12.

A TRIFLE ON PROJECTILES.

[*Philosophical Magazine*, XI. (1856), pp. 450—453.]

IN teaching the subject of projectiles *in vacuo*, the following solution has presented itself to me of a question not wholly without practical interest, namely, of determining the angle of projection to give the best range in the most general case, namely, when a gun is fired upon a slope at a given vertical height above the slope. The solution is not wholly either without theoretical interest in point of method, as leading to a result of some little complexity in maxima and minima by very simple calculations, and without the aid of the differential calculus. Therefore I venture to submit it to the readers of the *Philosophical Magazine*. In the next number of the Magazine I hope to have leisure to lay before them a subject of much greater interest, also belonging to the theory of projectiles, showing how, by the oblique action of gravity combined with the earth's rotation, a pendulum suitably adjusted may be caused to advance in a westerly direction, and so the earth be made the means of impelling a light carriage without any visible motive force, or any influence of magnetism.

To this pendulum I give, for reasons which will be apparent when the matter is more clearly set forth, and in contradistinction to the ordinary fixed or circular pendulum on the one hand, and to Foucault's free or spherical pendulum on the other, the name of the *Cylindrical* or *Travelling* Pendulum. But to resume the business of this present communication: let us begin with determining the angle of projection to give the maximum range when a gun is fired from a point *in* a plane sloping at an angle i from the horizon.

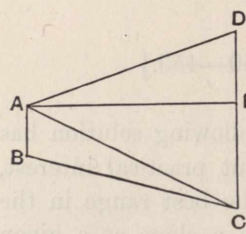
This question is most simply solved (the result itself is of course familiar to all who will read this paper) by resolving the velocity V, supposed to make an angle θ with the horizon, as also g, the accelerating force of gravity, each into two parts, V into $V\cos(\theta+i)$ and $V\sin(\theta+i)$, and g into $g\sin i$ and $g\cos i$, respectively parallel and perpendicular to the plane of the slope.

The time of flight is of course found by looking to the perpendicular part of the velocity and of gravity alone, and is evidently $2\dfrac{V\sin(\theta+i)}{g\cos i}$, which call τ; the range will evidently be

$$\frac{V\cos\theta\,.\,\tau}{\cos i}, \text{ that is, } \frac{V^2}{g\cos i}.\{\sin(2\theta+i)+\sin i\}.$$

Hence the best angle of range for this case is found by making $2\theta+i=90°$, $\theta=\frac{1}{2}(90°-i)$.

Now let us proceed to apply this result to the general case, as in the figure below, where BC is the slope upon which the range is to be measured, A the point of projection, AD the direction which gives the maximum range upon the slope, and BC the actual extent of this range; then I say AD is the direction which would give also the best range upon the slope AC. Since if, with the given velocity of projection, any other direction than AD would give a better range upon AC, the path corresponding to such direction must evidently cut BC at a point beyond C in that line in order to strike a point beyond C in the line AC.



Hence if we draw the horizontal line AE, we know by the preceding case that the angle $DAE=\frac{1}{2}CAB$*.

Let $CAB=\phi$, which is to be found; also let $AB=h$, and the inclination of BC to $AE=i$, h and i being given; and let t = time of flight, then

$$CAD=(90°-\phi)+\frac{\phi}{2}$$

$$=90°-\frac{\phi}{2}.$$

Hence also $$ADC=180°-\phi-\left(90°-\frac{\phi}{2}\right)=90°-\frac{\phi}{2}.$$

Hence $$\tfrac{1}{2}gt^2=CD=AC=h\frac{\sin ABC}{\sin ACB}$$

$$=\frac{h\cos i}{\cos(i+\phi)};$$

* This equation, and the isoscelism of the principal triangle of the figure to which it leads, would not readily present themselves to notice in the direct method of seeking the maximum range. It is for the sake of this pleasing geometrical relation, not unmixed perhaps with a desire of exhibiting the simple yet delicate turn of reasoning, the agreeable little point of method (a fly embalmed in amber) contained in the immediately preceding paragraph, that I have thought this trifle worth preserving in the pages of the Magazine.

and $$v\cos\frac{\phi}{2}t = AE = \frac{h\sin\phi\cos i}{\cos(i+\phi)}.$$

Hence eliminating t, we have

$$\frac{v^2}{gh\cos i} = \frac{(\sin\phi)^2}{1+\cos\phi}\cdot\frac{1}{\cos(i+\phi)} = \frac{1-\cos\phi}{\cos(i+\phi)}.$$

If $i=0$, that is, if the gun is fired from the top of a battery commanding a level plain, we have simply

$$\sec\phi = 1 + \frac{v^2}{gh},$$

which gives ϕ the double of the angle of elevation.

In other cases we may make $\phi + i = \psi$, we have then

$$\frac{1-\cos(\psi - i)}{\cos\psi} = \frac{1}{\cos\psi} - \frac{\sin\psi}{\cos\psi}\sin i - \cos i = \frac{v^2}{gh}\sec i.$$

Let $$\left(1+\frac{v^2}{gh}\sec^2 i\right)\cot i = \cot\epsilon;$$

then $$\frac{\sin i}{\sin\epsilon}\cos(\psi-\epsilon) = 1,$$

$$\cos(\psi-\epsilon) = \frac{\sin\epsilon}{\sin i},$$

or $$\cos(\phi + i - \epsilon) = \frac{\sin\epsilon}{\sin i},$$

from which ϕ, the double of the angle of elevation, may be determined.

Calling $\frac{\sin\epsilon}{\sin i} = \cos\mu$, and taking ϕ_1, ϕ_2 as the two values of ϕ, we have

$$\phi_1 + i - \epsilon = \mu,$$

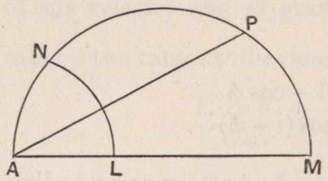
$$\phi_2 + i - \epsilon = 360^\circ - \mu.$$

ϕ_1, ϕ_2 correspond to the angles of projection *down and up* the slope respectively, the one affording what in an *algebraical* sense is a maximum, and the other a minimum, but of course, *arithmetically* speaking, both giving maximum values of the range.

Thus when $h=0$, so that $\sin\epsilon = 0$, $\mu = 90^\circ$, and $\frac{1}{2}(\phi_2 - \phi_1)$ is a right angle, as may easily be verified.

It may be worth while to exhibit the geometrical construction for the case of firing from a gun in position commanding a *horizontal plane*.

Let A be the position of the gun, LN a portion of a circle of radius AL which represents the height of the gun above the plain, LM twice the height due to the velocity of projection, ANM a semicircle on AM, P the point in it bisecting the arc MN, then (abstraction made of the resistance of the air) AP is the elevation at which the gun must be pointed to give the greatest range on the plain below, for sec $2PAM$ obviously $= 1 + \frac{(\text{velocity of ball})^2}{g\,.\,AL}$.



Suppose a sea battery as much as 300 feet* above the water, and a cannon-ball projected at the low rate of 1200 feet per second (which is less than that of a common musket-ball), we should have twice the height due to the velocity of projection equal to 44720, and therefore

$$\sec 2\alpha = \frac{44720}{1200} + 1$$

$$= 38{,}2666,$$

and consequently $$2\alpha = 88^\circ\ 30'\ 9''$$

or $$\alpha = 44^\circ\ 15'\ 5'',$$

differing very little from 45°; showing that certainly in a non-resisting medium, and in all probability in air, the height of the point of fire above the plane which it commands will very little indeed influence, under any conceivable circumstances of practice, the angle of elevation which gives the best range.

[* The succeeding calculation uses 1200.]