26.

OUTLINES OF SEVEN LECTURES ON THE PARTITIONS OF NUMBERS.

[*Proceedings of the London Mathematical Society*, XXVIII. (1897), pp. 33—96.]

PREFACE.

THESE outlines appertain to lectures delivered by Prof. Sylvester at King's College, London, during the year 1859. The outline of each lecture was printed shortly before its delivery and handed to those in attendance, and a few copies also were privately circulated. They are now published for the first time. The Professor's attention was called away shortly afterwards to another department of mathematics, with the result that his researches on compound partitions were never published. As the lectures constitute the only serious attempt that has ever been made to deal with the subject, and as copies of the outlines are very scarce, Prof. Sylvester has yielded to the suggestion made to him in regard thereto by the Council of the London Mathematical Society, so far as to assent to their publication in the *Proceedings*, with all their imperfections on their heads. The present state of his health and the long lapse of time combine to render any revision upon the part of the Professor impossible. He desires it to be known that he cannot vouch for the correctness of all that appears in the notes, and that they were prepared in a hand-to-mouth manner during the process of investigation between the lectures, and that it is only on the opinion of the Council urgently expressed to him that the work should not entirely perish that he has consented at this late hour to the publication.

The Council desires to acknowledge the assistance it has derived from Prof. H. W. Lloyd Tanner, of University College, Cardiff, who kindly placed his annotated copy of the outlines at its disposal, and also to Mr R. F. Scott, of St John's College, Cambridge, who presented a copy to the Society.

FIRST LECTURE*.

Introductory Remarks.

Resolution of an integer into parts.

Resolution of an integer into parts limited in number.

Resolution of a number into parts limited in magnitude.

Euler's law of reciprocity, viz.,

As many ways as an integer n can be resolved into parts not exceeding m in number, so many ways can it be resolved into parts not exceeding m in amount.

Ferrers' Proof. Example. $n=5$, $m=3$,

111	11	11	1	
11	11	1	1	may be read as
	1	1	1	3, 2; 2, 2, 1; 2, 1, 1, 1; 1, 1, 1, 1, 1;
		1	1	or 2, 2, 1; 3, 2; 4, 1; 5.
			1	

Cayley's application of this law to the calculation of groups of symmetric functions.

Example. To find Σx^5, $\Sigma x^4 y$, $\Sigma x^3 y^2$, $\Sigma x^2 y^2 z$, where x, y, z are roots of

$$x^3 - p_1 x^2 + p_2 x - p_3 = 0,\quad p_1 \cdot p_1 \cdot p_1 \cdot p_1 \cdot p_1,\quad p_2 \cdot p_1 \cdot p_1 \cdot p_1,\quad p_2 \cdot p_2 \cdot p_1,\quad p_3 \cdot p_2$$

will be linear functions of the quantities to be found.

Euler, Waring, Paoli, De Morgan, Warburton, Herschel, Kirkman, Ferrers, Cayley, in connexion with question of resolution.

The resolution of a number into parts is the problem of ascertaining the different modes of composing n with the elements

$$1, 2, 3, \ldots \text{ up to } n.$$

General problem of *simple* partition is to find in how many ways a given number n can be composed of given elements $a, b, c, \ldots k$.

General problem of *binary* partition is to find in how many ways the couple m, m' can be composed of the couples $a, a', b, b', c, c', k, k'$.

Statement of problem under form of equations.

Denumeration and denumerant defined.

Denumerant of $U=0$ same as that of $kU=0$.

Denumerant of $U=0$, $V=0$ same as that of

$$kU + lV = 0,\quad k'U + l'V = 0.$$

* Delivered at King's College, London, on the 6th June, 1859.

Coefficient groups and constant group defined.

How the resolution of an equation or system of equations with any *real* coefficients may be made to depend on the inverse problem of the centre of gravity of a system of points.

Example. A system of two equations.

Origin, coefficient points, primary defined.

Total of coefficient points is called a cluster.

Coefficient points may be denoted by the variables to which they belong.

Weighted cluster.

Weight of primary assumed to be positive unity.

If primary and cluster balance about the origin, the weights at the several points of cluster will satisfy the given system of equations.

Linear cluster; plane cluster; solid cluster.

The cluster origin and primary may be considered apart from the axes used in the construction.

Ray cluster; axis of cluster defined.

Derivative of an equation-system. An equation-system really consists of the universe of its derivatives.

How this universe is contained in the geometrical representation of the system.

A principal derivative of a binary system is the equation resulting from the elimination of any *one* of its variables.

A principal derivative of a ternary system is the equation resulting from the elimination of any *two* of its variables.

Universe or Plexus of Principal Derivatives.

How to construct geometrically the principal derivatives by aid of the cluster, primary, and origin.

(1) For binary system.

(2) For ternary system.

We can thus perform the process of elimination geometrically.

If more than the regular number of variables can be eliminated simultaneously out of the system, this will be evidenced in the plane cluster by three or more points lying in a line, and in the solid cluster by four or more points lying in a plane*.

* The general polyhedron *in solido* analogous to the polygon *in plano* is a polyhedron with triangular faces exclusively.

An equation is said to be homonymous when the coefficients of the variables are all positive or all negative.

It may be congruous or incongruous.

Example. $2x + 3y + 4z = 10$ congruous,

$2x + 3y + 4z = -10$ incongruous.

An *omni-positive* solution of an equation or system means a solution in which the variables are all positive.

An *omni-negative* solution is one in which the variables are all negative.

A *homonymous* solution is one which is either omni-positive or omni-negative.

An equation or equation-system may be *definite* or *indefinite*.

Indefinite when homonymous solutions can be found wherein the variables may be made indefinitely great.

Definite when the variables cannot be made indefinitely great in any homonymous solution.

The equations $ax - by = m$ and $ax + by - cz = m$, where $a, b, c, \ldots m$ are any real positive quantities whatever, are indefinite.

The character as to definite or indefinite depends only on the coefficients, and not on the constant term.

A single equation to be definite must be homonymous.

A system of equations to be definite must admit of a homonymous derivative.

If it admit of one, it must admit of an infinite number of such.

Definiteness and indefiniteness of systems depend only on the relative values of coefficients, and not on the constant terms.

Hence, the relative position of origin and cluster must suffice geometrically to determine this character.

Definition of boundary of a plane or solid cluster of points.

Lemma. The centre of gravity of any weighted cluster is contained inside the boundary, and may be made to lie at any point within it by a due adjustment of the relative magnitudes of the weights at the several points.

Theorem. If the origin lies within the cluster, the system is indefinite; if outside, definite.

In Fig. 1 the centre of gravity of the cluster may be brought to the position g or g' as near as we please to O on either side of it in a line with PO, and, the sum of the weights

$$x + y + z + t + u + v + w + \omega$$

being $\frac{PO}{gO}$ or $-\frac{PO}{g'O}$, may be made indefinitely great either on the positive or negative side of zero.

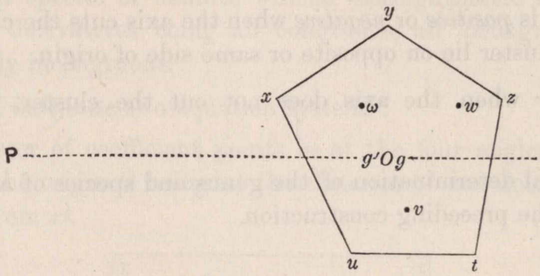


Fig. 1.

In Fig. 2, if origin is at O, Σx will lie between $\frac{PO}{gO}$ and $\frac{PO}{g'O}$; if origin is at O', Σx will lie between $-\frac{PO'}{gO'}$ and $\frac{PO'}{g'O'}$ *; if at O'', the system cannot by any system of weights, all positive or all negative, be made to balance the weight at P about the origin.

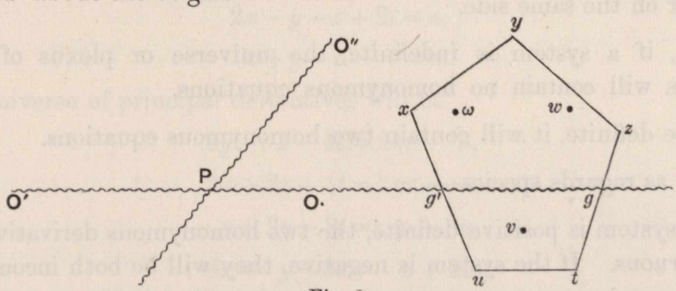


Fig. 2.

The same method is applicable to points *in solido*.

Indefinite systems in general admit of homonymous solutions of both kinds.

The only case of exception is when the origin is in the contour of cluster.

Definite systems admit only of solutions of one kind.

* Hence it may easily be shown that the greatest and least values of Σx in any definite system of equations

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots = m,$$
$$a_1'x_1 + a_2'x_2 + a_3'x_3 + \ldots = m',$$
$$a_1''x_1 + a_2''x_2 + a_3''x_3 + \ldots = m'',$$

will be the greatest and least values of ρ deduced successively from all the equations that can be formed after the type of the following:—

$$\begin{vmatrix} a_1, & a_2, & a_3, & 1 \\ a_1', & a_2', & a_3', & 1 \\ a_1'', & a_2'', & a_3'', & 1 \\ m, & m', & m'', & \rho \end{vmatrix} = 0,$$

and in like manner we may derive from the geometrical method a simple rule for determining algebraically the maxima and minima values of each separate variable.

Three Species of Definite Systems.

The system is *positive* or *negative* when the axis cuts the cluster according as primary or cluster lie on opposite or same side of origin.

It is *neuter* when the axis does not cut the cluster. (For example, origin O''.)

An analytical determination of the genus and species of a system may be deduced from the preceding construction.

Binary System.

In the indefinite case, if we draw lines from origin to every point in cluster, each such ray divides the cluster into two parts.

In the definite case there are two extreme rays leaving all the points in the cluster on the same side.

Hence, if a system is indefinite, the universe or plexus of principal derivatives will contain no homonymous equations.

If it be definite, it will contain two homonymous equations.

Again, as regards species—

If the system is positive definite, the two homonymous derivatives will be both congruous. If the system is negative, they will be both incongruous.

If the system be neuter, the homonymous will be one congruous, the other incongruous.

Ternary System.

If the system is indefinite, all the planes through the origin and any two points of the cluster divide the cluster into two parts.

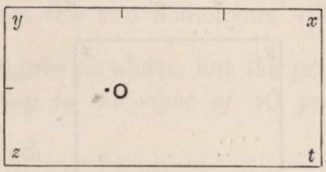
If it be definite, the bounding planes of the pyramid formed by joining the origin with each point of the cluster will leave the other points of cluster all on one side.

Hence, when the system is indefinite, the plexus of principal derivatives will contain no homonymous equations; when it is definite, there will be some homonymous derivatives, and the number cannot be less than three or greater than the number of variables. For, if we project the cluster from the origin on a plane cutting the rays all on the same side of origin, the number of sides in contour of this projection will be the number of planes in the pyramid, and n points in a plane cannot form a figure bounded by less than three nor more than n sides*.

The different species of definite will be distinguishable by the homonymous principal derivatives being all congruous, all incongruous, or partly congruous, partly incongruous.

Examples of *indefinite* two-equation systems:

Let the cluster of coefficient points be at the four angles of a parallelogram x, y, z, t, the origin O being at the distance of one unit from yz, yx, zt, and two units from xt.



The system of equations will be

$$2x - y - z + 2t = n,$$
$$x + y - z - t = m.$$

The universe of principal derivatives will be

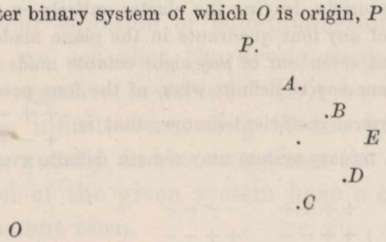
$$3y - z - 4t = 2m - n,$$
$$3x - 2z + t = m + n,$$
$$-x + 2y - 3t = m - n,$$
$$4x + y - 3z = n + 2m,$$

all of which are heteronymous or indefinite, showing that the system is indefinite.

* Thus we see in like manner that the number of homonymous principal derivatives to a definite quaternary system of n variables is some number intermediate to 4 and q (where q is the number of faces in a triangular polyhedron with n summits), that is, $2n - 4$.

This would be difficult to prove by a direct analytical process.

N.B. In any neuter binary system of which O is origin, P the primary and $ABCDE$ the



cluster, all the triangles OPA, OPB, &c., following the same order of rotation will represent determinants of the same sign.

This cannot be the case for definite positive or negative, or for indefinite systems.

Hence the neuter case may be recognised by the determinants, obtained by conjugating *in situ* each coefficient group in succession with the constant group, never changing sign.

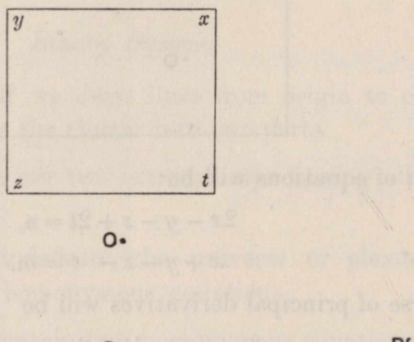
If y, z, x, t were a square, y, t as well as x, z would be brought into line with O; equations would become

$$x - y - z + t = m,$$
$$x + y - z - t = n,$$

and, on account of these two syzygies, there would be only two principal derivatives, viz.:

$$2y - 2t = n - m,$$
$$2x - 2z = n + m.$$

Examples of Definite Systems.*



Positive Case. Take x, y, z, t at the angles of a square, two units each way (breadth and depth).

Let the origin O be at an equal distance from z and t, and from x and y and *the primary P in a line with xt.* The system referred to OP and OQ at right angles to OP as axes of moment gives rise to the equations

$$x - y - z + t = 0,$$
$$(1 + c)x + (1 + c)y + cz + ct = m,$$

* In order that a system may be definite the points of the cluster, whether in line, plane, or solid, must be all *in front* to an eye at the origin. In the last two cases accordingly, a line or a plane may be drawn through the origin, leaving the cluster entirely on one side. Now, as a line in a plane will cut three out of any four quadrants in the plane made by two intersecting lines, and a plane *in solido* will cut seven out of any eight octants made by three intersecting planes, it follows that a binary system *may* be definite when of the four possible combinations of signs affecting the terms of the several coefficient groups, that is, $\begin{matrix} + + - - \\ + - + - \end{matrix}$, three are found among the several groups, and so a ternary system may remain definite even when out of the 8 possible combinations of signs

$$\begin{matrix} + + + + & - - - - \\ + + - - & + + - - \\ + - + - & + - + - \end{matrix}$$

all but one are found among the several coefficient groups.

We may then safely infer that, in general, all but one of the possible combinations of sign may occur in the coefficient groups of any system without the system necessarily ceasing to be definite. But, if all possible combinations occur, the system will be necessarily indefinite.

c being the distance of O from zt, and m its distance from P. The extreme rays being Oz and Ot, the two homonymous principal derivatives will be the two resultants in respect to z and t, that is,

$$(1+2c)\,x+y+2ct=m,$$
$$x+(1+2c)\,y+2cz=m,$$

both of which are congruous.

Negative Case. Figure the same as the preceding, but *position of P reversed* (that is, *passed through origin to an equal distance from it on the other side*). The equations will be as above, with the exception of m becoming negative, so that the two homonyms will be incongruous.

Neuter Case. Same figure as above, but *the primary P moved horizontally through d to P′ lying to the right of zO produced.* This condition implies that $\frac{m}{d}<c$ or $m<cd$.

The two equations now become

$$x-y-z+t=d,$$
$$(1+c)\,x+(1+c)\,y+cz+ct=m,$$

and the two homonyms are

$$(1+2c)\,x+y+2ct=m+cd,$$
$$x+(1+2c)\,y+2cz=-(cd-m),$$

of which the first is congruous, the second incongruous, thereby indicating that the system is neuter.

The determinants

$$\begin{vmatrix} 1 & d \\ 1+c & m \end{vmatrix} \quad \begin{vmatrix} -1 & d \\ 1+c & m \end{vmatrix} \quad \begin{vmatrix} -1 & d \\ c & m \end{vmatrix} \quad \begin{vmatrix} 1 & d \\ c & m \end{vmatrix}$$

that is $\quad m-cd-d, \quad -m-d-cd, \quad -m-cd, \quad m-cd,$

being all negative, would also have served to prove the system to be neuter.

Scholium. If our equation or equation-system be now supposed to be integer equations, we see that the denumerant will be in all cases zero, if the system be negative or neuter. If it be indefinite, the denumerant in general will be infinite (according to the known theory of numbers), but it may be zero, namely, in the case where the coefficients of any equation or derived equation of the given system have a common factor which is not a factor of the constant term.

The plexus of principal derivatives affords an absolute criterion for determining whether the denumerant of a given indefinite system of equations is infinite or zero.

SECOND LECTURE*.

Definition of denumerant recalled,

$$\text{Д} U, \quad \text{Д}(U, V), \quad \text{Д}(U, V, W),$$

used as implicit symbols of denumeration.

$$\text{Д}\, U \text{ in its explicit form } \frac{n\,;}{a, b, c, \ldots l\,;}$$

$$\text{Д}(U, V) \quad ,, \quad ,, \quad \frac{n, n'\,;}{a, a'\,; b, b'\,; c, c'\,; \ldots; l, l'\,;}$$

&c. &c.

Numeratives and denominatives defined.

Herschel's symbol r_n explained.

Its value as a linear function of nth powers of the rth roots of unity.

$$r_n \text{ in the new theory will be replaced by } \frac{n\,;}{r\,;}$$

Observation. $$\frac{n\,;}{r\,;} + \frac{n-1\,;}{r\,;} + \ldots \frac{n-(r-1)\,;}{r\,;} = 1.$$

More generally,

$$\frac{n\,;}{r'r\,;} + \frac{n-r\,;}{r'r\,;} + \frac{n-2r\,;}{r'r\,;} + \ldots \frac{n-(r'-1)\,r\,;}{r'r\,;} = \frac{n\,;}{r\,;}$$

for, if $\frac{n}{r}$ is fractional, so is $\frac{n}{r}-1$, $\frac{n}{r}-2$, ...,

and *a fortiori*,

$$\frac{1}{r'}\cdot\frac{n}{r}, \quad \frac{1}{r'}\left(\frac{n}{r}-1\right), \quad \frac{1}{r'}\left(\frac{n}{r}-2\right), \ \ldots \ \frac{1}{r'}\left(\frac{n}{r}-r'-1\right),$$

and, if $\frac{n}{r}$ is integer, one and only one of the above quantities will be an integer.

What $E\left(\frac{n}{r}\right)$ is commonly used to denote; Herschel's notation $\frac{n}{\overline{\overline{r}}}$.

$$E\left(\frac{n}{r}\right) = \frac{(n-r)\,;}{1, r\,;}$$

Examples. $$x + 2y = 7, \quad E\left(\frac{7}{2}\right) = 3,$$

$$x + 3y = 8, \quad E\left(\frac{8}{3}\right) = 2,$$

$$x + 5y = 15, \quad E\left(\frac{15}{5}\right) = 3.$$

* Delivered at King's College, London, on the 9th June, 1859.

N.B. In the partition theory, zero always counts as a *positive* integer*.

Of course the residue of n to modulus r is

$$n - r \times \frac{(n-r);}{1, r;};$$

but it may also be expressed as a binary denumerant.

Simplest class of indeterminate equations

$$x_1 + x_2 + \ldots + x_r - n = 0,$$

$$\frac{n}{1} = 1, \quad \frac{n}{1, 1} = n + 1, \quad \frac{n}{1, 1, 1} = \frac{(n+1)(n+2)}{2}, \text{ \&c.}$$

Generally $\mathit{\Omega}\, U$ in above equation is coefficient of t^n in $\dfrac{1}{(1-t)^r}$.

The denumerant of the equation

$$ax_1 + ax_2 + \ldots + ax_r - n = 0,$$

$$\frac{n;}{a;} \times \left\{\frac{n+a}{a} \cdot \frac{n+2a}{2a} \cdots \frac{n+(r-1)a}{(r-1)a}\right\}.$$

Provisional Method of Simple Denumeration.

Any simple denumerant may be expressed in terms of denumerants of the class last treated of.

Example 1. $\qquad x + 2y = n;$

x must be of the form 2ξ or $2\xi + 1$, two suppositions mutually exclusive.

Hence the denumerant of the given equation is the sum of those of the two equations,

$$2\xi + 2y = n \quad \text{and} \quad 2\xi + 2y = n - 1.$$

Hence $$\frac{n;}{1, 2;} = \frac{n;}{2, 2;} + \frac{n-1;}{2, 2;}$$

$$= \frac{n;}{2;} \times \frac{n+2}{2} + \frac{n-1;}{2;} \times \frac{n+1}{2}.$$

But $$\frac{n;}{2;} + \frac{n-1;}{2;} = 1.$$

Hence $$\frac{n;}{1, 2;} = \frac{2n+3}{4} + \left\{\frac{n;}{2;} - \frac{(n-1);}{2;}\right\}\frac{1}{4}.$$

Observe that $$\left\{\frac{n;}{2;} - \frac{(n-1);}{2;}\right\}\frac{1}{4} = (-)^n\,\frac{1}{4}.$$

* Consequently, the equation $ax + by + cz \ldots = 0$ has the denumerant *unity*, and is not neuter; there being in fact no neuter cases for *simple* partition.

Example 2. $x+3y=n$;

x must be of the form 3ξ, or $3\xi+1$, or $3\xi+2$.

Hence
$$\frac{n;}{1,3;}=\frac{n;}{3,3;}+\frac{(n-1);}{3,3;}+\frac{(n-2);}{3,3;}$$
$$=\frac{n;}{3;}\cdot\frac{n+3}{3}+\frac{(n-1);}{3;}\cdot\frac{n+2}{3}+\frac{(n-2);}{3;}\cdot\frac{n+1}{3}$$
$$=\frac{1}{3}\left[(n+2)+\left\{\frac{n;}{3;}-\frac{(n-2);}{3;}\right\}\right].$$

Example 3. To find the denumerant of
$$x+2y+4z=n.$$

4 is the least common multiple of 1, 2, 4.

x is either 4ξ, $4\xi+1$, $4\xi+2$, or $4\xi+3$,

$2y$ is either 4η, or $4\eta+2$,

$4z$ is $4z$.

Thus there are eight cases, each giving rise to an equation of the form
$$4\xi+4\eta+4z+c=n.$$

I combine together those in which the constant on the left side is either the same, or leaves the same residue when divided by 4.

Thus, we obtain
$$\frac{n;}{1,2,4;}=\frac{n;}{4,4,4;}+\frac{n-4;}{4,4,4;}$$
$$+\frac{n-1;}{4,4,4;}+\frac{n-5;}{4,4,4;}$$
$$+2\,\frac{n-2;}{4,4,4;}+2\,\frac{n-3;}{4,4,4;}$$

and observing that
$$\frac{n-4;}{4;}=\frac{n;}{4;},\quad \frac{n-5;}{4;}=\frac{n-1;}{4;}$$

we obtain
$$\frac{n;}{1,2,4;}=\frac{n;}{4;}\left\{\frac{(n+4)(n+8)}{4.8}+\frac{n(n+4)}{4.8}\right\}$$
$$+\frac{n-1;}{4;}\cdot\left\{\frac{(n+3)(n+7)}{4.8}+\frac{(n-1)(n+3)}{4.8}\right\}$$
$$+\frac{n-2;}{4;}\cdot\frac{(n+2)(n+6)}{4.4}$$
$$+\frac{n-3;}{4;}\cdot\frac{(n+1)(n+5)}{4.4}$$
$$=\frac{1}{16}\left\{\frac{n;}{4;}(n^2+8n+16)+\frac{n-1;}{4;}(n^2+6n+9)+\frac{n-2;}{4;}(n^2+8n+12)\right.$$
$$\left.+\frac{n-3;}{4;}(n^2+6n+5)\right\}.$$

By aid of the identities,

$$\frac{n;}{4;}+\frac{n-1;}{4;}+\frac{n-2;}{4;}+\frac{n-3;}{4;}=1,$$

$$\frac{n;}{4;}+\frac{n-2;}{4;}=\frac{n;}{2;},\quad \frac{n-1;}{4;}+\frac{n-3;}{4;}=\frac{n-1;}{2;},$$

the above equation becomes

$$\frac{n;}{1,\,2,\,4;}=F+\left(\frac{n;}{2;}-\frac{n-1;}{2;}\right)G+\left(\frac{n;}{4;}+\frac{n-1;}{4;}-\frac{n-2;}{4;}-\frac{n-3;}{4;}\right)H,$$

where $F=\frac{1}{16}\left(n^2+7n+\frac{21}{2}\right);\quad G=\frac{2n+7}{32},\quad H=\frac{1}{8}.$

F will be the mean value of the transcendental function $\frac{n;}{1,\,2,\,4;}$.

The mean value of any simple denumerant, by virtue of the theorem discovered by the lecturer, is always expressible directly as an algebraical function of n, and of the quantities $a_1, a_2, \ldots a_r$ left perfectly indefinite.

Observe that in the multipliers of G and H the sums of the coefficients are all zero.

General direct method of expressing every simple denumerant under a simple form is furnished by theorem above referred to.

The method above given substantially consists in making the denumeration of $a_1x_1+a_2x_2\ldots+a_rx_r=n$ depend on finding all the solutions of the congruence

$$a_1u_1+a_2u_2+a_3u_3\ldots+a_ru_r-u_{r+1}=0 \text{ to modulus } K,$$

K being the least common multiple of $a_1, a_2, \ldots a_r$, and $u_1, u_2, \ldots u_r$ being all limited to be positive integers less than K, but u_{r+1} being left indefinite.

Thus the numbering of the solutions in positive integers of an equation can be brought to depend upon finding the solutions themselves of a congruence in positive integers.

Euler's Method of Generating Fractions.

The denumerant of

$$ax+by+cz+\ldots+lt=n$$

is the coefficient of t^n in the expansion of

$$\frac{1}{(1-t^a)(1-t^b)\ldots(1-t^l)}$$

expanded in ascending powers of t.

Proof that the product of the series generated by $\frac{1}{1-t^a} \cdot \frac{1}{1-t^b}$, &c., gives $\frac{n\,;}{a, b, c, \ldots l\,;}$ as the coefficient of t^n.

Note that when $n=0$ the coefficient of t^n is 1.

Thus we see that the denumerant of $x_1+x_2+\ldots+x_r=n$ is the coefficient of t^n in $\frac{1}{(1-t)^r}$ as already found.

Necessity of attending to the *order* of terms in the denominators of generating fractions; $\frac{1}{p-q}$ and $\frac{1}{-q+p}$ distinguished; $\frac{1}{p\sim q}$ may be used to signify one or the other of the two previous forms, the choice being left subject to ulterior determination.

Euler's generating fraction continues to hold good even when any of the coefficients become negative, the expansion becoming *indefinite*.

Example. The denumerant of $x-y=n$ is generated by the product of $\frac{1}{1-t}$ by that of $\frac{1}{1-t^{-1}}$, that is to say, of the series

$$1+t\ +t^2\ +t^3\ +\ldots\ ad\ inf.,$$

by the series
$$1+t^{-1}+t^{-2}+t^{-3}+\ldots\ ad\ inf.$$

This product will consist of an ascending and descending branch, and the coefficients of every term in each branch will be infinite, showing that the denumerant of $x-y=n$ is infinite for all integer values of n whether positive or negative.

The cognate forms to a generating fraction defined.

Their number, if there are r factors in the denominator, is 2^r.

In above example $\frac{1}{(1-t)(1-t^{-1})}$ generates a double indefinite development, but the cognate form $\frac{1}{(1-t)(-t^{-1}+1)}$ will generate a series in which the indices of t ascend from 1 to ∞, and the coefficients for any finite value of an index remain finite.

So in general for $\frac{1}{(1-t^a)(1-t^{-b})}$; the coefficient of t^n in a cognate form to this is the coefficient of t^{n-b} in $\frac{1}{(1-t^a)(1-t^b)}$ with the sign changed. So again the coefficient of t^n in a cognate form to

$$\frac{1}{(1-x^a)\ldots(1-x^b)(1-x^{-c})(1-x^{-d})}$$

will be the coefficient of t^{n-c-d} in

$$\frac{1}{(1-x^a)\dots(1-x^b)(1-x^c)(1-x^d)},$$

and so on.

Every generating fraction to a single equation contains two cognate forms (of which itself may be one), which admit of development in series with *finite* coefficients. One of these will be purely an ascending, the other purely a descending, series.

The coefficient of t^n in the ascending development I call the connumerant of the equation.

The connumerant is always finite; it may be positive or negative; when the coefficients are all positive the connumerant and denumerant are identical.

The meaning of the symbol $\frac{n;}{a_1, a_2, \dots a_r;}$ extended and modified.

Rule for transforming a connumerant with some or all of its denominatives negative into one with all its denominatives positive.

Why connumerants are necessary.

When the numerative is a negative quantity the connumerant by virtue of the definition is always zero.

The denumerant of a binary system of equations

$$ax + by + cz + \dots = m,$$
$$a'x + b'y + c'z + \dots = m',$$

is the coefficient of $t^m t'^{m'}$ in

$$\frac{1}{(1-t^a \cdot t'^{a'})(1-t^b \cdot t'^{b'})(1-t^c \cdot t'^{c'})\dots}.$$

Unnecessariness of the limitation imposed by Euler upon the signs of the coefficients.

How to exhibit geometrically, the limiting ratios to the values of the indices of t and t' which can appear in the development of the Eulerian fraction containing t and t'.

Hence we see that the series generated by such an Eulerian may consist of a single branch, or of two branches, or of three branches.

So the Eulerian of a definite ternary system developed may have any number of branches from one to seven inclusive.

Definition. A determinate series is one in which none of the coefficients of terms at a finite distance from the origin become infinite in value. A determinate generating function is one which generates a determinate series.

Then $\frac{1}{(1-t)^2}$ is determinate, but $\frac{1}{(1-t)(t-1)}$ indeterminate.

So, again, $\frac{1}{(1-t)(u-1)(ut-1)}$ is determinate, but

$\frac{1}{(1-t)(u-1)(t-u)}$ indeterminate.

Denumerative function distinguished from denumerant.

Reversal of a point in a cluster defined.

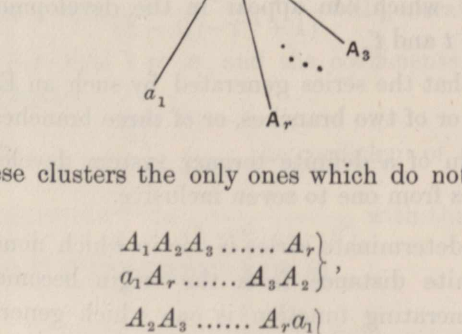
If we change any factor of an Eulerian of the second order

$$\frac{1}{1-t^a \cdot t'^{a'}} \text{ into } \frac{1}{-t^a \cdot t'^{a'}+1},$$

the equation-system by the denumeration of which the coefficient of $t^m t'^{m'}$ may be calculated undergoes a change in its constant terms as well as in its coefficients; but it is only the change in the latter which can influence the character of the system as to being definite or indefinite, and consequently the character of the coefficients of the developed Eulerian as to being finite or infinite.

Hence, it is easy to show geometrically that $2r$ out of the 2^r cognate forms to an Eulerian fraction of the 2nd order will give rise to series with finite coefficients.

For if there be r variables the total number of cognate forms will correspond to the 2^r clusters consisting of A_1 or a_1 (its reverse), combined with A_2 or a_2 its reverse, with A_3 or a_3 its reverse, and so on.



Now of all these clusters the only ones which do not enclose the origin are the pairs

$$\left.\begin{matrix} A_1 A_2 A_3 \ldots\ldots A_r \\ a_1 A_r \ldots\ldots A_3 A_2 \end{matrix}\right\},$$

$$\left.\begin{matrix} A_2 A_3 \ldots\ldots A_r a_1 \\ a_2 a_1 A_r \ldots\ldots A_3 \end{matrix}\right\},$$

and so on, there being as many pairs of clusters outside the origin as there are points $A_1, A_2, \dots A_r$.

In like manner an Eulerian fraction of the 3rd order and with r factors in its denominator will admit of as many cognate pairs of forms generating series with finite coefficients as there are combinations of r elements, 2 and 2 together, that is, $\frac{r \cdot r-1}{2}$ pairs, and so on, for any order whatever.

Hence it would not be possible without further specification to extend the definition of connumerants (if it were wished to do so) from simple equations to equation-systems.

Happily the necessity for the consideration of such does not arise, as it will be shown that denumerants of all orders may be expressed in terms of *simple* connumerants.

By the connumerant to

$$-ax - by - cz + dt + eu + \&c. = K,$$

I shall understand the expression

$$\frac{K;}{-a, -b, -c, d, e, \dots;}.$$

This connumerant will be the same save as to sign (which is or is not to be changed, according as the number of negative coefficients $-a, -b, -c$ is odd or even) as the denumerant of

$$a(x+1) + b(y+1) + c(z+1) + dt + eu + \&c. = K.$$

THIRD LECTURE*.

Reduction.

Reduction explained.

Reduction in partitions analogous to elimination in equations.

A prime group defined. Examples.

Syzygy of variables; predicable also (elliptically) of groups.

In a plane cluster, syzygy is evinced by two or more points being in a line with the origin.

In a solid cluster, by three or more points being in the same plane with the origin.

* Delivered at King's College, London, on June 16th, 1859.

Analytical condition of two groups in a binary system being in syzygy is that the determinant formed by their coefficients vanishes.

Analytical condition of three variables in a ternary system being in syzygy is same as above; and so in general.

If $ab' - a'b = 0$,

$$\frac{a}{b} = \frac{a'}{b'};$$

and, if a, b is a prime group, and also a', b', either

$$\begin{matrix} a = a' \\ b = b' \end{matrix} \text{ or } \begin{matrix} a = -a' \\ b = -b'. \end{matrix}$$

On the latter supposition, the system would be indefinite (for the origin would either lie *on* the contour of the cluster or *within* it).

Hence two non-identical prime groups cannot be in syzygy.

The same will be true of three non-identical prime groups in a ternary system.

If, in a definite binary system, each of a certain set of groups is a prime group, and no two of the groups the same, the system will be asyzygetic so far as this set of groups or their variables is concerned.

Importance of the case of equal, that is, identical, coefficients or coefficient groups.

The symmetric functions of the roots of indeterminate equations may be expressed as denumerants to equations or equation-systems with equal coefficients or coefficient groups.

Scheme: its definition as collective name for cluster and primary.

Scheme: linear, plane, or solid.

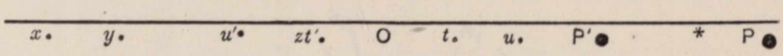


Fig. 1.

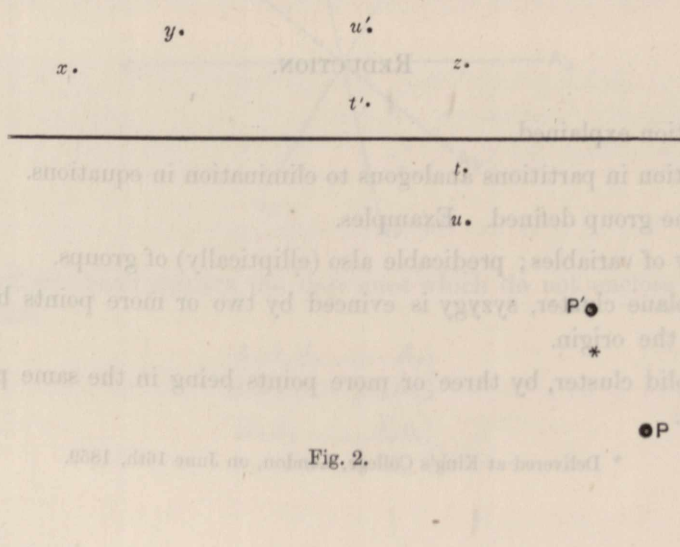


Fig. 2.

Centre: Axis: Balancing plane of scheme: denumerant of a linear scheme in respect to a given centre; of a plane scheme in respect to a given centre or axis; of a solid scheme in respect to a centre, axis, or plane.

Connumerant of a linear scheme in respect to a given centre; of a plane scheme in respect to a given axis*; of a solid scheme in respect to a given plane.

Notice the algebraical sign of the *connumerant*, which is positive or negative, according as an even number (including *zero* as one) or an odd number of transpositions of cluster-points is *transposed*.

Definitions of rays and planes of cluster recalled and applied to schemes.

The term beam substituted for ray of the primary.

The theory of the reduction of binary systems of equations may be geometrically stated.

Network. In line, plane, or solid.

Nodes and nodal lines.

Prime *point* or prime *ray* in network corresponds to prime groups of coefficients.

Prime *couples* or prime *planes* in network correspond to prime double-groups of coefficients, meaning a pair of groups whose minor determinants form a prime group.

Anticipatory statement, namely—

The denumerant of a plane scheme in respect to a given centre is the sum of its connumerants with respect to each in succession of the rays which lie on either side (*chosen at will*) *of the beam, provided that all the rays on the side so chosen are prime rays, and the points to which they are drawn are no two of them coincident. If these conditions are satisfied on both sides of the beam, each of the segments of the ray-cluster into which it is divided by the beam will give a distinct solution, and the two sums of connumerants appertaining respectively to the rays in either cluster will be equal to one another.*

Observe that, if the system be neuter, all the rays will be on one side of the beam, and there will be but one solution.

The denumerant of a solid scheme in respect to a given centre (corresponding to a ternary system of equations), which satisfies analogous conditions to the preceding, will be shown later on in the course to be expressible in very similar terms (cluster-planes being substituted for cluster-rays), with this remarkable difference, however, that, in lieu of a single dichotomous

* If the motion of P should carry it to P' on the opposite side of the axis, the transformed centre and primary will be brought to lie on one side of the axis, and consequently the cluster must have contrary signs to the primary, in order to balance about the axis; and, as there will thus be no omni-positive solution, the connumerant in that case will become zero. If P is sufficiently remote, this cannot take place.

division of the planes, there will be a considerable number of such, each of which will or may furnish a distinct pair of solutions.

The formation of these dichotomies involves the consideration of the doctrine of *normal orders*, or orders of *perspective sequence*—a branch of the doctrine of free geometry to which allusion was made in the opening address.

The problem of partitions stated as a problem in plane or solid network.

A system of equations in $x, y, z, \dots u$ may be denoted by $S(x, y, z, \dots u)$, or, when more convenient, by S alone, with implied reference to $x, y, z, \dots u$.

Resultants of systems. $R_x S$, where S is binary, defined. $R_{x,y} S$, where S is ternary, defined.

$R_x S$ is the equation which expresses that the coefficient cluster and primary of S balance about the axis Ox. This will remain good for a ternary system, so that $R_x S$ will then denote a specific binary system, that which corresponds to *projection* of cluster and primary on a plane through the origin perpendicular to O_x. $R_{x,y} S$ will denote that the centre of gravity of the cluster and primary of S is in the plane xy.

Interpretation of $\text{Ꭰ} R_x S$ when S is binary. Interpretation of the same when S is ternary.

$\text{Ꭰ} R_x S$ in the latter case is perfectly definite just as much as in the former, although the modes of expressing $R_x S$ are infinitely varied.

If S' is what S becomes when we write in S, $fx + g$, or more generally ϕx, in place of x we may denote S' symbolically by $\frac{\phi x}{x} S$.

Note that $$R_x S = R_x \left(\frac{\phi x}{x} S\right)$$

$$\left(\frac{\phi x}{x} \cdot \frac{\phi y}{y} \cdot \frac{\phi z}{z} \dots S\right) \text{ explained.}$$

Order of operative symbols $\frac{\phi x}{x}$, $\frac{\phi y}{y}$, &c. is indifferent. The denumerant of any principal derivative $R_x S$, if homogeneous, will furnish a superior limit to the denumerant of S; for all the solutions of F must be solutions of $R_x S$.

Hence, to the denumerant of a definite binary system we can always, by *simple* denumeration, obtain two superior limits; to the denumerant of a ternary system, some number of superior limits, between 3 and n inclusive, such number depending upon the morphological character of the system (as will hereafter be explained).

Examples of superior limits to binary denumerants.

Examples of superior limits to ternary denumerants:—

1. By means of principal *simple* derivatives.
2. By means of principal derivative *binary systems*.

Hereafter we shall find that when the coefficient groups are all prime groups, and none of them alike, these two limits are the respective first terms of two distinct finite series of connumerants, each of which expresses the value of the denumerant of the given binary systems.

Lemma. If the x group in any system is a prime group, any omni-positive integer solution of $R_x S$ is in general an omni-positive integer solution either of S or of $\underset{x}{\overset{-x}{=}} S$.

Proof in case of binary system.

Proof in case of ternary or ultra-ternary system.

How an exception arises when the solution of $R_x S$, substituted in S, makes $x = 0$.

Were it not for this exception, the equation following would always subsist for any variable x corresponding to a prime group, namely,

$$\mathfrak{A}S + \mathfrak{A}\left(\underset{x}{\overset{-x}{=}} . S\right) = \mathfrak{A}RS.$$

The number of omni-positive solutions of system $S(x, y, z, \ldots v)$, subject to the condition $x > k$, is the denumerant of

$$S\{(x+k), y, z, \ldots v\}.$$

Thus, if $x > 0$, for x we must substitute $1 + x$. Hence the true equation which connects the denumerants referred to *without* exception is

$$\mathfrak{A}S + \mathfrak{A}\underset{x}{\overset{-1-x}{=\!=\!=}}(S) = \mathfrak{A}R_x S;$$

or, if we please,

$$= \mathfrak{A}R_x \underset{x}{\overset{-1-x}{=\!=\!=}} S.$$

In future I shall denote

$$-x-1 \text{ by } \bar{x},$$
$$-y-1 \text{ by } \bar{y},$$

and so on.

We may therefore write

$$\mathfrak{A}S = \mathfrak{A}R_x \underset{x}{\overset{\bar{x}}{=}} S - \mathfrak{A}\left(\underset{x}{\overset{\bar{x}}{=}} . S\right).$$

Now let the y group be also a prime group; we shall have

$$D\underset{x}{\overset{\bar{x}}{=}} S = S\mathfrak{A}R_y\left(\underset{x}{\overset{\bar{x}}{=}}\underset{y}{\overset{\bar{y}}{=}} S\right) - \mathfrak{A}\left(\underset{x}{\overset{\bar{x}}{=}}\underset{y}{\overset{\bar{y}}{=}} S\right);$$

therefore $$\mathfrak{A}S = \mathfrak{A}R_x \underset{x}{\overset{\bar{x}}{=}} S - \mathfrak{A}R_y\left(\underset{x}{\overset{\bar{x}}{=}}\underset{y}{\overset{\bar{y}}{=}} S\right) + \mathfrak{A}\left(\underset{x}{\overset{\bar{x}}{=}}\underset{y}{\overset{\bar{y}}{=}} S\right);$$

and so, if the z group be a prime group,

$$\mathrm{C}S = \mathrm{C}R_x \frac{\overline{x}}{x} S - \mathrm{C}R_y \left(\frac{\overline{x}}{x}\frac{\overline{y}}{y} S\right) + \mathrm{C}\left(R_z \frac{\overline{x}}{x}\frac{\overline{y}}{y}\frac{z}{z} S\right) - \mathrm{C}\left(\frac{\overline{x}}{x}\frac{\overline{y}}{y}\frac{\overline{z}}{z} S\right),$$

and so on to any extent.

Extensions of this Equation to Systems of Equations of a Higher Order than the first indicated.

This I call the process of eduction. The question above indicated is always *true*, amounting in fact to the assertion of identity as regards the *solutions* themselves (not merely their number) of the systems on one side of the equation and those on the other. But, although true, it will be *nugatory* if any of the systems become indefinite, for then in general their denumerants will be infinite in magnitude.

The above equation applies to systems of any order. Its application will be first studied in respect to binary systems.

By continuing the process of eduction through a sufficient number of steps, we shall find that the equation $R_t\left(\frac{\overline{x}}{x}\frac{\overline{y}}{y}\cdots\frac{\overline{z}}{z}\frac{\overline{t}}{t}\right) S$ will become at length *incongruous*. Its denumerant will then vanish.

When this is the case, *a fortiori*, the denumerant of the system $\frac{\overline{x}}{x}\frac{\overline{y}}{y}\cdots\frac{\overline{z}}{z}$ will vanish. And thus the series is brought to a close, and the denumerant of S expressed entirely in terms of *simple* denumerants.

FOURTH LECTURE*.

THEORY OF EDUCTION (*continued*).

Process of eduction exemplified. Suppose the system $S(x, y, z, t)$; then

$$\mathrm{C}S = \mathrm{C}R_x S - \mathrm{C}\frac{\overline{x}}{x} R_y S + \mathrm{C}\frac{\overline{y}}{y}\cdot\frac{\overline{x}}{x} R_z S - \mathrm{C}\frac{\overline{z}}{z}\cdot\frac{\overline{y}}{y}\cdot\frac{\overline{x}}{x} R_t S + \mathrm{C}\left(\frac{\overline{t}}{t}\cdot\frac{\overline{z}}{z}\cdot\frac{\overline{y}}{y}\cdot\frac{\overline{x}}{x}\right) S;$$

but, if S is definite positive, $\frac{\overline{t}}{t}\cdot\frac{\overline{z}}{z}\cdot\frac{\overline{y}}{y}\cdot\frac{\overline{x}}{x} S$ is definite negative. Hence its denumerant is zero, and

$$\mathrm{C}S = \mathrm{C}R_x S - \mathrm{C}\frac{\overline{x}}{x} R_y S + \mathrm{C}\frac{\overline{y}}{y}\frac{\overline{x}}{x} R_z S - \mathrm{C}\frac{\overline{z}}{z}\frac{\overline{y}}{y}\frac{\overline{x}}{x} R_t S.$$

The same equation will subsist if S be definite neuter, but not if S be definite negative or indefinite.

* Delivered at King's College, London, on June 20th, 1859.

It is not necessary in general that *all* the coefficient groups should be prime groups, or *all* of them distinct from one another. Great importance of this observation.

Depression of order of denumerants by one degree.

Depression by several degrees :—(1) By successive eductions. (2) By one compound eduction.

Observe that successive eduction can only finally conduct to equations which are simple resultants of the original system, being resultants of its resultants.

Allusion to fundamental theorem for depression by two degrees, namely—

$$\text{\reflectbox{D}} R_{x.y} . S = \text{\reflectbox{D}} S + \text{\reflectbox{D}} . \underset{x}{\overset{\overline{x}}{=}} S + \text{\reflectbox{D}} . \underset{y}{\overset{\overline{y}}{=}} S + \text{\reflectbox{D}} . \underset{x}{\overset{\overline{x}}{=}} . \underset{y}{\overset{\overline{y}}{=}} S.$$

This equation is subject to the condition that the minor determinants of the matrix formed by the x and y coefficient groups conjoined shall form a prime group.

Observe the singular symbolical equations—

$$R_x = -\frac{1}{x}, \quad R_{x,y} = R_x . R_y = \frac{1}{x} \times \frac{1}{y}.$$

Notice that the lemma at p. [139] is true for systems of any order.

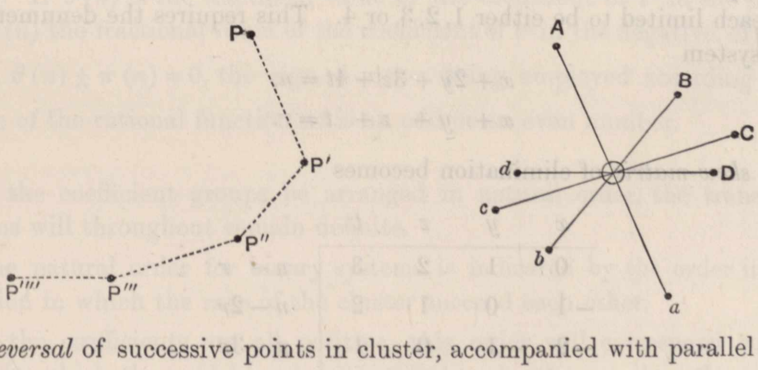
Problem of normal sequences stated. Its geometrical solution by means of the perspective to the cluster, (if a plane cluster) upon a line, (if a solid cluster) upon a plane, or (if a hypersolid cluster) upon a space.

Binary Systems.

Geometrical representation of the successive systems

$$S . \underset{x}{\overset{\overline{x}}{=}} S, \quad \underset{y}{\overset{\overline{y}}{=}} \underset{x}{\overset{\overline{x}}{=}} S, \text{ \&c.}$$

in which S is supposed to be definite and positive. (See figure.)



Reversal of successive points in cluster, accompanied with parallel motion of primary.

The systems $\frac{\bar{x}}{x}.S$, $\frac{\bar{y}}{y}\frac{\bar{x}}{x}.S$, &c. may be regarded as successive deformations of S, and then we may say the system, as it undergoes deformation, tends more and more to lose its positive character, until at length it becomes *neutral*, and immediately after changes into and continues *negative*.

The deformation may be commenced from either side. There are thus two courses of deformation.

If the primary were stationary, the deformed system in either course would become neuter after as many deformations as there are rays in S outside the beam, on that side of it from which the deformation proceeds.

The effect of the motion of the primary is to *accelerate* the tendency of the deformed system to become neuter.

If the primary be moved along the beam to a sufficient distance from the origin, the effect of such tendency will become at length insensible for such and for all greater distances.

The number of deformations in either course which may take place before the primary ray becomes denuded gives the number of terms in the development (by the eduction process) corresponding to that course.

When the primary is sufficiently remote, the sum of these numbers corresponding to the two courses of deformation will be the number of rays, that is, the number of variables in the system.

On account of the motion of the primary, the united sum may be *less* than this number; it cannot be greater.

Corollary. If all the groups are prime groups, the denumerant of a binary system may be expressed by a number of simple denumerants, *not greater* than half the number of variables in the system.

The question of partitions, given in number and species, is expressed by a binary system satisfying these conditions.

Example. To find the number of ways in which n can be made up of r values each limited to be either 1, 2, 3, or 4. This requires the denumeration of the system

$$x + 2y + 3z + 4t = n,$$
$$x + y + z + t = r.$$

The *skew-matrix* of elimination becomes

x	y	z	t	
0	1	2	3	$n - r$
-1	0	1	2	$n - 2r$
-2	-1	0	1	$n - 3r$
-3	-2	-1	0	$n - 4r$.

If $n < r$, the system is neuter, and the number required is 0.

If $n > r$ but $< 2r$, the number required is the connumerant $\dfrac{n-r;}{1, 2, 3;}$ which is the same in value as the sum of the three complementary connumerants.

If $n > 2r < 3r$, the number required is the sum of the connumerants

$$\frac{n-r;}{1, 2, 3;} + \frac{n-2r;}{-1, 1, 2;}, \text{ or, if we please, } \frac{4r-n;}{1, 2, 3;} + \frac{3r-n;}{-1, 1, 2;}.$$

If $n > 3r$ but $< 4r$, the number required is $\dfrac{4r-n;}{1, 2, 3;}$, which is the same in value as the sum of the three complementary denumerants.

If $n > 4r$, the system is again neuter, and its denumerant is zero.

By aid of the above expressions we may give the general analytical representation of the number of partitions of the kind proposed.

These expressions, it will be observed, are discontinuous, the particular one to be employed depending on the comparative values of n and r. It may be shown, however, that they, as it were, melt or modulate into one another, so not only at the mere limiting values of n, which separate the several formulæ, but also for a short distance beyond these limits, either of the conterminous expressions may be used indifferently.

The proof of this depends upon the proposition that the coefficient of t^n in the *ascending* expansion of

$$\frac{1}{(1-t^a)(1-t^b)(1-t^c)\dots(1-t^k)}$$

treated *as a function of* n remains fixed at zero, when n is made to become any negative number whose absolute value is inferior to

$$a+b+c+\dots+k.$$

The truth of this proposition follows immediately from a theorem of Mr Cayley relating to the development of any rational fraction $\dfrac{1}{\phi t}$ in its two forms. If $\theta(n)$ is the fractional value of the coefficient of t^n in the positive, and $\pi(n)$ the fractional value of the coefficient of t^n in the negative, expansion of $\dfrac{1}{\phi t}$, $\theta(n) \pm \pi(n) = 0$, the sign + or − being employed according as the degree of the rational function πt is an odd or an even number.

If the coefficient groups be arranged in natural order, the transformed systems will throughout remain definite.

The natural order for binary systems is indicated by the order in either direction in which the rays of the cluster succeed each other.

If the coefficients are all positive, this order will correspond with the order in which the variables must succeed each other, so that the ratios in each group of the coefficient out of one given equation with the corresponding coefficient out of the other may continually increase or decrease.

But, if the coefficients are positive and negative intermingled, the rule is that the determinants formed by the combination of any group with each in succession of those which follow must all bear the same sign; or, as we may express it, the algebraical sign arising from the contact of the groups, with due regard to antecedence, must be always the same.

If the system were indefinite, such a uniformity of signs could not be established by any arrangement of the groups whatever.

Example (1):—

$$\left.\begin{array}{l} x+y+z=50 \\ x+2y+3z=120 \end{array}\right\}.$$ In this system all the groups are prime groups.

First solution.

x	y	z	
0	1	2	70
−1	0	1	20
−2	−1	0	30.

Observe that the coefficients form a skew-matrix.

$$R_x \cdot \frac{\bar{x}}{x} \cdot S \text{ is } y+2z=70.$$

$$R_y \cdot \frac{\bar{y}}{y} \cdot \frac{\bar{x}}{x} \cdot S \text{ is } (x+1)+z=20, \text{ or } x+z=19.$$

$$R_z \cdot \frac{\bar{z}}{z} \cdot \frac{\bar{y}}{y} \cdot \frac{\bar{x}}{x} \cdot S \text{ is } 2(x+1)+(y+1)=-30, \text{ or } 2x+y=-33.$$

The desired denumerant will be $\dfrac{70;}{1,2;} - \dfrac{19;}{1,1;}$ which is

$$36-20=16.$$

Second solution.

z	y	x	
0	1	2	30
−1	0	1	−20
−2	−1	0	−70.

$$R_z \cdot \frac{\bar{z}}{z} \cdot S \text{ is } y+2x=30.$$

$$R_y \cdot \frac{\bar{y}}{y} \cdot \frac{\bar{z}}{z} \cdot S \text{ is } z+x=-21.$$

$$R_x \cdot \frac{\bar{x}}{x} \cdot \frac{\bar{y}}{y} \cdot \frac{\bar{z}}{z} \cdot S \text{ is } 2z+y=-73.$$

The desired denumerant will therefore be $\dfrac{30\,;}{1,\ 2\,;} = 16$, as before.

Observe, that $\mathit{Ǝ}S$ is the *sum* of the connumerants of R_xS, R_yS, R_zS, when these equations are written in the form in which they appear when the process of successive elimination is conducted by a uniform course of operations. This may be done in two ways, so as to give rise to two sets of equations differing from each other only in the signs. Observe, that the connumerant $L = \pm c$ is not the same as of $-L = \mp c$; of that one of these equations, in which the constant term is negative, the connumerant being zero, but of the other the connumerant being generally different from zero.

Corollary. The coefficient of $t^m \tau^\mu$ in

$$\frac{1}{(1 - t^a\tau^\alpha)(1 - t^b\tau^\beta)(1 - t^c\tau^\gamma)}$$

is the sum of the coefficients of

$$\rho^{a\mu - \alpha m} \text{ in } \frac{1}{(1 \sim \rho^{a\beta - b\alpha})(1 \sim \rho^{a\gamma - c\alpha})},$$

of

$$\rho^{b\mu - \beta m} \text{ in } \frac{1}{(1 \sim \rho^{b\alpha - a\beta})(1 \sim \rho^{b\gamma - c\beta})},$$

and of

$$\rho^{c\mu - \gamma m} \text{ in } \frac{1}{(1 \sim \rho^{c\alpha - a\gamma})(1 \sim \rho^{c\beta - b\gamma})},$$

subject to the interpretation that the preceding fractions are to be expanded all in terms of ascending, or all in terms of descending, powers of ρ, *provided* that the system

$$\left.\begin{aligned} ax + by + cz + dt &= m \\ \alpha x + \beta y + \gamma z + \delta t &= \mu \end{aligned}\right\}$$

is a definite non-negative system.

Allusion to Mr Cayley's proof of the proposition in this form. The proposition is, of course, general, that the denumerant of a definite binary system with r variables, in which the groups are all prime groups, admits of a double mode of representation as the sum of the connumerants of its principal derivatives. In the one mode of representation, only a certain number, at *utmost*, of these connumerants, say p, can differ from zero; in the other mode, only a certain number, at utmost, say q, can differ from zero; and, as we shall have $p + q = r$, these two modes of representation may be termed complementary to each other.

The denumerant of the system

$$\left.\begin{aligned} ax + by + cz + \ldots + dt &= m \\ x + \ \ y + \ \ z + \ldots + \ \ t &= \mu \end{aligned}\right\}$$

may therefore be represented in two modes as the sum of simple denumerants.

This is the system the denumeration of which constitutes the problem of the resolution of integers into parts given in number and species. See Euler's *Second Memoir on Partitions*.

The theorem of eduction may be put under the following form :—

$$\mathcal{C}S = \mathcal{C}R_x \frac{\bar{x}}{x} S - \mathcal{C}R_y \frac{\bar{y}}{y} \frac{\bar{x}}{x} S + \mathcal{C}R_z \frac{\bar{z}}{z} \frac{\bar{y}}{y} \frac{\bar{x}}{x} S \mp \&c.,$$

provided that S is a definite system in which the groups of coefficients appertaining to the several variables, or at least to so many of them as are included between x and the last of them which appears as a suffix in a non-zero term of the above expression, are all distinct prime groups.

2nd Example :—

$$\left.\begin{matrix} x + y + z = 10 \\ x + 2y + 3z = 40 \end{matrix}\right\}.$$

x	y	z			x	y	z
0	1	2	30	− 10	2	1	0
− 1	0	1	20	− 20	1	0	− 1
− 2	− 1	0	10	− 30	0	− 1	− 2.

The solution corresponding to the right-hand matrix is evidently 0, the system in fact being neuter. The left-hand matrix gives the solution—

$$\frac{30\,;}{1,\,2\,;} + \frac{20\,;}{-1,\,1\,;} + \frac{10\,;}{-2,\,-1\,;}$$

$$= \frac{30\,;}{1,\,2\,;} - \frac{19\,;}{1,\,1\,;} + \frac{7\,;}{2,\,1\,;}$$

$$= 16 - 20 + 4 = 0, \text{ as before.}$$

Illustrate effect of taking the variables in abnormal order.

3rd Example :—

$$2x + 5y + 2z - t = m,$$
$$x + y - 2z - 2t = m,$$

the system will also be neuter, and we shall have

$$\frac{m\,;}{6,\,9,\,3\,;} + \frac{2m\,;}{-3,\,6,\,3\,;} + \frac{4m\,;}{-9,\,-12,\,3\,;} + \frac{m\,;}{-3,\,-6,\,-3\,;} = 0,$$

or
$$\frac{m\,;}{3,\,6,\,9\,;} - \frac{2m-3\,;}{3,\,3,\,6\,;} + \frac{4m-21}{3,\,9,\,12\,;} - \frac{m-12}{3,\,3,\,6\,;} = 0.$$

Allusion to importance and fertility of theory of neuter systems.

Example of a denumeration of a binary system containing unprime groups :—

$$x + y + 4z + 3t = m,$$
$$3x + 2y + 6z + 3t = n,$$

$\frac{m}{n}$ being supposed to be intermediate between $\frac{1}{2}$ and $\frac{2}{3}$. The denumerant will be

$$\frac{3m-n\,;}{1,\,6,\,6\,;} - \frac{2m-n\,;}{-1\,;\,2\,;\,3\,;}.$$

Cæsura (definition of, and how determined), accidental and universal, distinguished.

FIFTH LECTURE*.

Eduction and Reduction.

The cæsura for equation-systems generally falls after that coefficient group subsequent to the introduction of which, in the eduction process, the depressed systems whose denumerants are to be taken *must* cease to be positive, so that they may be neglected. It is determined for binary systems by the relation of the ratios of the terms in the coefficient groups to that of the terms in the constant group; the determinant formed by the apposition of the constant group with any group on one side of the cæsura being positive, on the other side negative.

The point after which the terms in an eduction process can be neglected *may* (if the constant terms are sufficiently small) be attained before the cæsura is reached. Such a point may be termed a turning-point, or pause. There may thus (in the case of binary systems) be two turning-points or pauses on each side of the cæsura corresponding to the two courses of eduction, but either or both of them may, and in general will, coincide with the cæsura.

For greater simplicity, we may suppose the constant terms given in ratio only, and not in magnitude, so as to obviate the necessity of paying any attention to the accidental pauses as distinguishable from the cæsura. The cases where they are so distinguishable are always exceedingly limited in number. Their existence arises solely from the fact of the introduction of $-x-1$, $-y-1$, &c., and not $-x$, $-y$, &c., in lieu of x, y, in applying the method of eduction.

A *per-reducible* binary system is one in which *all* the coefficient groups are *prime* groups *distinct* from each other.

Being prime and distinct, none of them can be in syzygy. Such a system admits of a double process of eduction, giving rise in general to two distinct forms of solution. But it may happen, in some very special cases, that these two solutions are identical in form as well as in value.

Example. The system

$$\left.\begin{array}{l} x+3y+7z+9t=5i \\ x+2y+4z+5t=3i \end{array}\right\}$$

gives rise to the bordered matrix

$$\begin{array}{cccc|c} 0 & 1 & 3 & 4 & 2i \\ -1 & 0 & 2 & 3 & i \\ -3 & -2 & 0 & 1 & -i \\ -4 & -3 & -1 & 0 & -2i \\ \hline -2i & -i & i & 2i. & \end{array}$$

* Delivered at King's College, London, June, 1859.

Here each solution is the same, namely, $\dfrac{2i;}{1, 3, 4;} + \dfrac{i;}{-1, 2, 3;}$, meaning

$$\frac{2i;}{1, 3, 4;} - \frac{i-1;}{1, 2, 3}.$$

But, if the constant terms in the above system were $11i$ and $7i$ respectively, the bordered matrix would be

$$\begin{array}{|cccc|c} 0 & 1 & 3 & 4 & 4i \\ -1 & 0 & 2 & 3 & i \\ -3 & -2 & 0 & 1 & -5i \\ -4 & -3 & -1 & 0 & -8i \\ \hline -4i & -i & 5i & 8i & \end{array}$$

giving rise to the two equal sums of connumerants,

$$\frac{4i;}{1, 3, 4;} + \frac{i;}{-1, 2, 3;} \text{ and } \frac{8i;}{1, 3, 4;} + \frac{5i;}{-1, 2, 3;}.$$

In this example the matrix happens to be *persymmetrical*, which is the reason of the denumeratives being the same in each solution.

This is avoided in the example below of the system

$$\left.\begin{array}{l} x + 2y + z + t = 7i \\ x + 3y + 2z + 4t = 12i \end{array}\right\},$$

for which the bordered matrix is

$$\begin{array}{cccc|c} 0 & 1 & 1 & 3 & 5i \\ -1 & 0 & 1 & 5 & 3i \\ -1 & -1 & 0 & 2 & -2i \\ -3 & -5 & -2 & 0 & -16i \\ \hline -5i & -3i & 2i & 16i & \end{array}$$

giving rise to the two equivalent solutions

$$\frac{5i;}{1, 1, 3;} + \frac{3i;}{-1, 1, 5;} \text{ and } \frac{16i;}{2, 5, 3;} + \frac{2i;}{-2, 1, 1;},$$

meaning

$$\frac{5i;}{1, 1, 3;} - \frac{3i-1;}{1, 1, 5;} \text{ and } \frac{16i;}{2, 3, 5;} - \frac{2i-2;}{1, 1, 2;}.$$

A *simply-reducible* system is one for which the coefficient groups are prime and distinct on *one* side of the cæsura only.

Example:—

$$\left.\begin{array}{l} x + 2y = 4m \\ x + 4y = 5m \end{array}\right\}.$$

The eduction from the x side gives rise to the equation of $2y = m$, of which the denumerant is $\dfrac{m;}{2;}$. This is the true solution, whereas the eduction from the y side gives rise to the denumerant of $2x = 6m$, that is, 1, which is a false solution, owing to the group (2, 4) being a non-prime group.

If in a binary system the groups, which are either non-prime or repeated, or non-prime and repeated, represent ratios (between quantities given in algebraical sign) which are all less or all greater than the corresponding ratio of the constant terms, the system is still depressible by eduction commenced from that side of the system on which the groups of the kind mentioned do not fall.

Corollary. A single non-prime group, or a single sequence of any number of identical groups, can in no case hinder a binary system from being soluble by eduction.

The above remark is true also *à fortiori* for ultra-binary systems.

It should be noticed that $\left.\begin{matrix} a \\ 0 \end{matrix}\right\}$ is a non-prime group unless $a = \pm 1$. (For non-prime we may in future use the term composite.)

Example:— $\left.\begin{aligned} 3x + 2y + z + t &= i \\ 2z + 3t &= i \end{aligned}\right\}.$

Here the coefficient groups of x and y are both of them composite; but, the cæsura falling between y and z, the denumerant required will be the sum of the connumerants of the two resultants in respect to t and z, that is,

$$\frac{2i\,;}{1,\,6,\,9\,;} + \frac{i\,;}{-1,\,4,\,6\,;},$$

meaning

$$\frac{2i\,;}{1,\,6,\,9\,;} - \frac{i-1\,;}{1,\,4,\,6\,;}.$$

If a system is affected with composite or repeated groups on *each* side of the cæsura, its denumeration may be made to depend on systems where such groups exist on only one side of their respective cæsuras*.

Example:— $\left.\begin{aligned} 10x + 2y + 3z &= 5i \\ 15x + 4y + 9z &= 11i \end{aligned}\right\}$, which call S.

If we form a ternary system as follows:—

$$\left.\begin{aligned} 10x + 2y + 3z \quad &= 5i \\ 15x + 4y + 9z \quad &= 11i \\ px + qy + rz - t &= -m \end{aligned}\right\}, \text{ which call } S',$$

* In certain special cases the composite groups may be reduced in number by substituting a connective of the equations in lieu of one of them, as in the example

$$\left.\begin{aligned} 10t - 7z - 8y &= 5i \\ 9z + 2y &= 7i \end{aligned}\right\},$$

which is apparently irreducible, but which, put under the equivalent form

$$\left.\begin{aligned} 10t - 7z - 8y &= 5i \\ 15t - 6z - 11y &= 11i \end{aligned}\right\},$$

becomes simply-reducible.

where p, q, r, m are any positive integers whatever, it is apparent that the omni-positive solutions of S' may be found from the omni-positive solutions of S, and to each of the latter will correspond one, and only one, of the former. Hence the denumerant of S is the same as the denumerant of S', and, if p, q, r are so chosen that 10, 15, p; 2, 4, q; 3, 9, r are all prime groups, $\mathfrak{C}S'$, and therefore $\mathfrak{C}S$, may be made to depend on the denumerants of a certain set of new binary systems obtained by the eduction of S'. Thus let $p=1$, $q=1$, $r=1$, $m=0$, so that the auxiliary equation becomes

$$x+y+z-t=0.$$

R_xS' may be represented by

$$\left.\begin{aligned}10t-8y-7z&=5i\\15t-11y-6z&=11i\end{aligned}\right\},$$

R_yS' by

$$\left.\begin{aligned}2t+8x+z&=5i\\4t+11x+5z&=11i\end{aligned}\right\},$$

R_zS' by

$$\left.\begin{aligned}3t+7x-y&=5i\\9t+6x-5y&=11i\end{aligned}\right\},$$

R_tS' by

$$\left.\begin{aligned}10x+2y+3z&=5i\\15x+4y+9z&=11i\end{aligned}\right\}, \text{ being the original system } S.$$

It will be seen therefore that

$$R_xS';\quad \frac{\overline{x}}{x}R_yS';\quad \frac{\overline{y}}{y}\cdot\frac{\overline{x}}{x}R_zS';\quad\text{and}\quad \frac{\overline{z}}{z}\cdot\frac{\overline{y}}{y}\cdot\frac{\overline{x}}{x}R_tS'$$

respectively represent the systems following:—

$$\left.\begin{aligned}10t-8y-7z&=5i\\15t-11y-6z&=11i\end{aligned}\right\},\qquad(1)$$

$$\left.\begin{aligned}2t+z-8x&=5i+8\\4t+5z-11x&=11i+11\end{aligned}\right\},\qquad(2)$$

$$\left.\begin{aligned}3t+y-7x&=5i+6\\9t+5y-6x&=11i+1\end{aligned}\right\},\qquad(3)$$

$$\left.\begin{aligned}-10x-2y-3z&=5i+15\\-15x-4y-9z&=11i+28\end{aligned}\right\}.\qquad(4)$$

All these four systems are definite: in the first of them the natural order of the groups is

$$\begin{pmatrix}10 & -7 & -8\\15 & -6 & -11\end{pmatrix};$$

in the others the natural order is that in which they are written. The first two only will be definite-positive, the last will be definite-negative, and the last but one neuter if $i > 0$, negative if $i = 0$, and the denumerant required will be the difference between the denumerants of the two systems

$$\left.\begin{array}{l} 10p - 7q - \ 8r = \ 5i \\ 15p - 6q - 11r = 11i \end{array}\right\},$$

$$\left.\begin{array}{l} 2p + \ q - \ 8r = \ 5i + \ 8 \\ 4p + 5q - 11r = 11i + 11 \end{array}\right\}.$$

In each of these systems there is one, but only one, of the original non-prime groups, and no new ones have been introduced.

Consequently they admit of being depressed, and the final result will be an aggregate of simple denumerants.

If we had applied to S' a different course of eduction as follows :—

$$R_z S', \ \frac{\bar{z}}{z} R_y S', \ \frac{\bar{z}}{z}\frac{\bar{y}}{y} R_x S', \ \frac{\bar{z}}{z}\frac{\bar{y}}{y}\frac{\bar{x}}{x} R_t S',$$

it may easily be seen that all these systems likewise would be definite, and only the first of them definite-positive. Hence a second solution of the question will be $\mathfrak{C} R_z S'$, that is,

$$\frac{5i, \ 11i\,;}{3, 9\,; \ 7, 6\,; \ -1, -5\,;}\text{'}$$

which is of a depressible form, there being only one *affected* group (3, 9), and may be educed into a linear function of simple denumerants.

Dispersion process defined.

Cases which resist its application.

Theorem. The denumeration of any equation-system whatever may be made to depend upon the denumeration of systems that shall contain no composite groups, and at most only one set of repeated groups, and which will consequently be depressible.

Proof of this theorem in case of binary systems.

Definition of meaning of kG, and of $kG \pm lG'$, where G, G' represent any two coefficient groups of a system, and k, l are any two integers*.

Lemma 1. A system S, containing the coefficient group G, may be made to depend for its denumeration upon systems in each of which the coefficient groups are the same as in S, except that kG takes the place of G.

Lemma 2. A system S, containing the groups G and G', may be made to depend upon two systems, in one of which the coefficient groups are the same as in G, with the exception that H replaces G, and in the other the

* In a definite ternary system, where all the coefficient groups are prime groups, it may be shown that the only possible cases of syzygy are where $F = G$ or $F + G = H$ (F, G, H denoting coefficient groups of the system).

same as in G, with the exception that H replaces G', where H is $G - G'$ or $G' - G$.

Note that, if, instead of the coefficient group G in any definite system, first any other group H and then $-H$ be substituted, one *at least* of these substitutions must leave the deformed system definite.

Lemma 3 (*Corollary to Lemma* 1). Any equation-system may be made to depend for its denumeration on equation-systems in each of which one of the equations has all its coefficients positive units.

It follows from this lemma that, if a binary equation-system is free from syzygy (that is, from equalities of ratios between the coefficients of different variables), its denumeration may be made to depend upon that of systems which [their coefficient groups being all different and of the form $(a, 1)$] are per-reducible. But, if there be e sets of syzygies in the given system, there will be e sets of repetitions in the groups $(a, 1)$ in each of the deduced systems.

Lemma 4. In the case immediately above supposed, the e sets of syzygetic groups in the deduced systems may be replaced by e other syzygetic sets of groups of which all but one are of the form $(a, 1), (a, 1), \ldots ; (b, 1), (b, 1), (b, 1), \ldots$, &c., and that one of the form $(\sigma, 0) ; (\sigma, 0) ; (\sigma, 0), \ldots$.

Lemma 5. Any system of the form last supposed may (by virtue of Lemma 2) be replaced by two, in one of which $\pm (a - k\sigma, 1)$ takes the place of $(a, 1)$, and in the other $\pm (a - k\sigma, 1)$ takes the place of $(\sigma, 0)$, k being so chosen that $(a - k\sigma, 1)$ is distinct from every other coefficient group associated with it in the same system.

Lemma 6. Hence, by *repeated* application of this last process of replacement, the number of syzygetic groups in the deduced systems may be continually reduced until we arrive at systems in one class of which all the groups $(\sigma, 0)$ have disappeared, and in the other class of which all the syzygetic groups except those of the form $(\sigma, 0)$ have disappeared.

Lemma 7. Hence, so long as e is greater than 1, the deduced systems will eventually none of them contain more than $(e-1)$ sets of groups in syzygy, and thus we must eventually arrive at systems in none of which will be found more than a single set of groups in syzygy, which may be taken indifferently of the form $(a, 1)$ or $(a, 0)$.

Consequently the denumeration of every binary system, if free of syzygies, may be made to depend on the denumeration of per-reducible systems; and, if not free of syzygies, on the denumeration of simply reducible systems.

A similar demonstration may be extended to systems of a higher order than the second. Consequently every denumerant of an order higher than the first may be made to depend on denumerants of a lower order, and eventually upon simple denumerants.

Examples of Reduction of Persyzygetic Systems.

Let the given system S be

$$\left.\begin{array}{ll} x+y & =m \\ z+t & =n \end{array}\right\},$$

and suppose m not less than n. Making

$$x+z-u=0,$$

we obtain the systems

$$\left.\begin{array}{ll} u+y-z & =m \\ z+t & =n \end{array}\right\},$$

which is $R_x S'$, say T; and

$$\left.\begin{array}{ll} u+t+x & =(n-1) \\ x+y & =m \end{array}\right\},$$

which is $\overset{\overline{x}}{\underset{x}{=}} R_z S'$, say U.

Since $\overset{\overline{x}\,\overline{z}}{\underset{x\;z}{=\,=}} S'$ contains the equation

$$-x-z-n=2,$$

the eduction is complete, and the required denumerant $=\mathfrak{G}T-\mathfrak{G}U$.

T arranged in natural order becomes

$$\begin{array}{l} z+t \qquad\qquad\; =n, \\ -z \qquad +u+y=m, \end{array}$$

the cæsura falling between t and u. Accordingly we obtain

$$\mathfrak{G}T=\frac{n+m\,;}{1,1,1\,;}+\frac{m\,;}{-1,1,1\,;}=\frac{n+m\,;}{1,1,1\,;}-\frac{m-1\,;}{1,1,1\,;}$$

$$=\frac{(n+m+1)(n+m+2)}{2}-\frac{m(m+1)}{2}.$$

In like manner

$$\mathfrak{G}U=\frac{n-1\,;}{1,1,1\,;}-\frac{n-1-m\,;}{-1,1,1\,;}=\frac{n-1\,;}{1,1,1\,;}=\frac{n(n+1)}{2},$$

for $n-1-m$ is negative. And we have

$$\mathfrak{G}S=\frac{(n+m+1)(n+m+2)}{2}-\frac{m(m+1)}{2}-\frac{n(n+1)}{2}$$

$$=\frac{2nm+2m+2n+2}{2}=(m+1)(n+1),$$

which is evidently the correct answer, being the product of the denumerants of the two given equations taken independently.

Second Example:—
$$\left.\begin{aligned} x+y+\theta \quad &= m \\ \theta+z+t &= n \end{aligned}\right\}.$$

By the same method as above, we obtain

$$\text{ᗡ}S = \text{ᗡ}T - \text{ᗡ}U,$$

where T is
$$\left.\begin{aligned} z+t+\theta \quad\quad\quad &= n \\ -z \quad +\theta+u+y &= m \end{aligned}\right\},$$

and U is
$$\left.\begin{aligned} u+t+\theta+x \quad &= n-1 \\ \theta+x+y &= m \end{aligned}\right\},$$

the cæsura in T falling between θ and u, and in U between x and y. Hence

$$\text{ᗡ}T = \frac{m+n\,;}{1,\,1,\,1,\,1\,;} - \frac{m-1\,;}{1,\,1,\,1,\,1\,;} + \frac{m-n-3\,;}{2,\,1,\,1,\,1\,;},$$

and
$$\text{ᗡ}U = \frac{n-1\,;}{1,\,1,\,1,\,1\,;}.$$

Thus
$$\text{ᗡ}S = \frac{m+n\,;}{1,\,1,\,1,\,1\,;} + \frac{m-n-3\,;}{1,\,1,\,1,\,1\,;} - \frac{m-1\,;}{1,\,1,\,1,\,1\,;} - \frac{n-1\,;}{1,\,1,\,1,\,1\,;}\,{}^*.$$

Example of a composite group and a syzygy falling on opposite sides of the cæsura. *Problem*:—To express the residue of q in respect to p as a linear function of simple denumerants.

If we call x the required residue, we have

$$x+py=q, \quad x<p,$$

or
$$x+py=q,$$
$$x+z \;\; = p-1.$$

Hence the required residue is the denumerant of the system

$$py+t+u \quad = q-1,$$
$$t+u+z = p-2,$$

in which the coefficient groups are in natural sequence.

In its present form the system is irreducible, because the cæsura falls between y and t (observe that p, 0 is a non-prime group); but, by the method above given, the denumerant of this system, by virtue of the subsidiary equation

$$y+t=v,$$

becomes the difference between the denumerants of

$$\left.\begin{aligned} (1-p)\,t \quad +u+pv &= q-1 \\ t+z+u \quad\quad &= p-2 \end{aligned}\right\}$$

* The value of this expression will evidently be the sum

$$(m+1)(n+1)+mn+(m-1)(n-1)+\&c.+(m-n+1),$$

which is
$$\left(m+1-\frac{n}{3}\right)\frac{(n+1)(n+2)}{2}.$$

and of $\left.\begin{array}{r}(1-p)\,y+v+u+t=(q-1)+(p-1)=q+p-2\\ y+z+u+t=(p-2)-1 \qquad\quad =p-3\end{array}\right\}.$

The second system is neuter, for all the coefficient groups put in apposition with the constant group give determinants with negative values.

Hence the required expression is simply the denumerant of the first system, in which the cæsura falls between u and v. $(p, 0)$ being a composite group, the eduction must be commenced from the t side, and accordingly we obtain the series

$$\frac{(q-1)+(p-1)(p-2);}{(p-1),\,p,\,p;}+\frac{(q-1);}{-(p-1),\,1,\,p;}+\frac{q-p+1;}{-p,\,-1,\,p;}$$
$$=\frac{p^2-3p+q+1;}{p-1,\,p,\,p;}-\frac{q-p;}{1,\,p-1,\,p;}+\frac{q-2p;}{1,\,p,\,p;}$$

as the expression required for the residue of q in respect to p.

SIXTH LECTURE*.

Simple Partition.

Resolution of an integer into a defined number of parts.

With or without repetition.

$\dfrac{n-r;}{1,\,2,\,3,\,\ldots\,r;}$ expresses the r-ary partibility of n when repetitions are allowed;

$\dfrac{n-r\dfrac{r+1}{2};}{1,\,2,\,3,\,\ldots\,r;}$ the same when repetitions are excluded.

Example: $n=7$, $r=3$.

Proof of the above formulæ by Ferrers' method.

When n is great compared with r, these two functions approach to a ratio of equality.

The generating function for partitions without repetition.

Indefinite resolution of numbers with and without repetition.

Generating functions for both these kinds of indefinite resolutions.

Euler's *Series Mirabilis*, and its application.

Remark on indefinite *partition* with the elements 1, 2, 4, 8, &c.

Partition or composition. Partible number. Elements.

* Delivered at King's College, London, July 4th, 1859.

Construction of equation-system whose denumerant is the number of compositions of n with unrepeated elements.

The negation of the *possibility* (for integers) of the equation $x^i + y^i = z^i$ capable of being transformed into the affirmation of an analytical *identity* by the method of partitions.

Resolution of integers into a given number of parts, how treated by Sir John Herschel and others.

Formula of reduction $\dfrac{n\,;}{1,\,2,\,\ldots\,k\,;} = \dfrac{n-k\,;}{1,\,2,\,\ldots\,k\,;} + \dfrac{n-k\,;}{1,\,2,\,\ldots\,(k-1)\,;}$.

Objections to this method:—(1) As inductive instead of direct (besides being limited to a mere special case of partition). (2) As excessively prolix and unmanageable. (3) As leading to an amorphous result.

Mr Cayley's improved method. The true form of representation.

The lecturer's discovery of the general analytical solution.

The provisional method superseded.

Fundamental Theorem in Simple Partition.

Axiom:— $$\frac{n\,;}{a, b, c \ldots l\,;} = \frac{1}{abc \ldots l}\,\Sigma H_n(\alpha, \beta, \gamma, \ldots \lambda),$$

where $H_n(\alpha, \beta, \gamma, \ldots \lambda)$ indicates the sum of the homogeneous powers and products of $\alpha, \beta, \gamma, \ldots \lambda$ of the nth degree, and $\alpha, \beta, \gamma, \ldots \lambda$ are respectively roots of

$$x^a = 1,\; y^b = 1,\; z^c = 1,\; \ldots\; u^l = 1.$$

Example:— $$\frac{7\,;}{2,\,3\,;} = \tfrac{1}{6}\Sigma\,(\rho^7 + \rho^6\sigma + \ldots + \sigma^7),$$

where Σ includes six sums corresponding to the following six systems of values ρ, σ; namely,

$$1, 1;\; -1, 1;\; 1, \rho;\; -1, \rho;\; 1, \rho^2;\; -1, \rho^2,$$

ρ meaning a root of $$\rho^2 + \rho + 1 = 0.$$

In general,

$$H_n(p, q) = \frac{p^{n+1}}{p-q} + \frac{q^{n+1}}{q-p},$$

$$H_n(p, q, r) = \frac{p^{n+2}}{(p-q)(p-r)} + \frac{q^{n+2}}{(q-p)(q-r)} + \frac{r^{n+2}}{(r-p)(r-q)},$$

$$H_n(p, q, r, s) = \frac{p^{n+3}}{(p-q)(p-r)(p-s)} + \&\text{c.} + \&\text{c.} + \&\text{c.}$$

In applying this formula to the preceding axiom, several or all of the quantities p, q, r, &c., will become equal *inter se*, because the equations $a^{\alpha}=1$, $b^{\beta}=1$, $c^{\gamma}=1$, ... have the root unity in common, and will have other roots in common unless α, β, γ, ... are all prime to each other.

The value of $$\Sigma \frac{\phi p_1}{(p_1-p_2)(p_1-p_3)\dots(p_1-p_{e+1})},$$

when $$p_1=p_2=\dots p_{e+1},$$

is $$\frac{1}{1.2.3\dots e}\left(\frac{d}{dp}\right)^e \phi p.$$

Every distinct root of $x^m=1$, where m is the least common multiple of $a, b, c, \dots l$, furnishes a distinct expression to the sum and gives rise to a separate term, in the complete analytical expression for $\frac{n;}{a, b, c, \dots l;}$. Such a term is called a wave. Reason for this name.

There are as many waves as distinct factors in $a, b, c, \dots l$; every such factor as q giving rise to a term W_q.

If $a, b, c, \dots l$ become the series of natural numbers $1, 2, 3, \dots r$, the number of waves is r.

The value* of W_q for $\frac{n;}{a, b, c, \dots l;}$ is the coefficient of $\frac{1}{t}$ in

$$\frac{1}{abc\dots l}\Sigma\frac{(\rho e^t)^n}{[1-(\rho e^t)^{-a}][1-(\rho e^t)^{-b}]\dots[1-(\rho e^t)^{-l}]},$$

where ρ is a *primitive* root of $\rho^q=1$, that is, a root not belonging to $\rho^{q/i}=1$.

The only cases where the quantity under the sign of summation reduces to a single term is when $q=1$, for which case $\rho=1$, and when $q=2$, for which case $\rho=-1$.

W_1 considered. It is non-periodic. It is the coefficient of t^n in the development of $\frac{\phi(t)}{(1-t)^{\varrho}}$, when the Eulerian function

$$\frac{1}{(1-t^a)(1-t^b)\dots(1-t^l)}$$

is supposed capable of being represented under the form

$$\frac{\phi t}{(1-t)^{\varrho}}+\frac{\psi t}{\frac{1-t^a}{1-t}\frac{1-t^b}{1-t}\dots\frac{1-t^l}{1-t}}.$$

It is also the *mean* of the m algebraical forms, m being the least common

[* Cf. p. 91 above.]

multiple of $a, b, c, \dots l$, which represent $\dfrac{n\,;}{a, b, c, \dots l\,;}$ when n is made to go through the m forms

$$\frac{km\,;}{a, b, c, \dots l\,;},\quad \frac{km+1\,;}{a, b, c, \dots l\,;},\ \dots\ \frac{km+(m-1)\,;}{a, b, c, \dots l\,;}.$$

It is therefore the mean value of $\dfrac{n\,;}{a, b, c, \dots l\,;}$.

W_2 is $(-)^n B$, where B is the coefficient of $\dfrac{1}{t}$ in

$$\frac{1}{a\,b\,c \dots kl}\,\frac{1}{(1-e^{-at})(1-e^{-bt})\dots(1+e^{-kt})(1+e^{-lt})},$$

$a, b, \dots$ being the odd, and $\dots k, l$ the even, integers among $a, b, c, \dots k, l$.

If the first wave is A, $A+B$ will be the mean of $\dfrac{n\,;}{a, b, c, \dots l;}$ for even values of n, and $A-B$ the mean of the same for odd values of n.

Observe that the degree in n of W_1 is one unit less than the number of the elements $a, b, c, \dots l$; in the algebraical part of W_2 is one unit less than the number of even elements among $a, b, c, \dots l$, and in general W_q is one unit less than the number of elements which contain q as a factor.

Provisional notation co_{-1}, co_r explained.

The equations

$$co_l\,\phi(t) = co_{l+\omega}\,[t^{\omega}\,\phi(t)],\quad co_{2t}\,\phi(t^2) = co_t\,\phi(T)$$

identically true.

Mode of developing

$$\frac{(\rho e^t)^n}{[1-(\rho e^t)^{-a}]\,[1-(\rho e^t)^{-b}]\dots \text{to } i \text{ terms}},$$

under the form $\rho^n \,.\, e^{nt-R}$; where

$$R = \Sigma \log\,[1-(\rho e^t)^{-a}].$$

This an essential part of the theorem.

The expression for $\dfrac{1}{1-ke^u}$ in terms of u being known, $\log(1-ke^u)$ is known by integration from the identity

$$\frac{d}{du}\log(1-ke^u) = \frac{-ke^u}{1-ke^u} = 1-\frac{1}{1-ke^u},$$

so that in the first and second waves the only numerical constants to be determined are the numbers of Bernouilli.

Thus when $\rho = 1$, corresponding to W_1, R becomes

$$\Sigma \left\{\log(at) - \frac{at}{2} + \frac{B_1}{2^2} a^2t^2 - \frac{B_2}{2.3.4^2} a^4t^4 + \frac{B_3}{2.3.4.5.6^2} a^6t^6 - \&c.\right\}$$

$$= \log(a_1 a_2 \dots a_i) + i \log t - \tfrac{1}{2}\Sigma a \,.\, t + \frac{B_1}{2^2}(\Sigma a^2)\, t^2 - \frac{B_2}{2.3.4^2}(\Sigma a^4)\, t^4 \pm \&c.,$$

so that $nt - R$ becomes

$$-\log(a_1 a_2 \dots a_i) - i \log t + \left\{(n + \tfrac{1}{2}\Sigma a)\, t - \frac{B_1}{2^2}(\Sigma a^2)\, t^2 \pm \&c.\right\},$$

and
$$e^{nt-R} = \frac{t^{-i}}{a_1 a_2 \dots a_i} e^{\nu t - (B_1/2^2) s_2 t^2 + \&c.};$$

and, finally, $W_1 = \dfrac{1}{a_1 a_2 \dots a_i} co_{i-1} \{e^{\nu t - \frac{1}{24} s_2 t^2 + \frac{1}{2880} s_4 t^4 - \frac{1}{181440} s_6 t^6 \pm \&c.}\}$;

where
$$\nu = n + \tfrac{1}{2}\Sigma a.$$

In like manner, when $\rho = -1$, corresponding to W_2, if $a_1, a_2, \dots a_e$ are the even, and $b_1, b_2, \dots b_\omega$ the odd, elements,

$$R = \Sigma \log(1 - e^{-at}) + \Sigma \log(1 + e^{-bt}),$$

and we shall obtain

$$W_2 = \frac{(-)^n}{2^\omega a_1 a_2 \dots a_e} co_{e-1} \{e^{\nu t - \frac{1}{24}(s_2 + 3\sigma_2) + \frac{1}{2880}(s_4 + 15\sigma_4) \pm \&c.}\},$$

where
$$\nu = n + \tfrac{1}{2}(a_1 + a_2 + \dots + a_e + b_1 + b_2 + \dots + b_\omega),$$
$$s_l = \Sigma a^l, \quad \sigma_l = \Sigma b^l.$$

Calculation of Mean Values for any given Number of Elements.

Example 1. To find the mean value of $\dfrac{n\,;}{a, b, c, d\,;}$.

This will be the coefficient of t^3 in

$$\frac{1}{abcd}\{e^{\nu t} \times e^{-\frac{1}{24}s_2 t^2}\}$$

$$= \frac{1}{abcd} co_{+3} \left\{\begin{matrix} 1 + \nu t + \dfrac{\nu^2 t^2}{1.2} + \dfrac{\nu^3 t^3}{1.2.3} + \dots \\ \times \\ 1 - \dfrac{1}{24} s_2 t^2 \end{matrix}\right\}$$

$$= \frac{\nu}{abcd} co_{+1} \left\{\left(1 + \frac{\nu^2 t}{1.2.3}\right)\left(1 - \frac{1}{24} s_2 t_2\right)\right\}$$

$$= \frac{\nu}{abcd}\left\{\frac{\nu^2}{6} - \frac{s_2}{24}\right\}.$$

Example 2. To find the mean value of $\dfrac{n\,;}{a, b, c, d, e\,;}$.

This will be the coefficient of t^4 in

$$\frac{1}{abcde}\{e^{\nu t} \times e^{\frac{1}{24}s_2t^2} \times e^{\frac{1}{2880}s_4t^4}\}$$

$$= \frac{1}{abcde}\,co_{+2}\begin{Bmatrix} 1 + \dfrac{\nu^2}{1\,.\,2}\quad t + \dfrac{\nu^4}{1\,.\,2\,.\,3\,.\,4}\,t^2 \\ \times \\ 1 - \dfrac{1}{24}\quad s_2 t + \dfrac{1}{1152}\,{s_2}^2 t^2 \\ \times \\ 1 + \dfrac{1}{2880}\,s_4 t^2 \end{Bmatrix}$$

$$= \frac{1}{abcde}\left\{\frac{\nu^4}{24} - \frac{s_2}{24}\,\nu^2 + \left(\frac{{s_2}^2}{1152} + \frac{s_4}{2880}\right)\right\}.$$

Examples of Arithmetical Calculation of Simple Denumerants.

Example 1. To find the complete expression for $\dfrac{n\,;}{1, 2, 3\,;}$.

$$W_1 = \frac{1}{1\,.\,2\,.\,3}\,co_2\,\{e^{(n+3)\,t} \times e^{-\frac{14}{24}t^2}\}$$

$$= \frac{1}{1\,.\,2\,.\,3}\,co_1\left\{1 + \frac{(n+3)^2}{2}\,t\right\}\left\{1 - \frac{14}{24}\,t\right\}\ (\text{since } 1 + 4 + 9 = 14)$$

$$= \frac{1}{12}\left\{(n+3)^2 - \frac{7}{12}\right\};$$

$$W_2 = co_{-1}\frac{(-)^n\,e^{nt}}{(1+e^{-t})\,(1+e^{-3t})\,(1-e^{-2t})}$$

$$= \frac{(-)^n\,1}{8} + \frac{1}{8}\left\{\frac{n\,;}{2\,;} - \frac{n-1\,;}{2\,;}\right\},$$

$$W_3 = co_{-1}\,\Sigma\,\frac{\rho^n\,.\,e^{nt}}{(1-\rho^{-1}e^{-t})\,(1-\rho^{-2}e^{-2t})\,(1-e^{-3t})}\ (\text{where } \rho^2 + \rho + 1 = 0)$$

$$= \frac{1}{3}\left\{\frac{\rho^n}{(1-\rho)\,(1-\rho^2)} + \frac{\rho'^n}{(1-\rho')\,(1-\rho'^2)}\right\}$$

$$= \frac{1}{9}\,(\rho^n + \rho'^n)$$

$$= \frac{1}{9}\left\{2\,\frac{n\,;}{3\,;} - \frac{n-1\,;}{3\,;} - \frac{n+1\,;}{3\,;}\right\}.$$

Thus the complete analytical value of $\dfrac{n\,;}{1,2,3\,;}$ is

$$\frac{(n+3)^2}{12}-\frac{7}{144}+\frac{1}{8}\left\{\frac{n\,;}{2\,;}-\frac{n-1\,;}{2\,;}\right\}+\frac{1}{9}\left\{2\,\frac{n\,;}{3\,;}-\frac{n-1\,;}{3\,;}-\frac{n+1\,;}{3\,;}\right\}.$$

Since $$\frac{7}{144}+\frac{1}{8}+\frac{2}{9}=\frac{57}{144}<\frac{1}{2},$$

the arithmetical value of $\dfrac{n\,;}{1,2,3\,;}$ is the nearest integer to $\dfrac{(n+3)^2}{12}$, as had been early observed under a different form of statement by Mr De Morgan.

Example 2. The same process applied to $\dfrac{n\,;}{1,4,7\,;}$ will give

$$W_1=\frac{1}{2\times 4\times 7}\left\{\left(n+\frac{1+4+7}{2}\right)^2-\frac{1+16+49}{24}\right\}$$

$$=\frac{1}{56}\left\{(n+6)^2-\frac{11}{4}\right\},$$

$$W_2=\frac{(-)^n}{2^2\,.\,4}=\frac{1}{16}\left\{\frac{n\,;}{2\,;}-\frac{n-1\,;}{2\,;}\right\},$$

$$W_4=co_{-1}\,\Sigma\,\frac{i^n e^{nt}}{(1-i^{-1}e^{-t})(1-i^{-7}e^{-7t})(1-e^{-4t})}\ (\text{where } i^2+1=0)$$

$$=\frac{1}{4}\left\{\frac{i^n}{(1-i^3)(1-i)}+\frac{i'^n}{(1-i'^3)(1-i')}\right\}$$

$$=\frac{1}{8}\{i^n+i'^n\}$$

$$=\frac{1}{8}\left\{2\,\frac{n\,;}{4\,;}-2\,\frac{n-2\,;}{4\,;}\right\}$$

$$=\frac{1}{4}\left\{\frac{n\,;}{4\,;}-\frac{n-2\,;}{4\,;}\right\}.$$

$$W_7=co_{-1}\,\Sigma\,\frac{\theta^n e^{nt}}{(1-\theta^6e^{-t})(1-\theta^3e^{-nt})(1-e^{-7t})}$$

$$(\text{where } \theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1=0)$$

$$=\frac{1}{7}\,\Sigma\,\frac{\theta^n}{(1-\theta^3)(1-\theta^6)}$$

$$=\frac{1}{49}\,\Sigma\theta^n(1-\theta)(1-\theta^2)(1-\theta^4)(1-\theta^5)$$

$$=\frac{1}{49}\{\theta^n+\theta^{n+5}-2\theta^{n+1}-2\theta^{n+6}\}$$

$$=\frac{1}{7}\left\{\frac{n\,;}{7\,;}-2\,\frac{n+1\,;}{7\,;}-2\,\frac{n+4\,;}{7\,;}+\frac{n+5\,;}{7\,;}+2\,\frac{n+6\,;}{7\,;}\right\}.$$

Thus the complete value of $\dfrac{n\,;}{1,\,4,\,7\,;}$, expressed in terms exclusively of

$$\nu = n + 6,$$

is the following:—

$$\frac{1}{56}\left\{\nu^2 - \frac{11}{4}\right\} + \frac{1}{16}\left\{\frac{\nu\,;}{2\,;} - \frac{\nu-1\,;}{2\,;}\right\} + \frac{1}{4}\left\{\frac{\nu-2\,;}{4\,;} - \frac{\nu\,;}{4\,;}\right\}$$
$$+ \frac{1}{7}\left\{2\,\frac{\nu\,;}{7\,;} + \left(\frac{\nu+1\,;}{7\,;} + \frac{\nu-1\,;}{7\,;}\right) - \left(2\,\frac{\nu+2\,;}{7\,;} + 2\,\frac{\nu-2\,;}{7\,;}\right)\right\}.$$

The limiting values of the sum of the second, third, and fourth waves for any value of n will be

$$\frac{1}{16} + \frac{1}{4} + \frac{2}{7} = \frac{67}{112}$$

on the positive side, and

$$+\frac{1}{16} - \frac{1}{4} - \frac{2}{7} = \frac{-53}{112}$$

on the negative side.

Hence the difference between the exact value and $\dfrac{\nu^2}{56}$ must lie between $\dfrac{123}{224}$ and $\dfrac{-117}{224}$.

So that in the greatest number of cases the *nearest* integer to $\dfrac{(n+6)^2}{56}$ gives the value of $\dfrac{n\,;}{1,\,4,\,7\,;}$; and the result can never be in error by more than a single unit.

An analogous approximate form of representation can be made for the number of modes of composing an integer with any number of elements mutually prime to each other.

Observe in the foregoing expression that the form $\dfrac{\nu+i\,;}{q\,;}$ is always paired with $\dfrac{\nu-i\,;}{q\,;}$: and $\dfrac{\nu\,;}{q\,;}$ $\left(\text{which in the actual case under consideration is } \dfrac{\nu\,;}{7\,;}\right)$ affords no exception, for this may be expressed as

$$\frac{1}{2}\left\{\frac{\nu+0\,;}{q\,;} + \frac{\nu-0\,;}{q\,;}\right\}.$$

Neither does $\dfrac{\nu-r\,;}{2r\,;}$ $\left(\text{in the actual case } \dfrac{\nu-1\,;}{2\,;}\right)$, for this may be expressed as

$$\frac{1}{2}\left\{\frac{\nu-r\,;}{2r\,;} + \frac{\nu+r\,;}{2r\,;}\right\}.$$

The sign of the pairing may be positive or negative according to a rule which the exhibition of the result worked out in the following examples will render clear.

Example 3. The denumerant $\frac{n\,;}{1,\,3,\,5\,;}$, expanded in a similar manner, gives rise to the following expression:—

$$\frac{1}{30}\left\{(\nu)^2-\frac{35}{24}\right\}+\frac{2}{9}\left\{\frac{\nu+\frac{3}{2}\,;}{3\,;}+\frac{\nu-\frac{3}{2}\,;}{3\,;}-\frac{\nu+\frac{1}{2}\,;}{3\,;}-\frac{\nu-\frac{1}{2}\,;}{3\,;}\right\}$$
$$+\frac{1}{5}\left\{\frac{\nu+\frac{1}{2}\,;}{5\,;}+\frac{\nu-\frac{1}{2}\,;}{5\,;}-\frac{\nu+\frac{3}{2}\,;}{5\,;}-\frac{\nu-\frac{3}{2}\,;}{5\,;}\right\}.$$

And, since $$\frac{2}{9}+\frac{1}{5}+\frac{7}{144}=\frac{330}{720}<\frac{1}{2},$$

the arithmetical value of $\frac{n\,;}{1,\,3,\,5\,;}$ is always the *nearest integer* to $\frac{(2n+9)^2}{120}$.

This arithmetical mode of statement, how capable of extension to any set of elements following the natural order of the prime numbers, and to other cases.

Example 4. The denumerant $\frac{n\,;}{1,\,2,\,3,\,4,\,5,\,6,\,7\,;}$ in its expanded form is expressed by the following function of ν, which here represents $n+14$, namely,

$$\frac{1}{725760}\left\{\frac{\nu^6}{5}-35\nu^4+\frac{13419}{10}\nu^2-\frac{190325}{126}\right\}$$
$$+\frac{1}{768}\left\{\frac{\nu^2}{2}-\frac{77}{6}\right\}\left(\frac{\nu\,;}{2\,;}-\frac{\nu-1\,;}{2\,;}\right)$$
$$+\frac{1}{162}\left\{\frac{\nu+1\,;}{3\,;}-\frac{\nu-1\,;}{3\,;}\right\}\nu-\frac{5}{972}\left\{\left(\frac{\nu\,;}{3\,;}+\frac{\nu\,;}{3\,;}\right)-\left(\frac{\nu-1\,;}{3\,;}+\frac{\nu+1\,;}{3\,;}\right)\right\}$$
$$+\frac{1}{64}\left\{\left(\frac{\nu+2\,;}{4\,;}+\frac{\nu-2\,;}{4\,;}\right)-\left(\frac{\nu\,;}{4\,;}+\frac{\nu\,;}{4\,;}\right)\right\}$$
$$+\frac{1}{25}\left\{\left(\frac{\nu+1\,;}{5\,;}+\frac{\nu-1\,;}{5\,;}\right)-\left(\frac{\nu+2\,;}{5\,;}+\frac{\nu-2\,;}{5\,;}\right)\right\}$$
$$+\frac{1}{36}\left\{\left(\frac{\nu+3\,;}{6\,;}+\frac{\nu-3\,;}{6\,;}\right)-\left(\frac{\nu\,;}{6\,;}+\frac{\nu\,;}{6\,;}\right)+\left(\frac{\nu-2\,;}{6\,;}+\frac{\nu+2\,;}{6\,;}\right)-\left(\frac{\nu-1\,;}{6\,;}+\frac{\nu+1\,;}{6\,;}\right)\right\}$$
$$+\frac{1}{49}\left\{3\left(\frac{\nu\,;}{7\,;}+\frac{\nu\,;}{7\,;}\right)-\left(\frac{\nu+1\,;}{7\,;}+\frac{\nu-1\,;}{7\,;}\right)-\left(\frac{\nu+2\,;}{7\,;}+\frac{\nu-2\,;}{7\,;}\right)-\left(\frac{\nu+3\,;}{7\,;}+\frac{\nu-3\,;}{7\,;}\right)\right\}^{*}.$$

Observation. Provided that $\frac{-p\,;}{i\,;}$ shall be understood to signify the same thing as $\frac{p\,;}{i\,;}$, every wave in the above expansion remains entirely unaltered when ν becomes $-\nu$.

* The arithmetical value of $\frac{\nu-14\,;}{1,\,2,\,3,\,4,\,5,\,6,\,7\,;}$ is obviously the nearest integer to

$$\frac{1}{5760}\left\{\frac{\nu^6}{5}-35\nu^4+\frac{13419}{10}\nu^2\right\}+\frac{1}{1536}\left(\frac{\nu\,;}{2\,;}-\frac{\nu-1\,;}{2\,;}\right)\nu^2+\frac{1}{162}\left\{\frac{\nu+1\,;}{3\,;}-\frac{\nu-1\,;}{3\,;}\right\}\nu.$$

A priori view of the form of such expansions of $\dfrac{n\,;}{a, b, c, \ldots l\,;}$.

First, the algebraical part is, as it ought to be, a *homogeneous* function of n; $a, b, c, \ldots l$.

Secondly, the change of ν into $-\nu$ either leaves the expansion absolutely unaltered, or unaltered save as to algebraical sign.

This depends on the theory of the *denumerantive functions* as distinguishable from denumerants. The latter discontinuous quantities, the former continuous.

Binary denumerants have in general several functions attached to them, namely, one less than the number of their denominatives*.

All generated forms have arithmetical and functional values.

Example. The form u_n generated by

$$\frac{1}{(1-x)^2} = u_0 + u_1 x + \ldots u_n x^n + \&c.$$

Property of the denumerantive function $\phi(n, a, b, c, \ldots l)$ to $\dfrac{n\,;}{a, b, c, \ldots l\,;}$; namely, that

$$\phi(n, a, b, c, \ldots l) = \pm\, \phi(n', a, b, c, \ldots l),$$

if

$$n + n' = -a - b - c \ldots - l,$$

the + sign being used when the number of elements is odd, and the negative sign when it is even.

This explains the pairing of the terms observed in $\dfrac{n\,;}{1, 2, 3, \ldots 7\,;}$. Great importance of this fact of pairing.

The number of modes of dividing n into seven parts is represented by the above formula, namely, with repetitions and zero values of parts allowed by making

$$\nu = n + 14,$$

with repetitions and zero values disallowed by making

$$\nu = n - 14,$$

with repetitions allowed, but zero values disallowed, by making

$$\nu = n + 7.$$

And so in general with the values $n + \dfrac{(r-1)\,r}{4}$; $n - \dfrac{(r-1)\,r}{4}$; $n - r$ respectively substituted for ν.

Mr Kirkman's representation of partitions to the modulus 7.

* This is when the form zero is not counted as a function. Zero occurs as a form once only in simple, but twice over in binary denumerants.

Example 5. Expansion of $\dfrac{n:}{1, 2, 3, 4, 5, 6:}$ as a function of n.

$$\frac{1}{17280}\left\{\frac{\nu^5}{5}-\frac{91}{6}\nu^3+\frac{9191}{48}\right\}+\frac{1}{768}\left\{\frac{\nu+\frac{1}{2}:}{2:}-\frac{\nu-\frac{1}{2}:}{2:}\right\}\left(\nu^2-\frac{161}{12}\right)$$
$$+\frac{1}{162}\left\{\left[\left(\frac{\nu+\frac{3}{2}:}{3:}+\frac{\nu-\frac{3}{2}:}{3:}\right)-\left(\frac{\nu+\frac{1}{2}:}{3:}+\frac{\nu-\frac{1}{2}:}{3:}\right)\right]\nu+\left(\frac{\nu+\frac{1}{2}:}{3:}-\frac{\nu-\frac{1}{2}:}{3:}\right)\right\}$$
$$+\frac{1}{32}\left\{\left(\frac{\nu+\frac{3}{2}:}{4:}-\frac{\nu-\frac{3}{2}:}{4:}\right)+\left(\frac{\nu+\frac{1}{2}:}{4:}-\frac{\nu-\frac{1}{2}:}{4:}\right)\right\}$$
$$+\frac{1}{25}\left\{\left(\frac{\nu-\frac{1}{2}:}{4:}-\frac{\nu+\frac{1}{2}:}{4:}\right)+\left(\frac{\nu-\frac{3}{2}:}{4:}-\frac{\nu+\frac{3}{2}:}{4:}\right)\right\}$$
$$+\frac{1}{18}\left\{\left(\frac{\nu+\frac{3}{2}:}{6:}-\frac{\nu-\frac{3}{2}:}{6:}\right)+\frac{1}{36}\left(\frac{\nu+\frac{5}{2}:}{6:}-\frac{\nu-\frac{5}{2}:}{6:}\right)+\left(\frac{\nu+\frac{1}{2}:}{6:}-\frac{\nu-\frac{1}{2}:}{6:}\right)\right\},$$

where $$\nu=n+\frac{21}{2}.$$

Observe the substitution of the colon for the semicolon above and below the line in the fraction-form to distinguish a denumerantive function from a denumerantive proper. The arithmetical value of the foregoing is the nearest integer to the sum of its first, second, and third waves; and in the two latter it is only necessary to retain those terms which contain ν^2 and ν respectively.

On the expression for the number of waves when the denominatives of a denumerant or the elements of a partition are given.

On the blending of waves, and its advantages in some cases, as when the elements are all prime to each other without all being absolute primes.

Easy mode of deducing the fundamental theorem, by the application of a formula in the calculus of residues to the Eulerian. Its capital importance in the theory of partitions.

Close of the analytical portion of the course.

SEVENTH LECTURE*.

The representation of systems of linear equations by clusters of points recalled.

A single equation by a cluster of points in a line, of two simultaneous equations by a cluster of points in a plane, three simultaneous equations by a cluster of points in space—geometrical criterion between definite and indefinite systems.

The linear cluster which corresponds to a single equation is unique in form, not so the plane cluster which corresponds to two, or the solid cluster

* Delivered at King's College, London, July 11th, 1859.

which corresponds to three equations. One arbitrary parameter enters into the former, three into the latter.

All such clusters balance about their respective origins with the same weights at corresponding points.

Clusters so related might be termed homobaric.

Mechanical representation of the property of homobarism by a series of jointed parallelograms having two axes in common (see Fig. *A*). [Plate I. at the end of the volume.]

Criterion between definite and indefinite systems recalled.

"To determine the chance that three points thrown anywhere within a parallelogram may contain the centre."

Solution of this problem by theory of definite and indefinite binary equation-systems with three variables alluded to.

The chance is $\frac{5}{32}$ths against the points including the centre, whatever the form or dimensions of the parallelogram.

Hence the chance is $\frac{5}{32}$ths against three points capable of being taken anywhere in an indefinite plane including a *given* fourth point in the same plane.

But it would be incorrect to infer from this that the chance of some three out of four points (capable of being taken anywhere in an indefinite plane), including the fourth, is $\frac{5}{8}$; it will be much less than this.

Explanation of this seeming paradox. Geometrical and analytical modes of treating this second question alluded to.

Experimental method of verification. New game of odd and even.

The natural order of the variables in a single homonymous equation recalled.

Unless definite there is no natural order.

The importance of obtaining such natural orders to the theory of compound partition, namely, in applying the process of eduction. Example of natural and disturbed order.

General *analytical* condition of normal sequence.

Difficulty of seeing any natural order among the rays of a solid cluster; it will presently appear that such orders do exist, but that instead of one natural order there are several; the number depending (1) upon the number of points in the cluster, (2) upon the mode in which the rays are grouped, subject to the observation that the distinct modes of grouping in the view of this theory are always limited in number, and determinable *a priori*.

Passage by perspective from the grouping of rays in a plane to the grouping of points in a line; from the grouping of rays *in solido* to the grouping of points in a plane; and from the grouping of rays in plu-space to the grouping of points *in solido*.

Observe that in studying the character of an equation-system no attention need in the first instance be paid to the primary, because the process of eduction concerns the variables only, and not the constant terms in the system; by the act of taking the perspective, the origin no longer appears—thus, nothing is left but a perspective cluster or group of points, as many in number as the variables of the system.

Theorem. The number of classes of definite binary systems of linear equations for any given number of variables is *one*, because there is but one species of arrangement of a given number of points in a line. The number of classes of definite ternary systems of equations with r variables is the number of distinct modes of grouping together r points in a plane; the number of classes of definite quaternary systems of equations with r variables is the number of distinct modes of grouping together r points in space.

Observe the fact of space being made subservient through the method of perspective to systems of linear equations greater in number than the so-called dimensions of space.

In determining the natural order in a binary system, the perspective group may be substituted for the cluster, provided the line of projection cuts all the rays on the same side of the origin, so that a line through the origin parallel to the line of projection falls *outside* the cluster; but, if this condition is not observed, the order will be disturbed. (See Fig. *B*.) [Plate I. at the end of the volume.]

In like manner, for a ternary system, the plane of projection must be supposed to be drawn parallel to a plane through the origin external to the solid cluster.

Observation on Plane and Solid Groups, considered as representing definite Ternary and Quaternary Equation-systems respectively. If we suppose a ternary system of which one of the equations is of the form

$$x + y + z + \dots + u - 1 = 0,$$

the others being

$$ax + by + cz + \dots + lu - m = 0,$$

$$\alpha x + \beta y + \gamma z + \dots + \lambda u - \mu = 0,$$

such a system may evidently be represented by a group of points in a plane whose coordinates are

$$(a, \alpha)\,(b, \beta)\,(c, \gamma)\,\dots\,(l, \lambda);\ (m, \mu),$$

and

$$x, y, z, \dots u;\ -1,$$

will be the weights to be placed at these points respectively in order to *balance* each other.

Moreover, if we start with any *definite* ternary system, we may substitute for one of the equations in it a homonymous equation reducible to the form of

$$x' + y' + z' + \dots + u' - 1 = 0,$$

on taking $x', y', z', \dots u',$

all homonymous multiples of

$$x, y, z, \dots u.$$

Consequently, the *form* of the plexus of principal derivatives which depends essentially only on the relations of algebraical signs in the coefficients of these derivatives will be the same whether the system be considered as involving explicitly $x, y, z, \dots u$, or $x', y', z', \dots u'$, and, consequently, every definite ternary system of equations whatever may be represented in its essential properties of form by a plane group of points, in lieu of a solid cluster of rays. And in like manner, without going from hypersolidity to the solid, we see that any definite quaternary system may be represented by a group of points *in solido*.

The property indicated of the self-balancing group *in plano* being substitutable for the solid cluster with its centre of gravity at the origin may be deduced easily from this more general theorem, that if two groups of weighted points are in perspective about a given point G, and the weights at the corresponding points in the two groups are in the inverse ratios of their distances from G, if one of them has its centre of gravity at G, the other also will have its centre of gravity there. Hence, if one of these groups be considered as a derivative from the first, and all the points of the derivative group be brought to lie within the same plane, it must become self-balancing, since otherwise a plane group of statical points would have its centre of gravity outside the plane.

Notice that the geometrical construction for determining whether a system of equations is definite or indefinite would fail for a quaternary system, but the analytical method operative through the principal plexus continues to hold good.

Ternary Systems, and Plane Groups.

Imagine a sphere to be drawn with the origin of a solid cluster as its centre; the general arrangement of the points on the sphere will correspond to the arrangement of the points on the perspective plane, and, when convenient to do so, the one may be substituted for the other.

Illustration by examples with four and five points.

Recall eduction and the condition of its giving rise to definite systems, namely, that the systems deduced by the successive deformations of the given system shall remain definite, that is, external to the centre. If a group of points on a sphere be contained within the boundary of a hemisphere, the centre will be external to such group; but, if the bounding contour of the group formed by arcs of great circles cover more than half the sphere, the centre will be contained within the group. The effect upon the spherical perspective of a cluster representing a ternary system of equations due to the change of a variable x into $-x$ is to make the point corresponding to the coefficients of x pass to the opposite end of the diameter passing through it; such a change may be termed a *reversal* of the point, and the point so obtained the opposite of the original point. The problem of normal orders for ternary systems may therefore be stated geometrically as follows:—

A given number of points being contained within a hemisphere, to discover what orders of sequence of these points will possess this property that on the first, second, third, &c., to the last of them, one after the other undergoing *reversal*, the transformed group shall never occupy more than half the surface of the sphere.

From this it follows that, if

$$xyz \ldots tuv$$

be a normal sequence, $\qquad vut \ldots xyz$

will be so likewise, for, if we denote the opposite points to

$$xyz \ldots uv \text{ by } x'y'z' \ldots u'v',$$

respectively, it is clear that, if the groups

$$\left.\begin{array}{l} x'yz \;\ldots\; uv \\ x'y'z \;\ldots\; uv \\ x'y'z' \;\ldots\; uv \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ x'y'z' \;\ldots\; u'v \end{array}\right\}$$

are respectively contained, each within their own hemispheres, the groups

$$\left.\begin{array}{l} v'u \;\ldots\; zyx \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ v'u' \;\ldots\; zyx \\ v'u' \;\ldots\; z'yx \\ v'u' \;\ldots\; z'y'x \end{array}\right\}$$

will each also be contained in hemispheres opposite to the former, taken in reverse order.

The *contour* of a spherical group defined.

What is meant by a peripheral and what by an internal point to a group on a sphere.

Again, to obtain the law of normal sequences, we have the following propositions:—

(1) Any sequence $x, y, z, t, \ldots u, v, w$ of points in a sphere will be a normal order of sequence, provided the following condition is satisfied, namely, that, on joining those points with each other in the order of their succession by arcs of great circles, the broken line or spherical zigzag so formed shall be capable at every one of its angular points of being divided into two parts by a great circle which does not cut the line at any other point; for evidently in such case, if we draw a great circle through a point u, which does not cut any of the sides, or any other angle except u, the points x, y, z, t being all reversed, will lie together with v, w in a hemisphere bounded by the great circle so drawn.

(2) A normal sequence of points in a group cannot be bounded at either extremity by an interior point of the group. For, on joining the opposite of such exterior point with the closed figure surrounding that point, we evidently obtain a figure clasping the hemisphere bounded by the great circle perpendicular to the diameter through these points, and stretching into the hemisphere beyond. On the other hand, a normal order may always be commenced from any peripheral point in the group at pleasure, for, if u, v, y, z, x, t be any group contained within a hemisphere H, and u' a point in the contour, it is apparent that u, v, y, z, x, t, u' will also be contained within the same hemisphere H, so that, in fact, one way of characterizing a normal sequence would be as a sequence in which each point in turn becomes a peripheral point, alike when all the points preceding as when all the points following it are *reversed*.

(3) No arc joining y, z, two consecutive points in a normal sequence $x, y, z, t, u, v, w, \omega$, can cut uv any other such arc, for it is clear that, if yz crosses uv, x will be contained within the triangle $y'uv$ (y' meaning the opposite point to y), and, consequently, z will not be external to $tuvw\omega x'y'$, as it must be if the given order is normal.

(4) It follows also, as an immediate corollary from 2, that no point t can be contained within the contour of xyz or within that of $uvw\omega$.

Hence (5) it follows from (3) and (4) that the two spherical areas bounded respectively by the contours of xyz, $uvw\omega$ have no part whatever in common, and, consequently, may be separated by a great circle drawn through t.

Hence, combining the conclusions of (1) and (5), we arrive at the theorem that the sole necessary and sufficient condition for determining $x, y, z, t, u, v, w, \omega$ to be in normal sequence upon a sphere is that through any point as t a great circle can be drawn upon the sphere not cutting this line in any other point; and, consequently, the sole necessary and sufficient condition for a number of points in a plane group being in normal sequence is that the zigzag formed by drawing straight lines from any one point to the next in the sequence shall be capable of being cut in twain at any of its angles by a right line.

General definition of a diatomic line continuous or discontinuous in a *plane* or in *space* (see plate) [at the end of the volume].

The condition of normal sequence may be extended from plane to solid groups, that is, from ternary to quaternary equation-systems, the sole necessary and sufficient condition for determining a normal order of points, as well *in solido* as *in plano,* being that the zigzag following the succession of the points in the order shall be a *diatomic* line.

In order to depress a ternary system so as to make its denumerant depend upon binary denumerants, we must be able to form orders of normal sequence among its variables.

Every such order will or may furnish two distinct forms of solution, provided the requisite conditions of relative primeness and asyzygeticism are satisfied.

The easiest way of determining such normal orders is by means of diatomic lines drawn from point to point of the representative plane group.

Every ternary system may be identified by means of its principal plexus, as will presently be shown with some specific form of group, corresponding in number to the number of the variables in the system. It becomes necessary, therefore, to facilitate the solution of the problem of denumeration of ternary systems, to classify and register the distinct forms of arrangement of plane groups (and in like manner, in order to make denumerants of the fourth order depend upon those of the third, we must begin with classifying and registering the various dispositions of which a given number of points is susceptible in space).

Plane Groups.

For three points only one species of arrangement is possible, and all the orders are normal orders.

For four points two distinct arrangements only are possible, namely, of four points external to one another, or three points with a fourth point in the interior.

Morph defined—its geometrical and analytical meaning.

Exclusion of syzygetic cases.

The morph corresponding to the one case (see plate) [at the end of the volume] will be the following:—

$$xy : \quad yz : \quad zt : \quad tx :$$
$$x : z \quad y : t.$$

The morph corresponding to the other case (see plate) will be

$$\begin{matrix} zt: & xt: & yt: \\ y:z & z:x & x:y. \end{matrix}$$

Accordingly, the plexus of principal derivatives to every definite ternary system with four variables, omitting the constant terms, will be of one or the other of the two following forms:—

$$\left.\begin{matrix} ax+by=? & lx-mz=? \\ cy+dz=? & ny-pt=? \\ ez+ft=? & \\ gx+ht=? & \end{matrix}\right\} \text{1st form,}$$

$$\left.\begin{matrix} az+bt=? & \alpha y-\beta z=? \\ cx+dt=? & \gamma z-\delta x=? \\ ey+ft=? & \epsilon x-\phi y=? \end{matrix}\right\} \text{2nd form.}$$

The constants are of course not independent of each other, nor are the *signs* of the quantities left unexpressed, but symbolized by ?, independent. How to assign the laws of the dependence is not material to our immediate object.

By means of these two forms, we can by examination of the plexus of principal derivatives to a ternary system of linear equations with four variables, ascertain whether it corresponds to the form of group in Figs. 2, 3, or to that in Fig. 4.

The normal orders belonging to these groups respectively are shown in the plate [at the end of the volume].

If we proceed now to the groupings of five points, we shall find that the only essentially distinct arrangements are the following:—

Five points all external to each other.

Four points with one inside.

Three points with two inside.

No differences of any other kind than those above denoted will give rise to any difference in the character of the morph resulting. The three morphs corresponding to these three arrangements are the following:—

$$\left.\begin{matrix} ztu: & tux: & uxy: & xyz: & yzt: \\ zt:x & tu:y & ux:z & xy:t & yz:u \end{matrix}\right\} \text{1st form,}$$

$$\left.\begin{matrix} ztu: & xtu: & xyu: & yzu: \\ & zu:x & tu:y & \\ yz:t & xt:z & xy:t & xy:z \end{matrix}\right\} \text{2nd form,}$$

$$\left.\begin{matrix} ztu: & xtu: & ytu: \\ uz:y & uz:x & yu:x \\ yt:z & xt:z & xt:y \\ & xz:y & \end{matrix}\right\} \text{3rd form.}$$

And by simply observing whether the principal plexus has three, four, or five homonymous derivatives, the perspective representation of any definite ternary system of linear equations with five variables can be identified with one or the other of the Figures 5–7, 8–11, or 12–15. The diatomic zigzags expressing the normal orders to these figures are given in the plate.

So far we have found that the number of morphs has followed the progression 1 ; 2 ; 3 ; arising from the fact that there are no essential differences of position of a point or a couple of points within a triangle or of one point within a quadrilateral. Very different is the case for a figure of six points, corresponding to a definite ternary system, with six variables. The following cases arise :—

(1) The six points may be at the angles of a hexagon.

(2) Five of the six points may be at the angles of a pentagon, and the interior point may occupy any one of three essentially distinct regions, each such position giving rise to a distinct species of morph confined to the inner.

These three distinct kinds of position defined.

(3) Four of the points may be external, and the pair of points within occupy six distinct kinds of position.

These six different relative positions described.

Finally, three of the six points may be external, so that the figure may be viewed as a triangle within a triangle, and there will be six distinct relative positions of triangles so related.

These six different relative positions described.

There are thus in all sixteen different classes of arrangement of six points in a plane, and therefore sixteen different classes of ternary systems with six variables.

The normal orders for each of these sixteen cases have been completely worked out by Captain Noble, R.A., and the numbers for each figure attached to them in the plate (16–31).

On the arrangement of plane groups in natural families.

Classes belong to the same natural family which are capable of appearing simultaneously in the same course of the same process of eduction.

Any two classes of systems belong to the same family which may be obtained from one or other by altering the signs concurrently of one or more of such of the coefficient groups as may be so altered without the system ceasing to be definite. If in the morph of any class we make any letter pass from right to left of the colon, and *vice versa* throughout, and after such change the morph still contains characters of the form $xyz \ldots u$: corresponding to homonymous principal derivatives, the morph so obtained will belong to the same family with that from which it is derived. Thus from one morph all others of the same family may be derived by simple inspection and transposition.

Conversion defined.

Scales of derivation in general are divaricative, but for principal family are lineal.

Example four-, five-, and six-point systems.

How to determine *a priori* what letters in a given morph are convertible.

Examples in six-point systems.

The number of families for six-point arrangements is four.

The two classes of four-point systems, and the three classes of five-point systems, belong respectively to a single family. Proof.

Numerical and natural modes of classifying groups contrasted.

The principal class and principal family of any r-point group defined.

The tactical rule which serves to define any normal order in the principal class. Example in five-point system. Example in six-point system.

In four- and five-point systems all the classes belong to the principal family, there being no other.

The numerical system of arrangement in families gives rise to a new question in the partition of numbers.

Thus a seven-point system, and an eight-point system arranged after the numerical system, consist respectively of families which may be typified as follows :—

7 6, 1 5, 2 4, 3 3, 4 3, 3, 1

8 6, 2 5, 3 4, 4 4, 3, 1 3, 5 3, 4, 1 3, 3, 2

Theorem. All classes of the same family may be derived from one another by perspective projection.

Conversion balls and their use.

Theorem. Normal orders are orders of perspective sequence (see plate).

Application of perspective regions to finding normal orders by exhaustive method.

The position of the eye must be external to the group.

The entire plane outside the group may be divided into as many distinct perspective regions as there are normal orders (see plate).

A ship tacking along a diatomic zigzag is continually making angular way in reference to a point taken anywhere in some determinate region.

Normal orders of points in space are also orders of double perspective sequence, a line of view and planes of light being substituted for the point of view and rays of light.

Four-, five-, and six-point systems in space, like three-, four-, and five-point systems in planes, are reducible respectively to one, two, and three classes.

The classes of quaternary point-systems like those of ternary, and by the same method, may be arranged in natural families.

Reasons for believing a higher or more complex colligation of classes possible for quaternary systems.

Although the geometry of dispositions does not explicitly recognise distinctions grounded on magnitude, still the relations which it contemplates must admit of quantitative discrimination.

The *cæsura* in the *eduction* process following any normal order of the variables; how determined geometrically for ternary or quaternary systems by the principle of *denudation*; conformity of this with the rule for binary systems.

How the neutral region in a normal order which does not exist for binary arises for ternary and higher systems (see plate).

Example. The distances from each other of four points in a plane being given (six quantities connected by one equation) it must be possible to form one or more rational functions of these quantities of which the values as positive or negative must serve to discriminate between the two kinds of disposition in which four points may be grouped.

General character of the new geometry of disposition.

"It is the theory of permutation of space."—*Cayley.*