24.

ON THE PROBLEM OF THE VIRGINS, AND THE GENERAL THEORY OF COMPOUND PARTITION.

IN the *Opera Minora* of the great Euler, in the last page of his *second* memoir on the partition of numbers (Vol. I. p. 400), occur these words :— "Ex hoc principio definiri potest quot solutiones problemata quæ ab arithmeticis *ad regulam virginum* referri solent, admittunt ; hujusmodi problemata huc redeunt ut inveniri debeant numeri p, q, r, s, &c., ita ut his duabus conditionibus satisfiat,

$$ap + bq + cr + ds + \&\text{c.} = n, \quad \text{et} \quad \alpha p + \beta q + \gamma r + \delta s + \&\text{c.} = \nu \ ;$$

et jam quæstio est quot solutiones in numeris integris positivis locum sint habituræ ubi quidem tenendum est numeros a, b, c, $d \dots n$ et α, β, γ, $\delta \dots \nu$ esse integros" ; and he then proceeds to observe that the number in question is the coefficient of $x^m y^n$ in the expansion of the expression

$$\frac{1}{(1 - x^a y^\alpha)(1 - x^b y^\beta)(1 - x^c y^\gamma) \dots}$$

in terms of ascending positive powers of x and y.

Why the solution in integers of two simultaneous equations with an indefinite number of variables should be referred to "the rule of the Virgins" I am at a loss to conjecture, unless indeed it be supposed to have some mystical reference to the alligation or coupling of the coefficients of the two equations*. The problem in question may be otherwise stated as having

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* Professor De Morgan has kindly furnished me with the following information as to the use of this singular phrase :—

"I have seen this process cited as the rule of—Ceres, Series, Verginum, Virginum, Ceres and Virginum, Series and Virginum, Ceres and Verginum, Series and Verginum. I do not think any one of the eight is missing. I cannot find that Ceres is attended by any maidens, and I cannot guess who the ladies were. It is applied by the arithmeticians to the rule of *alligation* when of an indeterminate number of solutions—just Euler's problem which you quote." Mr De Morgan subsequently writes, "I forget whether they wrote Series or Ceries ; I think the latter" ; and adds a pleasant caution against indulging a passion for one of these algebraical virgins ; "for that though Jupiter did once animate a statue maiden at the prayer of an enamoured sculptor, yet even Jupiter himself could not impart a body to an algebraical abstraction."

for its object to discover the number of modes in which the couple m, n may be made up of the couples a, α; b, β; c, γ &c.

I need hardly remark that Euler's form of representation is no solution, but merely a transformation of the question. The problem in its most general form is to determine the number of modes in which a given set of conjoint partible numbers l_1, l_2, ... l_r can be made up simultaneously of the compound elements,

$$a_1, a_2, \ldots a_r; \quad b_1, b_2, \ldots b_r; \quad c_1, c_2, \ldots c_r; \text{ &c.}$$

The problem of simple partition has been already completely resolved by the author of this notice; but the resolution of the problem of double, and still more of multiple decomposition in general, seemed to be fenced round with insurmountable difficulties.

Let the reader imagine then with what surprise and joyful emotion, within a few days of despatching my previous paper on Partitions to this present Number of the Magazine, following out a train of thought suggested by the simple idea in that paper contained, I found myself led, as by a higher hand, to the marvellous discovery that the problem of compound partition in its utmost generality is capable of a complete solution—in a word, that this problem may in all cases be made to depend on that of simple partition. The theorem by which this is effected has been already confided to the great mathematical genius of England, and will be shortly committed to the 'Transactions' of one of our learned societies; for the present I shall confine myself to a disclosure of the general character of the theorem without going into any details. Thus, then, may the theorem be stated in general terms :—

Any given system of simultaneous simple equations to be solved in positive integers being proposed, the determination of the number of solutions of which they admit may in all cases be made to depend upon the like determination for one or more systems of equations of a certain fixed standard form. When a system of r *equations between* n *variables of the aforesaid standard form is given, the determination of the number of solutions in positive integers of which it admits may be made to depend on the like determination for*

$$\frac{n(n-1) \ldots (n-r+2)}{1 . 2 \quad \ldots \quad (r-1)}$$

single independent *equations derived from those of the given system by the ordinary process of elimination, with a slight modification; the final result being obtained by taking the sum of certain numerical multiples (some positive, others negative) of the numbers corresponding to those independent determinations. This process admits of being applied in a variety of modes, the resulting*

sum of course remaining unaltered in value whichever mode is employed, only appearing for each such mode made up of a different set of component parts*.

In the Problem of the Virgins, where but two equations are concerned, the equations are reduced to the standard form when the two coefficients of every the same variable in the two equations are prime to one another, and when no two pairs of coefficients have the same ratio; and for this problem the process is always limited to only two modes of application. The method, however, in a very important class of cases admits of being applied in *one*, and only *one* mode when these conditions are not strictly fulfilled.

Thus the virgins who appeared to Euler, but with their forms muffled and their faces veiled, have not disdained to reveal themselves to me under their natural aspect. Wonderful indeed has been the history of this theory of partitions. Notwithstanding that the immortal Euler had written two elaborate memoirs on the subject, that Paoli, and I believe other Italian mathematicians, had taken it up from another but less advantageous point of view, so completely had it fallen into oblivion, as far as the mathematicians of this country are concerned, that Sir John Herschel has written a memoir upon it, inserted in the *Philosophical Transactions*, without any reference to, and evidently in complete unconsciousness of, the labours of his predecessors, and subsequently Professor De Morgan, so justly celebrated for his mathematical erudition, in a paper in the *Cambridge and Dublin Mathematical Journal*, refers to the doctrine of partitions as being of quite recent creation. The importance of the subject in these later times has been vastly augmented by the magnificent applications which our great mathematical luminary has made of it to the doctrine of invariants.

* Since the above was in print, I have discovered a much more specific theorem, which, indeed, is to be regarded as the fundamental theorem in the doctrine of compound partition, and the basis of that given in the text. It is as follows :—*If there be* r *simultaneous simple equations between* n *variables (in which the coefficients are all positive or negative integers) forming a definite system (that is, one in which no variable can become indefinitely great in the positive direction without one or more of the others becoming negative), and if the* r *coefficients belonging to each of the same variable are exempt from a factor common to them all, and if not more than* r − 1 *of the variables can be eliminated simultaneously between the* r *equations, then the determination of the number of positive integer solutions of the given system may be made to depend on like determinations for each of* n *derived independent systems, in each of which the number of variables and equations is one less than in the original system.*

This reduction in general can be effected in a great but limited variety of modes. When only two equations, however, are concerned, the number of modes is always two, neither more nor less. So that in fact we are still navigating in the narrows, and have not fairly entered upon the wide ocean of the theory of compound partitions until we have passed the case of double partition. When the given system supposed definite is one of three equations between four variables, the number of modes of reduction is twelve or sixteen, according to that type out of two (to one or the other of which it must of necessity belong) under which the system falls. The theory of types applicable to any system of simultaneous simple equations with rational coefficients, here faintly shadowed forth, constitutes, I apprehend, a new and important branch in the theory of inequalities.

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Postscript. In the first instance I discovered the theorem above given by a method of induction, aided by an effort of imagination, and confirmed by numerous trials; but I have since obtained a very simple, although somewhat subtle general proof of it. Mr Cayley on his part, and independently, has also laid the foundation of a most ingenious and instructive method of demonstration entirely distinct from my own. I reason upon the equations, Mr Cayley upon the Eulerian generating function; but it was by operations performed upon this function that I was myself originally led to a perception of the transcendental analogies out of which I was enabled to evolve the law.

The very interesting case of the composition of a proposed integer out of elements given both in number and species, to which Euler has called particular attention, falls without preparation under the standard form; for this question is in fact merely that of determining the number of solutions of the binary system of equations,

$$ax + by + cz + \ldots + lw = m,$$
$$x + y + z + \ldots + w = \mu,$$

$a, b, c, \ldots l$ being supposed to be all different.

Thus, by way of very simple illustration, suppose it required to find in how many ways the number m can be made up of μ elements, limited to consist of the numbers 1, 2, 3. My method gives me at once the following solution. Call ν the number required. Then m must be not less than μ, and not greater than 3μ, or there will be no solutions. For all values of m between μ and 2μ, both inclusive,

$$\nu = \frac{m-\mu}{2} + \frac{3}{4} + (-1)^{\frac{m-\mu}{2}};$$

for all values of m between 2μ and 3μ, *still both inclusive*,

$$\nu = \frac{3\mu - m}{2} + \frac{3}{4} + (-1)^{\frac{m-\mu}{2}}.$$

It will be observed that when $m = 2\mu$, the two formulæ give the same value, so that either may be employed. Again, suppose we wish to express the number of modes of composition of m with the four elements 1, 2, 3, 4, the number of parts being μ, $\dfrac{m}{\mu}$ must be not less than 1 nor greater than 4, or there will be no solutions possible.

For all values of m from μ to 2μ inclusive,

$$\nu = \tfrac{1}{12}\left\{(m - \mu + 3)^2 - \tfrac{7}{6}\right\} + \tfrac{1}{8}(-1)^{m-\mu} + \tfrac{1}{9}(\rho^{m-\mu} + \rho'^{m-\mu}),$$

ρ, ρ' being the prime cube roots of unity.

For all values of m from 2μ to 3μ inclusive,

$$\nu = \frac{(m-\mu+3)^2}{12} - \frac{(m-2\mu+3)^2}{4} + \frac{73}{36}$$
$$+ \tfrac{1}{8}\{(-1)^{m-\mu} + (-1)^m\}$$
$$+ \tfrac{1}{9}\{\rho^{m-\mu} + \rho'^{m-\mu}\}.$$

Finally, for all values of m from 3μ to 4μ inclusive,

$$\nu = \tfrac{1}{12}\{(4\mu - m - 3)^2 - \tfrac{7}{6}\} + \tfrac{1}{8}(-1)^m + \tfrac{1}{9}(\rho^{m-\mu} + \rho'^{m-\mu}).$$

At the joining points (so to say) between the successive cases, viz. where $m = 2\mu$ or $m = 3\mu$, the contiguous formulæ give like results whichever of them is applied, so that the discontinuity in the form of the solution resembles that arising from the juxtaposition of different curves*. This discontinuity (in itself a remarkable phænomenon to be brought to light), far from being a reproach to the method employed, is to be regarded as a quality inherent in the subject matter under representation, and inexpugnable, as such, in the very nature of things.

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* The connexion between the contiguous formulæ is always closer than what is symbolized by the phrase used above. The curves must be regarded as not merely placed end to end, but to be, as it were, knit or spliced together through a certain finite portion of the extent of each of them. Thus the first and second formulæ in the text coincide [?] in value, not merely for $m = 2\mu$, but also for $m = 2\mu - 1$ and $m = 2\mu - 2$; and the second and third formulæ coincide, not merely for $m = 3\mu$, but also for $m = 3\mu + 1$ and $m = 3\mu + 2$. The adjacent curves have, so to say, in the instance above, the same tangents and circles of curvature at the points of union, so that we may be said to *modulate* from one formula into another. The *raison raisonnée* of this fact is easily explicable on *à priori* analytical principles.