse angle, B will now have to be rejected, and we must find a new centre as before, and so on continually. By this process we must inevitably at last exhaust all the given points; and the final circle so obtained will be the circle sought, unless the three points through which it has been drawn are distributed over the same semicircle, in which case the circle required is that described upon the chord joining the two extreme points as its diameter. The solution will evidently be *unique*, and (as already hinted at) merely require the construction upon the sphere either of a circle passing through a certain set of three out of all the given points, or else passing through only two of them, so as to be perpendicular to the radius bisecting their joining line.



If we imagine an india-rubber band (similar, we may suppose, in form to a "parlour quoit" but more elastic) having the faculty of maintaining its figure always circular, or which is more simple in the case before us, capable of maintaining itself in the same plane, and imagine this sufficiently stretched over the surface of the sphere to contain all the given points (represented by very minute pins' heads given upon it), this band will by its contraction upon the surface of the sphere, however originally placed, imitate the steps of Prof. Peirce's method of solution; and after (it may be) passing through and quitting successive sets of three points, come to a position of *geometrical equilibrium*, either when its circumference contains a triad of the given points lying at the angles of an acute-angled triangle, or a duad at the extremities of one of its diameters*.

The following observation, which constitutes a veritable theorem, and is presupposed in Prof. Peirce's solution, is very important:—"Any circle being found which, either passing through three of the given points such that no two of their joining lines form an obtuse angle, or which described upon the line joining two of the given points as a diameter, includes all the rest, is the minimum circle which contains all the points of the given cluster; so that one, and only one, circle exists satisfying the above *alternative* condition."

It may be instructive to proceed to the application of the method now fully explained to some of the more salient cases of inequality, it being understood that these cases are given to afford some general notion of the precision of the method, and by no means as specimens of such as it would be applied to in practice, for which the limits I shall suppose would be far too wide to furnish any useful result.

Example 1. x, y, z unlimited. Here the values of F, G, H, Q are the minor determinants of the matrix,

$$\begin{vmatrix} 1, & 0, & 0, & \bar{1} \\ 0, & 1, & 0, & \bar{1} \\ 0, & 0, & 1, & \bar{1} \end{vmatrix}$$

$F=G=H=1$, $Q=1$, and the linear approximation to $\sqrt{(x^2+y^2+z^2)}$ becomes $\dfrac{2}{\sqrt{3}+1}x + \text{\&c.}$, or $(\sqrt{3}-1)x+(\sqrt{3}-1)y+(\sqrt{3}-1)z$, or say

$$\cdot 73025x + \cdot 73025y + \cdot 73025z,$$

* The annexed is a more complete and, I think, a correct account of what would happen to the band under the supposed conditions. It will begin to move parallel to its own plane, and continue so to do until it comes in contact with one of the physical points (call it A) upon the surface of the sphere. Supposing that the position of equilibrium is not then attained by the band passing at the same moment through *one* other point at the opposite extremity of a diameter to A, or through *two* other of the given points forming a non-obtuse-angled triangle with A, it will begin to revolve (always contracting the while) about a tangent at A to its intersection with the sphere as an axis, until it meets a second of the given points, say B. If the line AB is a diameter of the band, *cadit quæstio*, the problem is solved. If not, the band will go on further contracting, revolving meanwhile round AB as an axis until either AB *becomes* a diameter in virtue of the contraction of the band's dimensions (and so the problem is solved), or else before this can take place the band is arrested at a third point C, either forming a non-obtuse-angled triangle with AB and so solving the problem, or else an obtuse-angled triangle with AB and lying exterior to the arc AB on one side of it or the other; on the latter supposition the line joining C with the extremity of AB nearest to it, will (it appears to me) form a new axis of rotation for the band, which will quit the further extremity of the old axis, and thus the motion will continue with an intermitting change of axes, until at last the band either finds out for itself an axis which in the course of the contraction becomes a diameter, or else brings the band into contact with a third point forming a non-obtuse-angled triangle with such axis, in either of which cases the minimum periphery is attained, the contraction comes to an end, and the problem is solved.

with a maximum proportional error

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}, \text{ or } 2-\sqrt{3} = \cdot 26895.$$

The corresponding error for $\sqrt{(x^2+y^2)}$ under the form $\cdot 8284x + \cdot 8284y$ is $\cdot 17160$, or about two-thirds of the one in question*.

Example 2. $z > \sqrt{(y^2+x^2)}$. Here the determining matrix is

$$\begin{vmatrix} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{vmatrix}$$

$$F = G = \sqrt{\tfrac{1}{2}} - \tfrac{1}{2} = \cdot 207107$$

$$H = \tfrac{1}{2}$$

$$Q = \tfrac{1}{2}$$

$$N^2 = F^2 + G^2 + H^2 = 1 - \sqrt{\tfrac{1}{2}} = \cdot 292893$$

$$N = \cdot 541196$$

$$N + Q = 1\cdot 041196 \qquad N - Q = \cdot 041196.$$

Thus the linear approximation becomes

$$\cdot 397825x + \cdot 397825y + \cdot 960430z,$$

with a maximum error $\cdot 039493$.

Example 3. $z > \sqrt{(y^2+x^2)}, y > x$. This is M. Poncelet's example (*Crelle*, Vol. XIII. p. 291). His a, b, c correspond respectively with my z, y, x; there are some misprints in line 6 of this page (in M. Poncelet's Memoir) which may perplex the reader; it is intended to stand thus:

$$\delta\sqrt{(a^2+b^2+c^2)} + \beta\delta'\sqrt{(b^2+c^2)} = \sqrt{(a^2+b^2+c^2)}\,.\left(\delta + \beta\delta'\sqrt{\frac{b^2+c^2}{a^2+b^2+c^2}}\right).$$

Here the determining matrix corresponds to the area ZKN (the coordinates of N being found from the equations $z^2 = x^2 + y^2$, $y = x$, $z^2 + x^2 + y^2 = 1$), and the matrix will be as subjoined.

* It would have been more exact to have treated this as a case of a circle to be drawn through four points, namely, Z the middle points of ZX, ZY and the middle or lowest point (in reference to Z) of the small circle drawn through these two, and having Z for its pole. But it is easily seen that the small circle drawn through the three former will contain the one last named, for the tangent of its circular radius will be $\sqrt{2} \times \tan\frac{45^\circ}{2}$, and consequently its summit will be further from Z than from the point in question. A similar remark applies to the subsequent and some other examples.

$$\left|\begin{array}{cccc} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{array}\right|.$$

$$F = \sqrt{\tfrac{1}{2}} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}} - \tfrac{1}{2} - \tfrac{1}{2} = 3\sqrt{\tfrac{1}{8}} - 1 = \cdot 060660$$

$$G = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \tfrac{1}{2} - \sqrt{\tfrac{1}{8}} = \cdot 146447$$

$$H = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \cdot 353553$$

$$Q = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \cdot 353553$$

$$N^2 = F^2 + G^2 + H^2 = \tfrac{17}{8} + \tfrac{3}{8} + \tfrac{1}{8} - 7\sqrt{\tfrac{1}{8}}$$

$$= \tfrac{21}{8} - \tfrac{1}{2}\sqrt{24 \cdot 5}$$

$$= 2 \cdot 625 - 2 \cdot 474874$$

$$= \cdot 150126$$

$$N = \cdot 387461, \quad N + Q = \cdot 741014, \quad N - Q = \cdot 033908.$$

The maximum error therefore is $\dfrac{33908}{741014} = \cdot 0457$, or about one-tenth less than that given by M. Poncelet's form.

$$\frac{2F}{N+Q} = \frac{6066}{37051} = \cdot 1637,$$

$$\frac{2G}{N+Q} = \frac{14645}{37051} = \cdot 3953,$$

$$\frac{2H}{N+Q} = \frac{35355}{37051} = \cdot 9542.$$

The last of these quantities is less, the first two greater, than the corresponding coefficients in M. Poncelet's form.

Examples 4 *and* 5. The inequality system, $\sqrt{(x^2 + y^2)} > z > y > x$, is represented by the triangle KNQ, and the corresponding determining matrix will be

$$\left|\begin{array}{cccc} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{array}\right|.$$

So, too, the inequality system, $\sqrt{(x^2 + y^2)} < z < y > x$, has for its locus the triangle ZKN, its determining matrix

$$\left|\begin{array}{cccc} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ 0, & 0, & 1, & \bar{1} \end{array}\right|.$$

It would be superfluous to go on multiplying numerical examples, that may be left to those who feel the want of the Tables which this method affords. If the limiting conditions were supposed to be $z > y$, $z > x$, this would correspond to the quadrilateral $ZK'OK$ in the last figure: it may easily be ascertained that a circle passing through $K'ZK$ would contain O, and would have its centre between N and Z. Hence by the application of Peirce's law, we know that the minimum circle in this case is that which can be drawn through $K'ZK$, and consequently the linear form and maximum error will be precisely the same as for the simpler case already considered, $z > \sqrt{(x^2+y^2)}$. On the other hand, if the conditions imposed were simply $z < x$, $z < y$ (conditions, be it remembered, far wider than ever would be admitted in practice), the limiting figure becomes XOY; and since $MO < MX$ or MY, the centre of the circle through XOY would fall under XY, so that the limiting circle in this case would be that having M for its pole; the linear substitutive form would not contain z, but would be the same as if z did not appear, namely $\cdot 96046x + \cdot 960467y$, with $\cdot 03954$ as the maximum proportional error. The same remark would apply to the system of conditions $z < \lambda x$, $z < \lambda y$ for any value of λ not inferior to $\sqrt{\frac{1}{2}}$.

The conditions $z > x$, $z > y$, $z < \sqrt{(x^2+y^2)}$ would correspond to the limiting area $KK'O$, which would give rise to the determining matrix,

$$\left|\begin{matrix} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{matrix}\right|.$$

The condition $z < \sqrt{(x^2+y^2)}$ would correspond to a limiting area, $KK'XY$. If KY be bisected in G, and $K'X$ in G', and $G'YGX$ intersect in H, it is obvious that a small circle may be described with H as its pole passing through all four points X, Y, K, K', which will be the minimum circle of limitation. To assign the determining matrix, we may take any three of these four points, as, for example, Y, X, K, which will give

$$\left|\begin{matrix} 0, & 1, & 0, & \bar{1} \\ 1, & 0, & 0, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{matrix}\right|.$$

This gives

$$Q = \sqrt{\tfrac{1}{2}} = \cdot 70711,$$
$$F = \sqrt{\tfrac{1}{2}},\ G = \sqrt{\tfrac{1}{2}},\ H = 1 - \sqrt{\tfrac{1}{2}} = \cdot 29289,$$
$$N^2 = \tfrac{5}{2} - \sqrt{2} = 1\cdot 085786,$$
$$N = 1\cdot 04200,$$
$$N + Q = 1\cdot 74911, \quad N - Q = \cdot 33489.$$

The linear approximation is accordingly

$$\cdot 8090x + \cdot 8090y + \cdot 3351z,$$

with a maximum proportional error $\cdot 1914$.

Finally, for $z > y$, $y > x$ the limiting triangle will be ZKO, the determining matrix

$$\begin{vmatrix} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{vmatrix}$$

$$F = \sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{3}} = \cdot 1297, \quad G = \sqrt{\tfrac{1}{3}}\,\{1 - \sqrt{\tfrac{1}{2}}\} = \cdot 1692,$$

$$H = \sqrt{\tfrac{1}{6}} = \cdot 4082,$$

$$N^2 = \tfrac{3}{2} - \sqrt{\tfrac{2}{3}} - \sqrt{\tfrac{2}{3}} = \cdot 21207,$$

$$N = \cdot 4605, \qquad N + Q = \cdot 8687,$$

$$Q = \sqrt{\tfrac{1}{6}} = \cdot 4082, \qquad N - Q = \cdot 0523.$$

The linear approximation is $\cdot 2986x + \cdot 3895y + \cdot 9397z$, with a maximum error $\cdot 06$ (more precisely $\cdot 0602$). This is a trifle beyond half as much again as the maximum error of the best linear approximation to $\sqrt{(x^2 + y^2)}$, subject to the limitation $x > y$, which (see Poncelet's Memoir, p. 280) is a little under $\cdot 04$.

Poncelet has shown that for $\sqrt{(x^2 + y^2)}$, when x, y are the coordinates of a point limited within a sector whose bounding radii make angles ϕ and ψ with the axis of X, the approximate linear form is

$$\frac{\cos \frac{1}{2}(\phi + \psi)}{\cos^2 \dfrac{\phi - \psi}{4}} x + \frac{\sin \frac{1}{2}(\phi + \psi)}{\cos^2 \dfrac{\phi - \psi}{4}} y,$$

with a maximum error $\tan^2 \dfrac{\phi - \psi}{4}$.

In like manner it follows immediately from the method given in the text, that if the summit of the limiting segment make angles λ, μ, ν with the axes of X, Y, Z, and its spherical radius be ρ, the approximate expression for $\sqrt{(x^2 + y^2 + z^2)}$ is

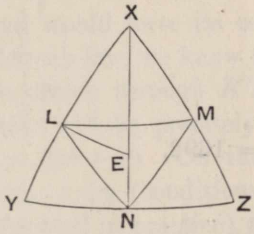
$$\frac{\cos \lambda}{\cos^2 \dfrac{\rho}{2}} x + \frac{\cos \mu}{\cos^2 \dfrac{\rho}{2}} y + \frac{\cos \nu}{\cos^2 \dfrac{\rho}{2}} z,$$

with a maximum error $\tan^2 \dfrac{\rho}{2}$, which expressions are the precise analogues of the former, as will immediately appear from the consideration that the summit of the spherical segment corresponds with the centre of the circular arc.

As an example of the use of these formulæ, suppose the given limits to be

$$x < \sqrt{(y^2 + z^2)}, \quad y < \sqrt{(z^2 + x^2)}, \quad z < \sqrt{(x^2 + y^2)}.$$

If we bisect the quadrants XY, YZ, ZX in L, N, M respectively, the variable point will be limited to lie in LMN, and the base of the corresponding segment will be the circle passing through LMN whose summit will be at E, the point where the perpendicular to XY at L and the arc bisecting the angle X meet.



Here then we have

$$\rho = LE, \quad \lambda = \mu = \nu = XE,$$

$$\tan\rho = \cos 45^\circ = \sqrt{\tfrac{1}{2}}, \qquad \cot\lambda = \sqrt{\tfrac{1}{2}},$$

$$\cos\rho = \sqrt{\tfrac{2}{3}}, \quad \cos^2\frac{\rho}{2} = \tfrac{1}{2}\{1 + \sqrt{\tfrac{2}{3}}\}, \quad \cos\lambda = \sqrt{\tfrac{1}{3}},$$

$$\tan^2\frac{\rho}{2} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}.$$

Hence the linear approximation is

$$\frac{2}{\sqrt{3}+\sqrt{2}}(x+y+z) = 2(\sqrt{3}-\sqrt{2})\{x+y+z\}$$

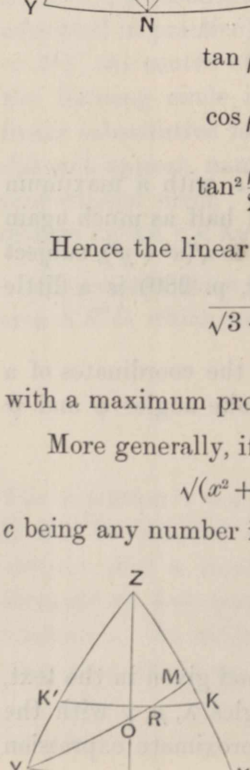
$$= {\cdot}6356744\,(x+y+z),$$

with a maximum proportional error $5 - \sqrt{24} = {\cdot}10102$.

More generally, if we assume the system of conditions

$$\sqrt{(x^2+y^2)} > cz, \quad \sqrt{(y^2+z^2)} > cx, \quad \sqrt{(z^2+x^2)} > cy,$$

c being any number intermediate between 1 and $\sqrt{2}$, if in the figure annexed, we take $\tan ZK = \tan ZK' = c$, and join KK' by a small circle intersecting YM, which bisects ZX, in R, O remaining still the summit of XZY, it is easy to perceive that the limiting area will be included within the triangular space cut out between KK' and the two other analogous small circles; λ, μ, ν will remain the same as before, and OR will represent ρ. Accordingly we have from the quadrantal triangle ZYR,



$$\cos ZR = \sin RY \cos RYZ,$$

that is
$$\sin RY = \sqrt{\frac{2}{c^2+1}};$$

therefore
$$RY = \tan^{-1}\sqrt{\frac{2}{c^2-1}},$$

$$\tan\rho = \tan RO = \tan(RY - OY) = \frac{\sqrt{\left(\frac{2}{c^2-1}\right)} - \sqrt{2}}{1 - \frac{2}{\sqrt{(c^2-1)}}}$$

$$= \sqrt{2}\left\{\frac{1-\sqrt{(c^2-1)}}{\sqrt{(c^2-1)}-2}\right\}.$$

When $c=\surd 2$, this vanishes; and when $c>\surd 2$, the conditions become incompatible.

The equations $\tan\phi=\sqrt{\dfrac{2}{c^2-1}}$, or $\cos 2\phi=\dfrac{c^2-3}{c^2+1}$, and

$$\rho=\phi-\tan^{-1}\surd 2=\phi-54^\circ\,44',$$

are well adapted for logarithmic computation. Suppose

$$c=\tfrac{4}{5},\quad \cos 2\phi=-\tfrac{11}{25}=-\cdot 44,\quad 2\phi=180^\circ-63^\circ\,54'=116^\circ\,6',$$

$$\phi=58^\circ\,3',\quad \rho=3^\circ\,19',$$

giving a maximum error $\tan(1^\circ\,39'\,30'')^2=\cdot 0008375$. The linear form corresponding to this is

$$\frac{2\surd\frac{1}{3}}{1+\cos\rho}\{x+y+z\}=\cdot 5778x+\cdot 5778y+\cdot 5778z.$$

If $c<1$, the formula changes; the limiting area, from a triangle, becoming a hexagon through all the angles of which a circle will admit of being drawn, which circle will give the limiting segment. ρ becomes the third side of a spherical triangle of which the other two sides are $\tan^{-1}\surd 2$ and $\tan^{-1}c$ respectively, and the included angle 45°; so that

$$\cos\rho=\sqrt{\frac{1}{3(1+c^2)}}+\sqrt{\frac{c}{3(1+c^2)}}=(1+\surd c)\sqrt{\frac{1}{3(1+c^2)}},$$

and the maximum error, that is $\tan^2\dfrac{\rho}{2}$, becomes

$$\frac{\surd\{3(1+c^2)\}-1-\surd c}{\surd\{3(1+c^2)\}+1+\surd c}.$$

The only real difficulty in extending M. Poncelet's method in the manner pursued in the above unpretending study, consisted in forming a clear preconception of the mode in which any given system of limits require for the purpose in view to be regarded, namely, as enveloped, so to say, in a single condition (no wider than absolutely necessary) expressed by a linear equation between the given surd function and the variables which enter into it.

I may in conclusion just observe that if the relative values of the variables be limited, not by a system of conditions giving rise to a polygonal area of limitation, but by a condition expressed by the positivity of a single homogeneous function of the variables of any degree, the variable point will then be limited by the intersection of the sphere with a cone, and we should have to solve a preliminary geometrical problem of circumscribing a spherical curve by the least possible circle,—a question which I have neither leisure nor inclination to discuss, but to which I believe Mr Cayley has paid some attention.

Before taking final leave of my readers and the subject, I devote a word to the *inverse case* of *Three Rectangular Forces.* This is the case where the resultant and two of the rectangular components are given, and it is the third component which is to be expressed linearly in terms of them. In this case an approximate expression is to be found for $\sqrt{(z^2-y^2-x^2)}$, and the geometrical locus which replaces the sphere becomes an equilateral hyperboloid of revolution of two sheets.

If the variable point be supposed to be limited to a segment of one sheet of the hyperboloid cut off by the plane $Ax+By+Cz=1$, the discriminant of $z^2-y^2-x^2$ being 1, and its polar reciprocal of the same form as itself, the approximate linear form of the surd becomes

$$\frac{2Cz}{\sqrt{(C^2-B^2-A^2)}+1}+\frac{2By}{\sqrt{(C^2-B^2-A^2)}+1}+\frac{2Ax}{\sqrt{(C^2-B^2-A^2)}+1},$$

with a maximum proportional error

$$\frac{1-\sqrt{(C^2-B^2-A^2)}}{1+\sqrt{(C^2-B^2-A^2)}}.$$

To *envelope*, however, any given arbitrary system of inequalities between the coordinates x, y, z on the hyperboloid within a single condition,

$$Ax+By+Cz-1>0,$$

becomes a geometrical problem of somewhat greater difficulty than the corresponding one for the sphere, and I do not propose to enter upon the discussion of it here.

I shall content myself, as M. Poncelet has done in the corresponding case *in plano*, with exhibiting a single numerical application of the method.

Suppose the given limits to be defined by the equations

$$z^2>\tfrac{3}{2}(y^2+x^2),\quad y>x.$$

Here it is obvious that the *enveloping condition* will be expressible by means of the equation to a plane drawn through three points on the hyperboloid, the coordinates of one of which are found by writing

$$y=0,\quad x=0;$$

of a second by writing

$$z^2-\tfrac{3}{2}y=0,\quad x=0;$$

and of the third by writing

$$z^2-\tfrac{3}{2}(y^2+x^2)=0,\quad y-x=0;$$

and for all three

$$z^2-y^2-x^2=1.$$

Hence we obtain the matrix

$$\begin{vmatrix} 1, & 0, & 0, & \bar{1} \\ \sqrt{3}, & \sqrt{2}, & 0, & \bar{1} \\ \sqrt{3}, & 1, & 1, & \bar{1} \end{vmatrix}.$$

And if we call the minors obtained by leaving out the first, second, third, fourth columns respectively H, G, F, Q, the linear form becomes

$$\frac{2Hz}{\sqrt{(H^2-G^2-F^2)}+Q}+\frac{2Gy}{\sqrt{(H^2-G^2-F^2)}+Q}+\frac{2Fx}{\sqrt{(H^2-G^2-F^2)}+Q},$$

with a maximum error

$$\frac{Q-\sqrt{(H^2-G^2-F^2)}}{Q+\sqrt{(H^2-G^2-F^2)}}.$$

And since

$$Q=\sqrt{2},\quad H=\sqrt{2},\quad -G=\sqrt{3}-1,\quad -F=(\sqrt{2}-1)(\sqrt{3}-1),$$

we have

$$\sqrt{(H^2-G^2-F^2)}=1{\cdot}1714 \text{ and } Q=1{\cdot}4142,$$

so that the representative form becomes $1{\cdot}093z-{\cdot}566y-{\cdot}089x$, with a maximum relative error of about ·094.